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Generalized Invertibility in a Corner Ring

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Abstract. Let R be a ring with unity and let $a, g \in R$ be such that a is regular. In this article, the generalized invertibility of $ag + 1 - aa^-$ are investigated in term of the generalized invertibility of elements in a corner ring. As applications, several equivalent conditions on the Drazin invertibility of product and difference of idempotents are obtained. Moreover, we present the equivalent conditions for the existence of Moore-Penrose inverse in a ring with involution.

1. Introduction

Throughout this paper, R is an associative ring with unity. Given an element $a \in R$, a is (von Neumann) regular if there exists $b \in R$ such that $a = aba$. In this case, the element b is called an inner inverse of a and we will denote it by a^-. By $a\{1\} = \{b \in R : aba = a\}$ we denote the set of all inner inverses of a. Let $*$ be an involution (anti-isomorphism of degree 2) on R. That is, the involution satisfies $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in R$. If x satisfies $axa = a$ and $(ax)^* = ax$, then x is a $\{1,3\}$-inverse of a. If y satisfies $aya = a$ and $(ya)^* = ya$, then y is a $\{1,4\}$-inverse of a. The standard notions of group, Drazin and Moore-Penrose inverse can be referred to the literature [5, 10]. From now on, $R^\#$, R^D and $R^\dagger$ stand for the set of all group invertible elements, the set of all Drazin invertible elements and the set of all Moore-Penrose invertible elements of R, respectively.

A motivation for this research appeared in [7]. There, the authors investigated the (Drazin) invertibility of $ag + 1 - aa^-$ for $a, g \in R$ when a is regular. If we set $e = aa^-$ and $b = ag$, then

$$t = ag + 1 - aa^- = eb + 1 - e. \tag{1}$$

In [9], the relation between generalized invertible elements of eRe and $eRe + 1 - e$ was obtained. It should be stressed that the set

$$eRe + 1 - e = \{exe + 1 - e : x \in R\},$$

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is a (multiplicative) semigroup. The subrings of the form eRe are called corner rings. In section 3, the Drazin and group invertibility of $ag + 1 - aa^-$ are investigated in term of the generalized invertibility of elements in a corner ring. As applications, several equivalent conditions on the Drazin invertibility of product and difference of idempotents are obtained. In section 4, we consider the Moore-Penrose invertibility in a corner ring. Moreover, we present the equivalent conditions for the existence of Moore-Penrose inverse in a ring with involution.

2. Preliminaries

In this section, we will introduce some lemmas which will play an important role in the forthcoming section. Let $e \in R$ be an idempotent. The group U_e of e-units in the corner ring eRe is given by $U_e = \{exe : exeR = eR, Rexe = Re\}$. We can link elements in U_e and invertible elements in $eRe + 1 - e$.

Lemma 2.1. *[1] Let R be a ring with unity and $e \in R$ be an idempotent. Then, for all $x \in R$,*

$$exe \in U_e \text{ if and only if } exe + 1 - e \text{ is invertible if and only if } ex + 1 - e \text{ is invertible.}$$

Lemma 2.2. *[9] Let $a \in R$ and $e \in R$ be an idempotent. Then the following statements are equivalent:*
(i) eae is Drazin invertible in eRe.
(ii) $eae + 1 - e$ is Drazin invertible in R.

Lemma 2.3. *[9] Let R be a ring with involution $*$, and let $a, e \in R$ be such that $e^2 = e^* = e$. Then, for all $x \in R$, the following statements are equivalent:*
(i) eae is Moore-Penrose invertible in eRe.
(ii) $eae + 1 - e$ is Moore-Penrose invertible in R.

Lemma 2.4. *(i) [2, Theorem 3.6]*[Jacobson lemma] *Let $a, b \in R$. If $1 - ab$ is (group) Drazin invertible with $ind(1 - ab) = k$, then $1 - ba$ is (group) Drazin invertible with $ind(1 - ba) = k$ and*

$$(1 - ba)^D = 1 + b((1 - ab)^D - (1 - ab)^\pi r)a,$$

where $r = \sum_{i=0}^{k-1}(1 - ab)^i$.
(ii) [4, Cline's Formula] *Let $a, b \in R$ and ab is Drazin invertible. Then ba is Drazin invertible too and $(ba)^D = b((ab)^D)^2 a$.*

3. Drazin Invertibility in a Corner Ring

Patricio in [7, Theorem 3.1] have considered the (Drazin) invertibility of the element $ag + 1 - aa^-$ when a is regular. In what follows, we provide new proofs of some results in [7] in term of the Drazin invertibility of elements in a corner ring. It is well known that $x \in R$ is Drazin invertible if and only if $x^k \in x^{k+1}R \cap Rx^{k+1}$ for some $k \in \mathbb{N}^+$, where $\mathbb{N}^+$ denote the set of all positive integer numbers.

Theorem 3.1. *[7, theorem 3.1] Let $a, g \in R$ be such that a is regular with an inner inverse a^-. The element $ag + 1 - aa^-$ is Drazin invertible in R if and only if $(ag)^k a \in (ag)^{k+1}aR \cap R(ag)^{k+1}a$ for some $k \in \mathbb{N}^+$.*

Proof. In view of (1.1), Lemma 2.2 and Lemma 2.4, one can see that $ag + 1 - aa^-$ is Drazin invertible in R if and only if ebe is Drazin invertible in eRe.

As a matter of fact, ebe is Drazin invertible in eRe if and only if $(ebe)^k \in (ebe)^{k+1}Re \cap eR(ebe)^{k+1}$ for some $k \in \mathbb{N}^+$. We note that

$$(ebe)^k \in (ebe)^{k+1}Re \cap eR(ebe)^{k+1} \text{ if and only if } (ag)^k a \in (ag)^{k+1}aR \cap R(ag)^{k+1}a.$$

Indeed, if $(ebe)^k \in (ebe)^{k+1}Re \cap eR(ebe)^{k+1}$, there exist $x, y \in R$ such that $(ebe)^k = (ebe)^{k+1}xe = ey(ebe)^{k+1}$. That is,

$$(ag)^k aa^- = (ag)^{k+1}aa^- xaa^- = aa^- y(ag)^{k+1}aa^-.$$

Premultiplication by a gives $(ag)^k a = (ag)^{k+1}aa^- xa = aa^- y(ag)^{k+1}a$, and thus,

$$(ag)^k a \in (ag)^{k+1}aR \cap R(ag)^{k+1}a.$$

Conversely, if $(ag)^k a \in (ag)^{k+1}aR \cap R(ag)^{k+1}a$, then

$$(ag)^k aa^- \in (ag)^{k+1}aRa^- \cap R(ag)^{k+1}aa^-.$$

It gives $(ebe)^k \in (ebe)^{k+1}eaRa^- \cap R(ebe)^{k+1}$, and then $(ebe)^k \in (ebe)^{k+1}R \cap R(ebe)^{k+1}$. This shows that $(ebe)^k \in (ebe)^{k+1}Re \cap eR(ebe)^{k+1}$, as desired. □

As we known, if $a \in R$ is regular, then $a + 1 - aa^-$ is invertible if and only if a is group invertible (See [9]). From Theorem 3.1, set $g = 1$, we obtain the following corollary.

Corollary 3.2. *Let $a \in R$ be regular with an inner inverse a^-. Then $a + 1 - aa^- \in R^D$ if and only if $a \in R^D$.*

Lemma 3.3. *[9] Let $a \in R$ and $e \in R$ be an idempotent. Then the following statements are equivalent:*
(i) eae is group invertible in eRe.
(ii) $eae + 1 - e$ is group invertible in R.

Remark 3.4. *It is worth to mention that, if $e \in R$ be an idempotent, eae is group invertible in eRe if and only if eae is group invertible in R. From Lemma 3.3 and Lemma 2.4 (i), if a is regular and let a^- be an arbitrary inner inverse of a, we set $e = aa^-$, then one can obtain that*

$$a^2a^- \in R^\sharp \Longleftrightarrow a^2a^- + 1 - aa^- \in R^\sharp \Longleftrightarrow a + 1 - aa^- \in R^\sharp$$

Next, we will give a counter example to show that $a^2a^- \in R^\sharp \not\Leftrightarrow a \in R^\sharp$. It also implies that $a + 1 - aa^- \in R^\sharp \not\Leftrightarrow a \in R^\sharp$.

Example 3.5. *Set $a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Note that $a^2 = 0$ and then a^2a^- is group invertible. Choose $a^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and it is easy to check that $s = a + 1 - aa^- = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ is not invertible in R, this leads to $a \notin R^\sharp$.*

In view of Corollary 3.2 and Remark 3.4, one can see that

$$a^2a^- \in R^\sharp \Longleftrightarrow a + 1 - aa^- \in R^\sharp \Longrightarrow a \in R^D$$

But next example show that the converse is not true, in general.

Example 3.6. *Set $a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to check that $a^3 = 0$ and $a^D = 0$. We can choose $a^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and thus, $x = a^2a^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Choose $x^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then $x + 1 - xx^- = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is not invertible in R, this leads to $x \notin R^\sharp$, that is, $a^2a^- \notin R^\sharp$.*

As an application, in what follows, p and q always mean two arbitrary idempotents in a ring R. In [3, proposition 3.1], several equivalent conditions on the Drazin invertibility of $1 - pq$ are given. As a matter of fact, it is a direct consequence of Lemma 2.2. Firstly, it is easy to check that $prp \in (pRp)^D$ if and only if $prp \in R^D$ for any $r \in R$.

Theorem 3.7. *The following statements are equivalent:*
(1) $1 - pq \in R^D$, (2) $p - pq \in R^D$, (3) $p - qp \in R^D$, (4) $1 - pqp \in R^D$, (5) $p - pqp \in R^D$,
(6) $1 - qp \in R^D$, (7) $q - qp \in R^D$, (8) $q - pq \in R^D$, (9) $1 - qpq \in R^D$, (10) $q - qpq \in R^D$.

Proof. Note that item (5) $p - pqp \in R^D$ if and only if $p(1-q)p \in (pRp)^D$. By Lemma 2.2, it is equivalent to $p(1-q)p + 1 - p \in R^D$, that is, (4) $1 - pqp \in R^D$ holds. By Lemma 2.4 (i), it is equivalent to (1) $1 - pq \in R^D$. For item (2) and (3), since (5) $p(1-q)p \in R^D$ holds, we can check them directly by Lemma 2.4 (ii). Similarly, we obtain that (6), (7), (8), (9) and (10) hold. □

In [3, Theorem 3.4], it is proven that $p(p-q)p \in R^D$ if and only if $p - q \in R^D$. In the following, we extend the result to the $p(p-q)^n p \in R^D$ case. We need to give some elementary and known results which play an important role in the next theorem.

Lemma 3.8. *[5] Let $a \in R$. Then a is Drazin invertible if and only if a^m is Drazin invertible for some (any) integer m.*

Lemma 3.9. *[5] Let $a, b \in R^D$ with $ab = ba$. Then $ab \in R^D$ and $(ab)^D = b^D a^D$.*

Theorem 3.10. *The following statements are equivalent:*
(i) $p - q \in R^D$.
(ii) $p(p-q)^n p \in R^D$ *for any* $n \geqslant 1$.
(iii) $p(p-q)^n p \in R^D$ *for some* $n \geqslant 1$.

Proof. Let us first observe that $p(p-q)^2 = p - pqp = (p-q)^2 p$. Then we have $p(p-q)^{2k} = (p-q)^{2k} p$ for any integer $k \geqslant 1$. Thus, we claim that

$$p(p-q)^{2k-1}p = p(p-q)^{2k}p. \tag{2}$$

We now proceed by induction on k. If $k = 1$, then it is clear that $p(p-q)^2 = p(p-q)p = (p-q)^2 p$. Assume that the result is true for some $k \geqslant 1$. This implies that $p(p-q)^{2k-1}p = p(p-q)^{2k}p$, and thus

$$\begin{aligned} p(p-q)^{2k+1}p &= p(p-q)^{2k-1}(p-q)^2 p = p(p-q)^{2k-1}p(p-q)^2 p \\ &= p(p-q)^{2k}p(p-q)^2 p = p(p-q)^{2k+2}p. \end{aligned}$$

$(i) \Rightarrow (ii)$ By Lemma 3.8, $p - q \in R^D$ if and only if $(p-q)^k \in R^D$ for any $k \geqslant 1$. From (3.1) and Lemma 3.9, we obtain $p(p-q)^{2n-1}p = (p-q)^{2n}p = p(p-q)^{2n} = p(p-q)^{2n}p$ is Drazin invertible. This implies that $p(p-q)^n p \in R^D$ whenever n is odd or even integer.

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (i)$ By (3.1), there exists an even number n such that $p(p-q)^n p \in R^D$. This gives that $p(p-q)^{2k}p \in R^D$ for some $k \in \mathbb{N}$. Note that

$$(p-q)^2 = p(1-q) + q(1-p).$$

Set $A = p(1-q)$ and $B = q(1-p)$. By $Bp = 0$ and $AB = BA = 0$, it is easy to get

$$p(p-q)^{2k}p = p(A^k + B^k)p = pA^k p = A^k p.$$

From Lemma 2.4 (ii) and Lemma 3.8, we have $p(p-q)^{2k}p \in R^D \Longleftrightarrow A \in R^D$. By Theorem 3.7, one can see

$$A \in R^D \Longleftrightarrow B \in R^D.$$

Then $A + B \in R^D$ and $(A+B)^D = A^D + B^D$ since $AB = BA = 0$. That implies that $(p-q)^2 \in R^D$ and $p - q \in R^D$. □

4. Moore-Penrose Invertibility in a Corner Ring

In what follows, R denotes an associate ring with unity and involution $*$. Moore-Penrose invertibility in a corner ring is considered in this section. Moreover, we present the equivalent conditions for the existence of Moore-Penrose inverse in R.

Lemma 4.1. *[7, corollary 3.1] Let $a, g \in R$ be such that a is regular with an inner inverse a^-. Then $ag + 1 - aa^-$ is invertible if and only if $a \in agaR \cap Raga$.*

Proof. Set $b = ag$ and $e = aa^-$. From Lemma 2.1, it implies that $ag + 1 - aa^-$ is invertible if and only if $e \in ebeR \cap Rebe$. That is, $aa^- \in agaa^-R \cap Ragaa^-$. We claim that $aa^- \in agaa^-R \cap Ragaa^-$ is equivalent to $a \in agaR \cap Raga$. Indeed,
" $\Rightarrow$ " $aa^- = agaa^-x = yagaa^-$ for some $x, y \in R$. Then $a = agaa^-xa = yaga$, so $a \in agaR \cap Raga$.
" $\Leftarrow$ " $a = agat = saga$ for some $s, t \in R$. Then $aa^- = agaa^-ata^- = sagaa^-$, so $aa^- \in agaa^-R \cap Ragaa^-$. □

Proposition 4.2. *Let $a \in R$ be regular. Then the following statements are equivalent:*
(i) $a \in R^\dagger$.
(ii) $a \in aa^*aR \cap Raa^*a$.
(iii) $a \in aa^*R \cap Ra^*a$.

Proof. $(i) \Leftrightarrow (ii)$ Note that $a \in R^\dagger$ if and only if $u = 1 + aa^* - aa^-$ is an unit of R, where a^- is an arbitrary inner inverse of a (See [6, Theorem 1.1]). From Lemma 4.1, it is easy to get $a \in R^\dagger$ if and only if $a \in aa^*aR \cap Raa^*a$.

$(ii) \Rightarrow (iii)$ It is clear.

$(iii) \Rightarrow (ii)$ There exist $r_1, r_2 \in R$ such that $a = aa^*r_1 = r_2a^*a$. It implies that $a = a(r_2a^*a)^*r_1 = aa^*ar_2^*r_1$ and $a = r_2(aa^*r_1)^*a = r_2r_1^*aa^*a$. So, we have $a \in aa^*aR \cap Raa^*a$. □

In the following, we give new characterizations for an element a to have Moore-Penrose inverse.

Theorem 4.3. *Let $a \in R$. The following statements are equivalent:*
(i) $a \in R^\dagger$.
(ii) $a \in aa^*aR$.
(iii) $a \in Raa^*a$.

Proof. $(i) \Rightarrow (ii)$ See Proposition 4.2.

$(ii) \Rightarrow (i)$ Since $a \in aa^*aR$, there exists $x \in R$ such that $a = aa^*ax$ and $a^* = x^*a^*aa^*$. Note that

$$a^*ax = (x^*a^*aa^*)ax = (x^*a^*aa^*ax)^* = (a^*ax)^*$$

It gives that $a = aa^*ax = a(a^*ax)^* = ax^*a^*a \in aRa$. So, we obtain that a is regular. Meanwhile, from $a = ax^*a^*a$, one can see that $a = ax^*a^*a = ax^*(x^*a^*aa^*)a \in Raa^*a$. By Porposition 4.2, we get $a \in R$ is Moore-Penrose invertible.

$(i) \Leftrightarrow (iii)$ The proof is similar to $(i) \Leftrightarrow (ii)$. □

Remark 4.4. *If $a \in aa^*R$ (or $a \in Ra^*a$), then $a \in R$ is $\{1,4\}$-invertible (or $\{1,3\}$-invertible). Form $a \in aa^*R$, there exists $x \in R$ such that $a = aa^*x$. Then $x^*a = x^*aa^*x$, this gives that $(x^*a)^* = x^*a$. So, we have $a = aa^*x = a(x^*a)^* = ax^*a$.*

Proposition 4.5. *Let a be $\{1,3\}$-invertible and $a^{(1,3)}$ be a $\{1,3\}$-inverse of a. Then the following are equivalent:*
(i) $w = agaa^{(1,3)} + 1 - aa^{(1,3)} \in R^\dagger$.
(ii) $aga \in Ragaa^{(1,3)}(ag)^*aga$.

Proof. Set $ag = b$ and $aa^{(1,3)} = e$. Then the element

$$w = agaa^{(1,3)} + 1 - aa^{(1,3)} = ebe + 1 - e.$$

By Lemma 2.3, $w \in R^\dagger$ if and only if $ebe \in (eRe)^\dagger$. As a matter of fact, when $e^2 = e = e^*$, $ebe \in (eRe)^\dagger$ if and only if $ebe \in R^\dagger$. By Theorem 4.3, it implies that $agaa^{(1,3)} \in Ragaa^{(1,3)}(ag)^*agaa^{(1,3)}$. It is equivalent to $aga \in Ragaa^{(1,3)}(ag)^*aga$. □

Recall that an element $a \in R$ is called EP [11], if $a \in R^\dagger \cap R^\#$ and $a^\dagger = a^\#$. Hence, we get

Corollary 4.6. *Let a be $\{1,3\}$-invertible and $a^{(1,3)}$ be a $\{1,3\}$-inverse of a. Then the following are equivalent:*
(i) $w = aa^* + 1 - aa^{(1,3)} \in R^\dagger$
(ii) aa^* *is EP.*

Proof. Set $g = a^*$ in item (ii) in Proposition 4.5. Then we can get $aa^*a \in Raa^*aa^*aa^*a$. Postmultiply by $a^{(1,3)}$, we get $aa^* \in Raa^*aa^*aa^*$. Note that $aa^* \in R^\#$ if and only if $aa^* \in Raa^*aa^*aa^*$. Moreover, we obtain that aa^* is EP by [8, Proposition 2]. □

Recall from [12] that a ring R is said to be $*$-reducing if, for any element $a \in R$, $a^*a = 0$ implies $a = 0$. Note that R is $*$-reducing if and only if the following implications hold for any $a \in R$: $a^*ax = a^*ay \Rightarrow ax = ay$ and $xaa^* = yaa^* \Rightarrow xa = ya$.

Remark 4.7. *Under the condition of corollary 4.6, if R is $*$-reducing, then we obtain that $w = aa^* + 1 - aa^{(1,3)} \in R^\dagger$ if and only if $a \in R^\dagger$. Set $x = a^*(aa^*)^\dagger$. Now $(xa)^* = [a^*(aa^*)^\dagger a]^* = a^*(aa^*)^\dagger a = xa$; $(ax)^* = aa^*(aa^*)^\dagger$ is self-adjoint; $xax = a^*(aa^*)^\dagger aa^*(aa^*)^\dagger = a^*(aa^*)^\dagger = x$. Finally, $axaa^* = aa^*(aa^*)^\dagger aa^* = aa^*$, and since R is $*$-reducing, we get $axa = a$.*

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References

[1] N. Castro-González, J.L. Chen, L. Wang, Further results on generalized inverses in rings with involution, Electron. J. Linear Algebra, 30 (2015) 118–134.
[2] N. Castro-González, C. Mendes-Araújo, P. Patrício, Generalized inverses of a sum in rings, Bull. Aust. Math. Soc., 82 (2010) 156–164.
[3] J.L. Chen, H.H. Zhu, The Drazin invertibility of product and difference of idempotents in a ring, Filomat, 28 (2014) 1133–1137.
[4] R. E. Cline, An application of representation for the generalized inverse of a matrix, MRC Technical Report 592, 1965.
[5] M.P. Drazin, Pseudo-inverses in associative rings and semigroups, Am. Math. Monthly, 65 (1958) 506–514.
[6] P. Patrício, The Moore-Penrose inverse of a companion matrix, Linear Algebra Appl., 437 (2012) 870–877.
[7] P. Patrício, R.E. Hartwig, Some regular sums. Linear Multilinear Algebra, 63 (2015) 185–200.
[8] P. Patrício, R. Puystjens, Drazin-Moore-Penrose invertibility in rings, Linear Algebra Appl., 389 (2004) 159–173.
[9] P. Patrício, R. Puystjens, Generalized invertibility in two semigroups of a ring, Linear Algebra Appl., 377 (2004) 125–139.
[10] R. Penrose, A generalized inverse for matrices, Proc. Camb. phil. Soc., 51 (1955) 406–413.
[11] J.J. Koliha, Elements of C^*-algebras commuting with their Moore-Penrose inverse, Studia Math., 139 (2000) 81–90.
[12] J.J. Koliha, P. Patrício, Elements of rings with equal spectral idempotents, J. Aust.Math. Soc., 72 (2002) 137–152.