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Fixed Points of a Finite Family of I-Asymptotically Quasi-Nonexpansive Mappings in a Convex Metric Space

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Abstract. In this paper, we study Ishikawa iterative scheme with error terms for a finite family of I-asymptotically quasi-nonexpansive mappings in a convex metric space. We established strong convergence theorems and their applications for the proposed algorithms in a convex metric space. Our theorems improve and extend the corresponding known results in Banach spaces.

1. Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $J = \{1, 2, ..., r\}$ the set of first r natural numbers. Denote by $F(T)$ the set of fixed points of T and by $F := (\bigcap_{i=1}^{r} F(T_i)) \cap (\bigcap_{i=1}^{r} F(I_i))$ the set of common fixed points of two finite families of mappings $\{T_i : i \in J\}$ and $\{I_i : i \in J\}$.

Definition 1.1. *Let X be a metric space and $T : X \to X$ be a mapping. The mapping T is said to be:*

1. *Nonexpansive if*

$$d(Tx, Ty) \leq d(x, y) \text{ for all } x, y \in X.$$

2. *Quasi-nonexpansive if $F(T) \neq \emptyset$ and*

$$d(Tx, p) \leq d(x, p) \text{ for all } x \in X \text{ and } p \in F(T).$$

3. *Asymptotically nonexpansive [1] if there exists $u_n \in [0, \infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} u_n = 0$ such that*

$$d(T^n x, T^n y) \leq (1 + u_n) d(x, y) \text{ for all } x, y \in X \text{ and } n \in \mathbb{N}.$$

4. *Asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists $u_n \in [0, \infty)$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} u_n = 0$ such that*

$$d(T^n x, p) \leq (1 + u_n) d(x, p) \text{ for all } x \in X, \ \forall p \in F(T) \text{ and } n \in \mathbb{N}.$$

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Remark 1.2. *From the above definition, it follows that if $F(T)$ is nonempty, then a nonexpansive mapping is quasi-nonexpansive, and an asymptotically nonexpansive mapping is asymptotically quasi-nonexpansive. But the converse does not hold (see, for example, [1–4]). It is obvious that if T is nonexpansive, then it is asymptotically nonexpansive with the constant sequence* $\{0\}$.

There are many concepts which generalize a notion of asymptotically nonexpansive mapping in Banach space. One of such concepts is I-asymptotically nonexpansive mapping defined by Temir and Gul *[8, 9]*. Let us give metric version of these mappings.

Definition 1.3. *Let X be a metric space and $T, I : X \to X$ be two mappings. T is said to be*

1. *I-asymptotically nonexpansive if there exists a sequence $\{v_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} v_n = 0$ such that*

$$d(T^n x, T^n y) \leq (1 + v_n) d(I^n x, I^n y)$$

for all $x, y \in X$ and $n \geq 1$.

2. *I-asymptotically quasi nonexpansive if $F(T) \cap F(I) \neq \emptyset$ and there exists a sequence $\{v_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} v_n = 0$ such that*

$$d(T^n x, p) \leq (1 + v_n) d(I^n x, p)$$

for all $x \in X$ and $p \in F(T) \cap F(I)$ and $n \geq 1$.

3. *I-uniformly Lipschitz if there exists $\Gamma > 0$ such that*

$$d(T^n x, T^n y) \leq \Gamma d(I^n x - I^n y), \ x, y \in X \text{ and } n \geq 1.$$

Remark 1.4. *It is obvious that, an I-asymptotically nonexpansive mapping is I-uniformly Lipschitz with* $\Gamma = \sup\{1 + v_n : n \geq 1\}$ *and an I-asymptotically nonexpansive mapping with $F(T) \cap F(I) \neq \emptyset$ is I-asymptotically quasi nonexpansive. However, the converse of these claims are not true in general. It is easy to see that if I is identity mapping, then I-asymptotically nonexpansive mappings and I-asymptotically quasi nonexpansive mappings coincide with asymptotically nonexpansive mappings and asymptotically quasi nonexpansive mappings, respectively.*

In 1970, Takahashi [5] introduced the concept of convexity in a metric space (X, d) as follows.

Definition 1.5. *[5] A convex structure in a metric space (X, d) is a mapping $W : X \times X \times [0, 1] \to X$ satisfying, for all $x, y, u \in X$ and all $\lambda \in [0, 1]$,*

$$d(u, W(x, y; \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space together with a convex structure is called a convex metric space. A nonempty subset C of X is said to be convex if $W(x, y; \lambda) \in C$ for all $(x, y; \lambda) \in C \times C \times [0, 1]$.

Definition 1.5 can be extended as follows: A mapping $W : X^3 \times [0, 1]^3 \to X$ is said to be a convex structure on X, if it satisfies the following condition:

For any $(x, y, z; a, b, c) \in X^3 \times [0, 1]^3$ with $a + b + c = 1$, and $u \in X$,

$$d(u, W(x, y, z; a, b, c)) \leq a d(u, x) + b d(u, y) + c d(u, z).$$

If (X, d) is a metric space with a convex structure W, then (X, d) is called a convex metric space.

Let (X, d) be a convex metric space. A nonempty subset C of X is said to be convex if $W(x, y, z; a, b, c) \in C$, $\forall (x, y, z) \in C^3$, $\forall (a, b, c) \in [0, 1]^3$ with $a + b + c = 1$.

It is easy to prove that every linear normed space is a convex metric space with a convex structure $W(x, y, z; a, b, c) = ax + by + cz$, for all $x, y, z \in X$ and $a, b, c \in [0, 1]$ with $a + b + c = 1$. But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [5] and, Gunduz and Akbulut [6]).

In 2009, Temir [8] introduced an iteration process for a finite family of I-asymptotically nonexpansive mappings in Banach space as follows.

Let K be a nonempty subset of X Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of I_i-asymptotically nonexpansive self-mappings and $\{I_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive self-mappings of K. Let $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1]$. Then the sequence $\{x_n\}$ is generated as follows:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n I_{i(n)}^{k(n)} y_n \\ y_n = (1-\beta_n)x_n + \beta_n T_{i(n)}^{k(n)} x_n \end{cases} \quad n \geq 1, \tag{1}$$

where $n = (k(n)-1)N + i(n)$, $i(n) \in \{1, 2, \ldots, N\}$.

Now, we transform iteration process (1) with error terms for a finite family of I-asymptotically quasi-nonexpansive mappings in convex metric spaces as follows:

Definition 1.6. *Let (X,d) be a convex metric space with convex structure W, $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings. Suppose that $\{u_n\}$ and $\{v_n\}$ are two bounded sequences in X and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\hat{\alpha}_n\}, \{\hat{\beta}_n\}$, $\{\hat{\gamma}_n\}$ are six sequences in $[0,1]$ such that $\alpha_i + \beta_n + \gamma_n = 1 = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n$ for $n \in \mathbb{N}$. For any given $x_1 \in X$, iteration process $\{x_n\}$ defined by,*

$$\begin{aligned} x_{n+1} &= W\left(x_n, I_i^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \\ y_n &= W\left(x_n, T_i^n x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\right), \ n \geq 1, \end{aligned} \tag{2}$$

where $n = (k-1)r + i$, $i = i(n) \in J$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Thus, (2) can be expressed in the following form:

$$\begin{aligned} x_{n+1} &= W\left(x_n, I_{i(n)}^{k(n)} y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), \\ y_n &= W\left(x_n, T_{i(n)}^{k(n)} x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\right), \ n \geq 1. \end{aligned}$$

Our purpose in the rest of the paper is to use the iteration process (2) to prove some strong convergence results for approximating common fixed points of a finite family of I-asymptotically quasi-nonexpansive mappings and a finite family of asymptotically quasi-nonexpansive mappings in a convex metric space.

In the sequel, we shall need the following lemma and proposition.

Lemma 1.7. *[7] Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying*

$$\sum_{n=0}^{\infty} b_n < \infty, \ \sum_{n=0}^{\infty} c_n < \infty, \ a_{n+1} = (1+b_n)a_n + c_n, \ n \geq 0.$$

Then

i) $\lim_{n\to\infty} a_n$ *exists,*

ii) if $\liminf_{n\to\infty} a_n = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Remark 1.8. *[10] It is easy to verify that Lemma 1.7 (ii) holds under the hypothesis* $\limsup_{n\to\infty} a_n = 0$ *as well. Therefore, the condition (ii) in Lemma 1.7 can be reformulated as follows:*

ii)′ if either $\liminf_{n\to\infty} a_n = 0$ *or* $\limsup_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Proposition 1.9. *Let (X,d) be a convex metric space with convex structure W, $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F := (\bigcap_{i=1}^r F(T_i)) \cap (\bigcap_{i=1}^r F(I_i)) \neq \emptyset$. Then, there exist a point $p \in F$ and sequences $\{k_n\}, \{l_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = \lim_{n\to\infty} l_n = 0$ such that*

$$d\left(T_i^n x, p\right) \leq (1+k_n) d\left(I_i^n x, p\right) \text{ and } d\left(I_i^n x, p\right) \leq (1+l_n) d(x,p)$$

for all $x \in K$, for each $i \in I$.

Proof. Since $\{T_i : i \in J\} : X \to X$ is a finite family of I-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ is a finite family of asymptotically quasi-nonexpansive mappings with $F := (\bigcap_{i=1}^r F(T_i)) \cap (\bigcap_{i=1}^r F(I_i)) \neq \emptyset$, there exist $p \in F$ and sequences $\{k_{in}\}$, $\{l_{in}\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_{in} = \lim_{n\to\infty} l_{in} = 0$ for each $i \in J$ such that

$$d\left(T_i^n x, p\right) \leq (1 + k_{in}) d\left(I_i^n x, p\right) \text{ and } d\left(I_i^n x, p\right) \leq (1 + l_{in}) d(x, p)$$

for each $x \in X$. Let $k_n = \max\{k_{in} : i \in J\}$ and $l_n = \max\{l_{in} : i \in J\}$. So, we have that $\{k_n\}, \{l_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = \lim_{n\to\infty} l_n = 0$. Hence, there exist $p \in F$ and $\{k_n\}, \{l_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = \lim_{n\to\infty} l_n = 0$ such that

$$d\left(T_i^n x, p\right) \leq (1 + k_n) d\left(I_i^n x, p\right) \text{ and } d\left(I_i^n x, p\right) \leq (1 + l_n) d(x, p)$$

for all $x \in K$, for each $i \in J$. □

2. Main Results

Lemma 2.1. *Let (X, d, W) be a convex metric space with convex structure W, $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\{x_n\}$ is as in (2) with $\{\gamma_n\}, \{\hat{\gamma}_n\}$ satisfying $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \hat{\gamma}_n < \infty$. If $\lim_{n\to\infty} d(x_n, F) = 0$ where $d(x, F) = \inf\{d(x, p) : p \in F\}$, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Let $p \in F$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in X, there exists $M > 0$ such that

$$\max\left\{\sup_{n\geq 1} d(u_n, p), \sup_{n\geq 1} d(v_n, p)\right\} \leq M.$$

Then we have from Proposition 1.9 and (2) that

$$\begin{aligned} d(y_n, p) &= d\left(W\left(x_n, T_i^n x_n, v_n; \hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n\right), p\right) \\ &\leq \hat{\alpha}_n d(x_n, p) + \hat{\beta}_n d\left(T_i^n x_n, p\right) + \hat{\gamma}_n d(v_n, p) \\ &\leq \hat{\alpha}_n d(x_n, p) + \hat{\beta}_n (1 + k_n) d\left(I_i^n x_n, p\right) + \hat{\gamma}_n M \\ &\leq \hat{\alpha}_n d(x_n, p) + \hat{\beta}_n (1 + k_n)(1 + l_n) d(x_n, p) + \hat{\gamma}_n M \\ &\leq \left(1 + \hat{\beta}_n (k_n + l_n + k_n l_n)\right) d(x_n, p) + \hat{\gamma}_n M \end{aligned} \tag{3}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d\left(W\left(x_n, I_i^n y_n, u_n; \alpha_n, \beta_n, \gamma_n\right), p\right) \\ &\leq \alpha_n d(x_n, p) + \beta_n d\left(I_i^n y_n, p\right) + \gamma_n d(u_n, p) \\ &\leq \alpha_n d(x_n, p) + \beta_n (1 + l_n) d(y_n, p) + \gamma_n M. \end{aligned} \tag{4}$$

Substituting (3) into (4),

$$\begin{aligned} d(x_{n+1}, p) &\leq \alpha_n d(x_n, p) + \beta_n (1 + l_n) d(y_n, p) + \gamma_n M \\ &\leq \alpha_n d(x_n, p) + \beta_n (1 + l_n)\left(1 + \hat{\beta}_n (k_n + l_n + k_n l_n)\right) d(x_n, p) + \beta_n (1 + l_n) \hat{\gamma}_n M + \gamma_n M \\ &\leq \alpha_n d(x_n, p) + \beta_n (1 + l_n) d(x_n, p) + \beta_n (1 + l_n) \hat{\beta}_n (k_n + l_n + k_n l_n) d(x_n, p) + (\beta_n (1 + l_n) \hat{\gamma}_n + \gamma_n) M \\ &\leq \left[1 + \beta_n l_n + \beta_n \hat{\beta}_n (1 + l_n)(k_n + l_n + k_n l_n)\right] d(x_n, p) + (\beta_n (1 + l_n) \hat{\gamma}_n + \gamma_n) M. \end{aligned}$$

Thus we obtain

$$d(x_{n+1}, p) \leq [1 + \kappa_n] d(x_n, p) + t_n \tag{5}$$

where $\kappa_n = \beta_n l_n + \beta_n \hat{\beta}_n (1 + l_n)(k_n + l_n + k_n l_n)$ and $t_n = (\beta_n (1 + l_n)\hat{\gamma}_n + \gamma_n) M$ with $\sum_{n=1}^{\infty} \kappa_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$. Hence, we have

$$d(x_{n+1}, F) \leq [1 + \kappa_n] d(x_n, F) + t_n \tag{6}$$

It follows from (6) and Lemma 1.7 that the $\lim_{n\to\infty} d(x_n, F)$ exists.

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence. In fact, since $\sum_{n=1}^{\infty} \kappa_n < \infty$, $1 + x \leq e^x$ for all $x \geq 0$, and (5), therefore we have

$$d(x_{n+1}, p) \leq \exp\{\kappa_n\} d(x_n, p) + t_n. \tag{7}$$

Hence, for any positive integers n, m, from (7) it follows that

$$\begin{aligned} d(x_{n+m}, p) &\leq \exp\{\kappa_{n+m-1}\} d(x_{n+m-1}, p) + t_{n+m-1} \\ &\leq \exp\{\kappa_{n+m-1}\}\left[\exp\{\kappa_{n+m-2}\} d(x_{n+m-2}, p) + t_{n+m-2}\right] + t_{n+m-1} \\ &= \exp\{\kappa_{n+m-1}\}\exp\{\kappa_{n+m-2}\} d(x_{n+m-2}, p) \\ &\quad + \exp\{\kappa_{n+m-1}\} t_{n+m-2} + t_{n+m-1} \\ &\leq \cdots \\ &\leq \exp\left\{\sum_{i=n}^{n+m-1} \kappa_i\right\} d(x_n, p) + \exp\left\{\sum_{i=n}^{n+m-1} \kappa_i\right\}\sum_{i=n}^{n+m-1} t_i \\ &\leq Q d(x_n, p) + Q \sum_{i=n}^{n+m-1} t_i, \end{aligned}$$

where $Q = \exp\left\{\sum_{i=n}^{n+m-1} \kappa_i\right\} < \infty$.

Since $\lim_{n\to\infty} d(x_n, F) = 0$ and $\sum_{n=1}^{\infty} t_n < \infty$, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that

$$d(x_n, F) < \frac{\varepsilon}{4(Q+1)}, \ \sum_{n=1}^{\infty} t_n < \frac{\varepsilon}{2Q}, \ \forall n \geq n_0.$$

Therefore there exists $p_1 \in F$ such that

$$d(x_n, p_1) < \frac{\varepsilon}{2(Q+1)}, \ \forall n \geq n_0.$$

Consequently, for any $n \geq n_0$ and for all $m \geq 1$ we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(x_n, p_1) \leq (1+Q) d(x_n, p_1) + Q \sum_{n=1}^{\infty} t_n \\ &\leq \frac{\varepsilon}{2(Q+1)}(1+Q) + Q\frac{\varepsilon}{2Q} = \varepsilon. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in X. □

Theorem 2.2. *Let (X, d, W) be a convex metric space with convex structure W, $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\{x_n\}$ is as in (2) with $\{\gamma_n\}$, $\{\hat{\gamma}_n\}$ satisfying $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \hat{\gamma}_n < \infty$. Then*

(*i*) $\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0$ *if $\{x_n\}$ converges to a unique point in F.*

(*ii*) *$\{x_n\}$ converges to a unique fixed point in F if X is complete and either* $\liminf_{n\to\infty} d(x_n, F) = 0$ *or* $\limsup_{n\to\infty} d(x_n, F) = 0$.

Proof. (i) Let $p \in F$. Since $\{x_n\}$ converges to p, $\lim_{n\to\infty} d(x_n, p) = 0$. So, for a given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$d(x_n, p) < \varepsilon \qquad \forall n \geq n_0.$$

Taking infimum over $p \in F$, we have

$$d(x_n, F) < \varepsilon \qquad \forall n \geq n_0.$$

This means $\lim_{n\to\infty} d(x_n, F) = 0$ so that

$$\liminf_{n\to\infty} d(x_n, F) = \limsup_{n\to\infty} d(x_n, F) = 0.$$

(ii) Suppose that X is complete and $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$. Then, we have from Lemma 1.7 (ii) and Remark 1.8 that $\lim_{n\to\infty} d(x_n, F) = 0$. From the completeness of X and Theorem 2.1, we get that $\lim_{n\to\infty} x_n$ exists. Put $\lim_{n\to\infty} x_n = q \in X$, we will prove that $q \in F$.

For any given $\varepsilon_1 > 0$, there exists a constant n_1 such that for all $n \geq n_1$, we have

$$d(x_n, q) < \frac{\varepsilon_1}{2(2+l_1)} \text{ and } d(x_n, F) < \frac{\varepsilon_1}{2(4+3l_1)}. \tag{8}$$

In particular, there exists a $s \in F$ and a constant $n_2 \geq n_1$ such that

$$d(x_{n_2}, s) < \frac{\varepsilon_1}{2(4+3l_1)} \tag{9}$$

For any I_i, $i \in J$, we obtain from (8) and (9) that

$$\begin{aligned} d(I_iq, q) &\leq d(I_iq, s) + d(s, I_ix_{n_2}) + d(I_ix_{n_2}, s) + d(s, x_{n_2}) + d(x_{n_2}, q) \\ &= d(I_iq, s) + 2d(I_ix_{n_2}, s) + d(s, x_{n_2}) + d(x_{n_2}, q) \\ &\leq (1+l_1)d(q, s) + 2(1+l_1)d(x_{n_2}, s) + d(s, x_{n_2}) + d(x_{n_2}, q) \\ &\leq (2+l_1)d(x_{n_2}, q) + (4+3l_1)d(x_{n_2}, s) \\ &\leq (2+l_1)\frac{\varepsilon_1}{2(2+l_1)} + (4+3l_1)\frac{\varepsilon_1}{2(4+3l_1)} = \varepsilon_1. \end{aligned}$$

Since ε_1 is arbitrary, so $d(I_iq, q) = 0$ for all $i \in J$; i.e., $I_iq = q$. This implies $q \in \bigcap_{i=1}^{k} F(I_i)$. Similarly, $q \in \bigcap_{i=1}^{k} F(T_i)$. Therefore, $q \in F$. □

3. Applications

Now, we give some applications of Theorem 2.2.

Theorem 3.1. *Let (X, d, W) be a complete convex metric space with convex structure W, $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\{x_n\}$ is as in* (2) *with $\{\gamma_n\}$, $\{\hat{\gamma}_n\}$ satisfying $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \hat{\gamma}_n < \infty$. Assume that the following two conditions hold.*

$$i)\ \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{10}$$

ii) the sequence $\{y_n\}$ in X satisfying $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ implies

$$\liminf_{n\to\infty} d(y_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(y_n, F) = 0. \tag{11}$$

Then $\{x_n\}$ converges to a unique point in F.

Proof. Using (10) and (11), we get

$$\liminf_{n\to\infty} d(x_n, F) = 0 \text{ or } \limsup_{n\to\infty} d(x_n, F) = 0.$$

Therefore, we obtain from Theorem 2.2 (ii) that the sequence $\{x_n\}$ converges to a unique point in F. □

Theorem 3.2. *Let (X, d, W) be a complete convex metric space with convex structure W, $\{T_i : i \in J\} : X \to X$ be a finite family of I_i-asymptotically quasi-nonexpansive mappings and $\{I_i : i \in J\} : X \to X$ be a finite family of asymptotically quasi-nonexpansive mappings satisfying $\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, I_i x_n) = 0$ with $F \neq \emptyset$. Suppose that $\sum_{n=1}^{\infty} k_n < \infty$, $\sum_{n=1}^{\infty} l_n < \infty$ and $\{x_n\}$ is as in (2) with $\{\gamma_n\}$, $\{\hat{\gamma}_n\}$ satisfying $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \hat{\gamma}_n < \infty$. If one of the following is true, then the sequence $\{x_n\}$ converges to a unique point in F.*

i) If there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$, $g(t) > 0$ for all $t \in (0, \infty)$ such that either $d(x_n, T_i x_n) \geq g(d(x_n, F))$ or $d(x_n, I_i x_n) \geq g(d(x_n, F))$ for all $n \geq 1$ (See Condition A' of Khan and Fukhar-ud-din [11]).

ii) There exists a function $f : [0, \infty) \to [0, \infty)$ which is right continuous at 0, $f(0) = 0$ and $f(d(x_n, T_i x_n)) \geq d(x_n, F)$ or $f(d(x_n, I_i x_n)) \geq d(x_n, F)$ for all $n \geq 1$.

Proof. First assume that (i) holds. Then

$$\lim_{n\to\infty} g(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, T_i x_n) = 0 \text{ or } \lim_{n\to\infty} g(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, I_i x_n) = 0.$$

Thus, $\lim_{n\to\infty} g(d(x_n, F)) = 0$; and properties of g imply $\lim_{n\to\infty} d(x_n, F) = 0$.

Now all the conditions of Theorem 2.2 are satisfied, therefore $\{x_n\}$ converges to a point of F.

Next, assume (ii) holds. In this case,

$$\lim_{n\to\infty} d(x_n, F) \leq \lim_{n\to\infty} f(d(x_n, T_i x_n)) = f\left(\lim_{n\to\infty} d(x_n, T_i x_n)\right) = f(0) = 0.$$

or

$$\lim_{n\to\infty} d(x_n, F) \leq \lim_{n\to\infty} f(d(x_n, I_i x_n)) = f\left(\lim_{n\to\infty} d(x_n, I_i x_n)\right) = f(0) = 0.$$

From above inequalities, $\lim_{n\to\infty} d(x_n, F) = 0$. Thus $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$. By Theorem 2.2, $\{x_n\}$ converges to a point of F. □

Remark 3.3. *Our theorems generalize and improve the corresponding results of Temir [8] (i) from Banach space setting to the general setup of convex metric space (ii) from Ishikawa iterative scheme to Ishikawa iterative scheme with error terms (iii) from a finite family of I_i-asymptotically nonexpansive mappings to a finite family of I_i-asymptotically quasi-nonexpansive mappings.*

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