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Sherman, Hermite-Hadamard and Fejér like Inequalities for Convex Sequences and Nondecreasing Convex Functions

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Abstract. In this paper, we prove Sherman like inequalities for convex sequences and nondecreasing convex functions. Thus we develop some results by S. Wu and L. Debnath [19]. In consequence, we derive discrete versions for convex sequences of Petrović and Giaccardi's inequalities. As applications, we establish some generalizatons of Fejér inequality for convex sequences. We also study inequalities of Hermite-Hadamard type. Thus we extend some recent results of Latreuch and Belaïdi [8]. In our considerations we use some matrix methods based on column stochastic and doubly stochastic matrices.

1. Introduction and Preliminaries

In this expository section we collect some relevant notation, terminology and facts in majorization theory and convex analysis.

Definition 1.1 ([9, p. 8]). Let $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ be two given sequences in $\mathbb{R}^m$. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ (written as $\mathbf{y} \prec \mathbf{x}$), if the sum of j largest entries of $\mathbf{y}$ does not exceed the sum of j largest entries of $\mathbf{x}$ for all $j = 1, 2, \ldots, m$ with equality for $j = m$.

That is, $\mathbf{y} \prec \mathbf{x}$ if

$$\sum_{i=1}^{j} y_{[i]} \le \sum_{i=1}^{j} x_{[i]} \quad \text{for } j = 1, 2, \ldots, m, \qquad \text{and} \qquad \sum_{i=1}^{m} y_{[i]} = \sum_{i=1}^{m} x_{[i]}.$$

Here $x_{[i]}$ and $y_{[i]}$ stand for the ith largest entry of $\mathbf{x}$ and $\mathbf{y}$, respectively.

Definition 1.2 ([12, p. 11], [4, pp. 72-73]). A function $f : I \to \mathbb{R}$ is said to be *convex* on an interval $I \subset \mathbb{R}$, if

$$f(\alpha x + (1 - \alpha)y) \le \alpha f(x) + (1 - \alpha) f(y) \quad \text{for } x, y \in I, \, \alpha \in [0, 1].$$

A function $f : I \to \mathbb{R}$ is said to be *concave* if $-f$ is convex.

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For properties and applications of convex functions, consult recent books [4, 9, 12].

Now we present some auxilary results. The following Hardy-Littlewood-Pólya-Karamata Theorem for a convex function is of great importance.

Theorem A. [9, pp. 92-93], [19, Lemma 1] *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in I^m$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in I^m$.*

Then

$$\mathbf{y} \prec \mathbf{x} \quad \textit{implies} \quad \sum_{i=1}^{m} f(y_i) \leq \sum_{i=1}^{m} f(x_i). \tag{1}$$

See e.g. [3, 6] for further inequalities for convex functions.

Definition 1.3 ([9, pp. 29-30]). An $m \times k$ real matrix $\mathbf{S} = (s_{ij})$ is called *column stochastic* if $s_{ij} \geq 0$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, k$, and all column sums of $\mathbf{S}$ are equal to 1, i.e., $\sum\limits_{i=1}^{m} s_{ij} = 1$ for $j = 1, 2, \ldots, k$.

An $m \times m$ real matrix $\mathbf{S} = (s_{ij})$ is called *doubly stochastic* if $s_{ij} \geq 0$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, m$, and all row and column sums of $\mathbf{S}$ are equal to 1, i.e., $\sum\limits_{j=1}^{m} s_{ij} = 1$ for $i = 1, 2, \ldots, m$, and $\sum\limits_{i=1}^{m} s_{ij} = 1$ for $j = 1, 2, \ldots, m$.

The next result is Birkhoff Theorem which gives a matrix characterization of majorization.

Theorem B. [9, p. 33] *Let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$ and $\mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbb{R}^m$. Then $\mathbf{y}$ is majorized by $\mathbf{x}$ if and only if there exists an $m \times m$ doubly stochastic matrix $\mathbf{S}$ such that*

$$(y_1, y_2, \ldots, y_m) = (x_1, x_2, \ldots, x_m)\mathbf{S}.$$

A more general result is Sherman Theorem for convex functions, as follows.

Theorem C. [2, 5, 18] *Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in I^m$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in I^k$, $\mathbf{a} = (a_1, a_2, \ldots, a_m) \in \mathbb{R}_+^m$ and $\mathbf{b} = (b_1, b_2, \ldots, b_k) \in \mathbb{R}_+^k$.*

If

$$\mathbf{y} = \mathbf{x}\mathbf{S} \quad \textit{and} \quad \mathbf{a} = \mathbf{b}\mathbf{S}^T \tag{2}$$

for some $m \times k$ column stochastic matrix $\mathbf{S}$, then

$$\sum_{j=1}^{k} b_j f(y_j) \leq \sum_{i=1}^{m} a_i f(x_i). \tag{3}$$

If f is concave, then the inequality (3) is reversed.

Remark 1.4. The main requirement (2) in Theorem C amounts to the notion of *weighted majorization* for pairs $(\mathbf{x}, \mathbf{a})$ and $(\mathbf{y}, \mathbf{b})$ (see [5]).

The reader is referred to [15] for applications of Sherman inequality.

The first example is to show the merits of Sherman inequality (3).

Example 1.5. Let $f : I \to \mathbb{R}$ be a convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ with $x_1, x_2, \ldots, x_n \in I$. Put $k = 1$ and $m \in \mathbb{N}$. Consider any $\lambda_i \geq 0$, $i = 1, 2, \ldots, m$, with $\sum\limits_{i=1}^{m} \lambda_i = 1$.

Let $\mathbf{S}$ be the $m \times 1$ column stochastic matrix of the form

$$\mathbf{S} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{pmatrix}$$

and $\mathbf{b}$ be the 1×1 matrix of the form

$$\mathbf{b} = (1).$$

Then it follows from (2) that

$$\mathbf{y} = \mathbf{x}\mathbf{S} = (\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m)$$

and

$$\mathbf{a} = \mathbf{b}\mathbf{S}^T = (1)(\lambda_1, \lambda_2, \ldots, \lambda_m) = (\lambda_1, \lambda_2, \ldots, \lambda_m).$$

In this situation Sherman inequality (3) reduces to the following Jensen inequality

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_m x_m) \le \lambda_1 f(x_1) + \lambda_2 f(x_2) + \ldots + \lambda_m f(x_m).$$

In what follows we also deal with *Fejér inequality* (4) for convex functions (see [1, 7, 10]).

Theorem D. [1] *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$ with $a < b$, and let $p : [a, b] \to \mathbb{R}$ be a non-negative integrable weight on I. Assume that p is symmetric about $\frac{a+b}{2}$. Then the following Fejér inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(t)\,dt \le \int_a^b f(t)p(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b p(t)\,dt. \tag{4}$$

In (4), by letting $p(t) = 1$ for $t \in [a, b]$, we get *Hermite-Hadamard inequality* (5). To be more precise, if $f : I \to \mathbb{R}$ is a continuous convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$ with $a < b$, then (see [4, p. 137], [11])

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{5}$$

In the sequel we have an interest in inequalities for *convex sequences*.

Definition 1.6 ([19, p. 526]). A sequence $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ is said to be *convex* if

$$z_i \le \frac{z_{i-1} + z_{i+1}}{2} \quad \text{for } i = 2, \ldots, n-1.$$

See [13, 17, 19] for applications of convex sequences.

By making use of inequality (1) for the continuous convex function $f(t) = \psi(\varphi_{\mathbf{z}}(t))$, where $\varphi_{\mathbf{z}} : [1, n] \to \mathbb{R}$ is given by

$$\varphi_{\mathbf{z}}(t) = \begin{cases} z_1 + (z_2 - z_1)(t-1), & t \in [1, 2), \\ z_2 + (z_3 - z_2)(t-2), & t \in [2, 3), \\ \ldots & \ldots \\ z_i + (z_{i+1} - z_i)(t-i), & t \in [i, i+1), \\ \ldots & \ldots \\ z_{n-1} + (z_n - z_{n-1})(t-n+1), & t \in [n-1, n], \end{cases} \tag{6}$$

Wu and Debnath [19] proved the following interesting result.

Theorem F. [19, Theorems 1 and 2] *Let $(z_1, z_2, \ldots, z_n) \in I^n$ be a convex sequence, and let $\psi : I \to \mathbb{R}$ be a continuous increasing convex function on I.*

Then for any $(p_1, p_2, \ldots, p_k) \prec (q_1, q_2, \ldots, q_k)$ ($1 \le p_i \le n$, $1 \le q_i \le n$, $p_i, q_i \in \mathbb{N}$, $i = 1, 2, \ldots, k$, $k \ge 2$), the following inequality holds

$$\sum_{i=1}^{k} \psi(z_{p_i}) \le \sum_{i=1}^{k} \psi(z_{q_i}). \tag{7}$$

In particular, if ψ is the identity function $\psi(t) = t$, $t \in I = \mathbb{R}$, then (7) reduces to

$$\sum_{i=1}^{k} z_{p_i} \leq \sum_{i=1}^{k} z_{q_i}. \tag{8}$$

The next two inequalities, due to Latreuch and Belaïdi [8, Theorem 1.4], can be viewed as a discrete counterpart of Fejér and Hermite-Hadamard inequalities (4)-(5).

Theorem G. [8, Theorem 1.4] *Let $(z_1, z_2, \ldots, z_n)$ be a convex sequence of real numbers and $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ be a positive sequence symmetric about $\frac{n+1}{2}$.*

Then the following inequalities hold

$$\frac{z_N + z_{n+1-N}}{2} \sum_{i=1}^{n} b_i \leq \sum_{i=1}^{n} b_i z_i \leq \frac{z_1 + z_n}{2} \sum_{i=1}^{n} b_i, \tag{9}$$

where $N = \left[\frac{n+1}{2}\right]$ is the integer part of $\frac{n+1}{2}$.

In particular, if $\mathbf{b} = \frac{1}{n}(1, 1, \ldots, 1)$ then (9) becomes

$$\frac{z_N + z_{n+1-N}}{2} \leq \frac{1}{n} \sum_{i=1}^{n} z_i \leq \frac{z_1 + z_n}{2}.$$

For odd n the above inequality reads as

$$z_{\frac{n+1}{2}} \leq \frac{1}{n} \sum_{i=1}^{n} z_i \leq \frac{z_1 + z_n}{2}.$$

For even n the index $t = \frac{n+1}{2}$ does not belong to the index set $\{1, 2, \ldots, n\}$. In this situation we need to interpolate the symbol z_t with $t \in [1, n] \setminus \{1, 2, \ldots, n\}$.

Namely, according to (6),

$$z_t = \psi_{\mathbf{z}}(t) = z_i + (z_{i+1} - z_i)(t - i),$$

provided that $t \in [i, i + 1)$ for some $i \in \{1, 2, \ldots, n\}$.

Consequently, for even n, $z_{\frac{n+1}{2}} = \frac{z_N + z_{n+1-N}}{2}$ (cf. (9)).

We end this section with a summary. In this paper we have three aims. The first is to prove a version of Theorem C for convex sequences (see Theorem 2.1). To do this, we follow a method due to Wu and Debnath given in [19].

The second aim is to extend Theorem F by relaxing the majorization assumption on sequences $\mathbf{p}$ and $\mathbf{q}$, and by introducing scalar coefficients before $\psi(z_{p_i})$ and $\psi(z_{q_i})$ in (7) and before z_{p_i} and z_{q_i} in (8). In doing so, we utilize Theorem 2.1 and Corollary 2.3. In consequence, we obtain discrete versions for convex sequences of Petrović and Giaccardi's inequalities [16].

Finally, our third aim is to present some applications of the obtained results (see Section 3). We are going to derive a generalization for convex sequences of Fejér and Hermite-Hadamard inequalities. Thus we extend some recent results by Latreuch and Belaïdi (see Theorem G).

In our considerations we apply column stochastic matrices according to Theorem C. Although in essence we are interested in double estimations (as in (4), (5) and (9)), we must divide our derivations into two separated cases: the first is for upper bound and the second is for lower bound. This is because the upper bound and lower bound are induced separately via Theorem C by two different column stochastic matrices. Constructing such matrices is the key for obtaining corresponding estimations.

2. Sherman Theorem for Convex Sequences

In Theorem 2.1 we demonstrate a Sherman type inequality (11) for convex sequences (cf. [18], see also [2, 5, 14]). This is a discrete counterpart of Theorem C in Section 1.

Theorem 2.1. *Let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in I^n$ be a convex sequence.*

Let $p_j, q_i \in \{1, 2, \dots, n\}$, $j = 1, 2, \dots, k$, $i = 1, 2, \dots, m$.

Assume that $\mathbf{p} = (p_1, p_2, \dots, p_k)$, $\mathbf{q} = (q_1, q_2, \dots, q_m)$, $\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbb{R}^m_+$ and $\mathbf{b} = (b_1, b_2, \dots, b_k) \in \mathbb{R}^k_+$.

If

$$\mathbf{p} = \mathbf{q}\,\mathbf{S} \quad \text{and} \quad \mathbf{a} = \mathbf{b}\,\mathbf{S}^T \tag{10}$$

for some $m \times k$ column stochastic matrix $\mathbf{S}$, then the following inequality holds

$$\sum_{j=1}^{k} b_j \psi(z_{p_j}) \le \sum_{i=1}^{m} a_i \psi(z_{q_i}). \tag{11}$$

In particular, if $\psi(t) = t$ for $t \in I = \mathbb{R}$, then (11) takes the form

$$\sum_{j=1}^{k} b_j z_{p_j} \le \sum_{i=1}^{m} a_i z_{q_i}. \tag{12}$$

Proof. The idea of this proof is to pass from a convex sequence to a convex function via a method given in [19]. Next, it is enough to employ Theorem C in Section 1.

Namely, consider the sequences $\mathbf{x} = \mathbf{q}$ and $\mathbf{y} = \mathbf{p}$. By (10) we get (2).

We recall the continuous convex function $\varphi_{\mathbf{z}} : [1, n] \to \mathbb{R}$ induced by the convex sequence $\mathbf{z}$ and defined by (6).

It is obvious that $\varphi_{\mathbf{z}}(r) = z_r$ for $r = 1, 2, \dots, n$. The entries of $\mathbf{x}$ and $\mathbf{y}$ are positive integers in the set $\{1, 2, \dots, n\}$. Therefore $\varphi_{\mathbf{z}}(p_j) = z_{p_j}$ for $j = 1, 2, \dots, k$, and $\varphi_{\mathbf{z}}(q_i) = z_{q_i}$ for $i = 1, 2, \dots, m$.

With the help of Theorem C applied to the convex function $f(t) = \psi(\varphi_{\mathbf{z}}(t))$ for $t \in [1, n]$, we obtain

$$\sum_{j=1}^{k} b_j \psi(z_{p_j}) = \sum_{j=1}^{k} b_j \psi\varphi_{\mathbf{z}}(p_j) \le \sum_{i=1}^{m} a_i \psi\varphi_{\mathbf{z}}(q_i) = \sum_{i=1}^{m} a_i \psi(z_{q_i}),$$

as required. □

Another (direct) proof of Theorem 2.1.

In light of (10) we can see that

$$(p_1, p_2, \dots, p_k) = (q_1, q_2, \dots, q_m)\mathbf{S}.$$

Hence we have $p_j = \sum_{i=1}^{m} s_{ij} q_i$, where $\mathbf{S} = (s_{ij})$ with $\sum_{i=1}^{m} s_{ij} = 1$, $j = 1, 2, \dots, k$, and $s_{ij} \ge 0$. By the convexity of the composition $f = \psi \circ \varphi_{\mathbf{z}}$, where $\varphi_{\mathbf{z}}$ is given by (6), we can write

$$f(p_j) = f\left(\sum_{i=1}^{m} s_{ij} q_i\right) \le \sum_{i=1}^{m} s_{ij} f(q_i) \quad \text{for } j = 1, 2, \dots, k.$$

Therefore

$$\sum_{j=1}^{k} b_j f(p_j) \le \sum_{j=1}^{k} b_j \sum_{i=1}^{m} s_{ij} f(q_i),$$

which means

$$\sum_{j=1}^{k} b_j f(p_j) \le \sum_{i=1}^{m} \sum_{j=1}^{k} b_j s_{ij} f(q_i).$$

However, $\mathbf{a} = \mathbf{b}\,\mathbf{S}^T$ by (10). So, we find that $a_i = \sum\limits_{j=1}^{k} b_j s_{ij}$, $i = 1, 2, \dots, m$.

In consequence, we deduce that

$$\sum_{j=1}^{k} b_j f(p_j) \le \sum_{i=1}^{m} \left(\sum_{j=1}^{k} b_j s_{ij} \right) f(q_i) = \sum_{i=1}^{m} a_i f(q_i).$$

In other words, since $\varphi_{\mathbf{z}}(r) = z_r$ for $r = 1, 2, \dots, n$ (see (6)), we have

$$\sum_{j=1}^{k} b_j \psi(z_{p_j}) = \sum_{j=1}^{k} b_j \psi\varphi_{\mathbf{z}}(p_j) \le \sum_{i=1}^{m} a_i \psi\varphi_{\mathbf{z}}(q_i) = \sum_{i=1}^{m} a_i \psi(z_{q_i}).$$

This completes the proof of inequality (11).

It is clear that (12) is a simple consequence of (11) for the identity function $\psi(t) = t$, $t \in I = \mathbb{R}$. □

Remark 2.2. On account of Theorem B, if $\mathbf{b} = (b_1, b_2, \dots, b_n) = (1, 1, \dots, 1)$, and $\mathbf{S}$ is doubly stochastic, then Theorem 2.1 becomes Theorem F in Section 1 due to Wu and Debnath [19].

Corollary 2.3. *Let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in I^n$ be a convex sequence.*

Let $p_j, q_i \in \{1, 2, \dots, n\}$, $j = 1, 2, \dots, k$, $i = 1, 2, \dots, m$.

Assume that $\mathbf{p} = (p_1, p_2, \dots, p_k)$ and $\mathbf{q} = (q_1, q_2, \dots, q_m)$.

If

$$\mathbf{p} = \mathbf{q}\,\mathbf{S} \tag{13}$$

for some $m \times k$ column stochastic matrix $\mathbf{S}$, then the following inequality holds

$$\sum_{j=1}^{k} \psi(z_{p_j}) \le \sum_{i=1}^{m} a_i \psi(z_{q_i}), \tag{14}$$

where a_i is the sum of all entries of the ith row of $\mathbf{S}$, $i = 1, \dots, m$.

Proof. We define

$$\mathbf{a} = (a_1, a_2, \dots, a_m) \in \mathbb{R}_+^m \quad \text{and} \quad \mathbf{b} = (1, 1, \dots, 1) \in \mathbb{R}_+^n.$$

It is easily seen that

$$(a_1, a_2, \dots, a_m) = (1, 1, \dots, 1)\mathbf{S}^T.$$

This and (13) imply (10). It is now sufficient to use Theorem 2.1. □

Remark 2.4. In the special case where the matrix $\mathbf{S}$ is doubly stochastic, Corollary 2.3 reduces to Theorem F. This is because (13) with double stochasticity of $\mathbf{S}$ means $\mathbf{p} \prec \mathbf{q}$ (by Birkhoff's Theorem B), and $a_i = 1$, $i = 1, \dots, n$.

We now illustrate Sherman type inequalities (11)-(14) for sequences by providing an example.

Example 2.5. *Petrović's inequality* [4, p. 123] states that if f is a real convex function defined on interval $[0,\infty)$, then

$$\sum_{i=1}^{k} f(x_i) \le f\left(\sum_{i=1}^{k} x_i\right) + (k-1)f(0) \tag{15}$$

for all $x_i \in [0,\infty)$, $i = 1,2,\ldots,k$.

Inequality (15) is a corollary to (1), because

$$(x_1, x_2, \ldots, x_k) \prec \left(\sum_{i=1}^{k} x_i, \underbrace{0,\ldots,0}_{k-1\ \ times}\right).$$

We shall show a corresponding result for a convex sequence as a consequence of Corollary 2.3.

To do so, let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$ and $\mathbf{z} = (z_0, z_1, \ldots, z_n) \in I^{n+1}$ be a convex sequence. Let $p_j \in \{0,1,\ldots,n\}$ for $j = 1,2,\ldots,k$ such that $\sum\limits_{j=1}^{k} p_j \le n$.

We set $\mathbf{p} = (p_1, p_2, \ldots, p_k)$ and $m = 2$, $\mathbf{q} = (q_1, q_2)$ with $q_1 = \sum\limits_{j=1}^{k} p_j$, $q_2 = 0$.

Since $p_j \in [q_2, q_1]$, $j = 1,2,\ldots,k$, for $\lambda_j = p_j / \sum\limits_{j=1}^{k} p_j \in [0,1]$ we have

$$p_j = \lambda_j q_1 + (1-\lambda_j) q_2. \tag{16}$$

By introducing the $2 \times k$ column stochastic matrix

$$\mathbf{S} = \begin{pmatrix} \lambda_1 & , & \lambda_2 & , & \ldots, & \lambda_k \\ 1-\lambda_1 & , & 1-\lambda_2 & , & \ldots, & 1-\lambda_k \end{pmatrix},$$

we get $\mathbf{p} = \mathbf{q}\,\mathbf{S}$ by (16).

It now follows from Corollary 2.3 that

$$\sum_{j=1}^{k} \psi(z_{p_j}) \le a_1 \psi(z_{q_1}) + a_2 \psi(z_{q_2}), \tag{17}$$

where a_i is the sum of all entries of the ith row of $\mathbf{S}$, $i = 1,2$, that is

$$a_1 = \sum_{j=1}^{k} \lambda_j = \sum_{j=1}^{k} \frac{p_j}{\sum\limits_{j=1}^{k} p_j} = 1,$$

$$a_2 = \sum_{j=1}^{k} (1-\lambda_j) = k - \sum_{j=1}^{k} \frac{p_j}{\sum\limits_{j=1}^{k} p_j} = k-1.$$

Finally, we deduce from (17) that

$$\sum_{j=1}^{k} \psi(z_{p_j}) \le \psi(z_{\sum_{j=1}^{k} p_j}) + (k-1)\psi(z_0).$$

This is a discrete version for convex sequences of Petrović's inequality (15).

In order to get a more general result, we take any $\mathbf{b} = (b_1, b_2, \dots, b_k) \in \mathbb{R}^k_+$ and apply Theorem 2.1. Thus we obtain

$$\sum_{j=1}^{k} b_j \psi(z_{p_j}) \le a_1 \psi(z_{\sum_{j=1}^{k} p_j}) + a_2 \psi(z_0), \tag{18}$$

where $\mathbf{a} = (a_1, a_2)$ is given by $\mathbf{a} = \mathbf{b}\,\mathbf{S}^T$, that is

$$a_1 = \sum_{j=1}^{k} b_j \lambda_j = \frac{\sum_{j=1}^{k} b_j p_j}{\sum_{j=1}^{k} p_j},$$

$$a_2 = \sum_{j=1}^{k} b_j (1 - \lambda_j) = \sum_{j=1}^{k} b_j - \frac{\sum_{j=1}^{k} b_j p_j}{\sum_{j=1}^{k} p_j}.$$

The result (18) is a discrete version for convex sequences of the *Giaccardi inequality* (see [17]).

3. Inequalities of Fejér and Hermite-Hadamard Types

We begin this section with a discussion of some generalizations of Fejér inequality. To this end, we employ Theorem 2.1 to generate some inequalities with $m = 2$ terms on their right-hand sides.

Theorem 3.1. *Let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in I^n$ be a convex sequence.*

Assume that $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{R}^n_+$.

Then the following inequality holds

$$\sum_{j=1}^{n} b_j \psi(z_j) \le a_1 \psi(z_1) + a_2 \psi(z_n), \tag{19}$$

where

$$a_1 = \frac{1}{n-1}[(n-1)b_1 + (n-2)b_2 + \ldots + 1 \cdot b_{n-1} + 0 \cdot b_n], \tag{20}$$

$$a_2 = \frac{1}{n-1}[0 \cdot b_1 + 1 \cdot b_2 + \ldots + (n-2)b_{n-1} + (n-1)b_n]. \tag{21}$$

In particular, if $\psi(t) = t$ for $t \in I = \mathbb{R}$, then (19) reduces to

$$\sum_{j=1}^{n} b_j z_j \le a_1 z_1 + a_2 z_n, \tag{22}$$

where a_1 and a_2 are given by (20)-(21).

Proof. We set $m = 2$ and $k = n$ and introduce index vectors $\mathbf{p}$ and $\mathbf{q}$ by

$$\mathbf{p} = (p_1, p_2, \dots, p_n) = (1, 2, \dots, n) \quad \text{and} \quad \mathbf{q} = (q_1, q_2) = (1, n).$$

Let $\mathbf{a} = (a_1, a_2)$ with a_1 and a_2 given by (20)-(21).

We shall find an $m \times k$ column stochastic matrix $\mathbf{S}$ such that $\mathbf{p} = \mathbf{q}\mathbf{S}$ holds as in (10).

Evidently, for each $i \in \{1, 2, \dots, n\}$, i can be expressed as a convex combination of 1 and n, i.e.,

$$i = \alpha_i \cdot 1 + \beta_i \cdot n, \tag{23}$$

where

$$\alpha_i = \frac{(n-1)-(i-1)}{n-1} \quad \text{and} \quad \beta_i = \frac{i-1}{n-1}. \tag{24}$$

It is clear that $\alpha_i \geq 0$ and $\beta_i \geq 0$ and $\alpha_i + \beta_i = 1$, since $1 \leq i \leq n$.

In order to show (23) with (24), observe that

$$\alpha_i \cdot 1 + \beta_i \cdot n = \frac{(n-1)-(i-1)}{n-1} \cdot 1 + \frac{i-1}{n-1} \cdot n$$
$$= \frac{(n-1)+(i-1)(n-1)}{n-1} = \frac{(i-1+1)(n-1)}{n-1} = i.$$

In light of (23)–(24) we see that

for $i = 1$,

$$1 = \frac{n-1}{n-1} \cdot 1 + \frac{0}{n-1} \cdot n,$$

for $i = 2$,

$$2 = \frac{n-2}{n-1} \cdot 1 + \frac{1}{n-1} \cdot n,$$

...,

for $i = n-1$,

$$n-1 = \frac{1}{n-1} \cdot 1 + \frac{n-2}{n-1} \cdot n,$$

for $i = n$,

$$n = \frac{0}{n-1} \cdot 1 + \frac{n-1}{n-1} \cdot n.$$

The above identities suggest the form of the required matrix $\mathbf{S}$. Namely, we define a 2-by-n column stochastic matrix $\mathbf{S}$ by

$$\mathbf{S} = \frac{1}{n-1} \begin{pmatrix} n-1, & n-2, & \dots, & 1, & 0 \\ 0, & 1, & \dots, & n-2, & n-1 \end{pmatrix}.$$

Then it follows that

$$\mathbf{p} = \mathbf{q}\,\mathbf{S},$$

that is

$$(1, 2, \dots, n) = (1, n) \cdot \frac{1}{n-1} \begin{pmatrix} n-1, & n-2, & \dots & 1, & 0 \\ 0, & 1, & \dots & n-2, & n-1 \end{pmatrix}.$$

Furthermore, by employing (20)-(21) it is not hard to check that

$$(a_1, a_2) = (b_1, \dots, b_n) \cdot \frac{1}{n-1} \begin{pmatrix} n-1, & 0 \\ n-2, & 1 \\ \vdots & \vdots \\ 1, & n-2 \\ 0, & n-1 \end{pmatrix}.$$

Thus we have

$$\mathbf{a} = \mathbf{b}\,\mathbf{S}^T.$$

We are now in a position to use Theorem 2.1 with $m = 2$, $k = n$, $p_j = j$ for $j = 1, \dots, k$, $q_1 = 1$ and $q_1 = n$. In consequence, we get inequality (19), as was to be proven.

Finally, by putting $\psi(t) = t$ for $t \in I = \mathbb{R}$, and applying (19) we obtain (22).

This completes the proof of Theorem 3.1. □

Throughout for any positive integer n we denote:

if n is odd,

$$N = N_1 = N_2 = \frac{n+1}{2} = \left[\frac{n+1}{2}\right],$$

and, if n is even,

$$N = N_1 = \frac{n}{2} = \left[\frac{n+1}{2}\right] \quad \text{and} \quad N_2 = \frac{n}{2} + 1.$$

We say that a sequence $\mathbf{b} = (b_1, \dots, b_n)$ is *symmetric* about $\frac{n+1}{2}$ if

$$b_1 = b_n, \quad b_2 = b_{n-1}, \quad \dots, \quad b_{N_1-1} = b_{N_2+1}, \quad b_{N_1} = b_{N_2}.$$

We also define

$$\epsilon_n = \begin{cases} \frac{1}{2}, & \text{when } n \text{ is odd,} \\ 1, & \text{when } n \text{ is even.} \end{cases}$$

In the next result we assume that the sequence $\mathbf{b} = (b_1, \dots, b_n)$ involved in the theorem is symmetric about $\frac{n+1}{2}$. This simplifies the formulas (20)-(21) to the form

$$a_1 = a_2 = \frac{1}{2}\sum_{i=1}^{n} b_i. \tag{25}$$

Theorem 3.2. *Let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in I^n$ be a convex sequence.*

Assume that $\mathbf{b} = (b_1, \dots, b_n) \in \mathbb{R}^n_+$ is symmetric about $\frac{n+1}{2}$.

Then the following Fejér type inequality holds

$$\sum_{j=1}^{n} b_j \psi(z_j) \le \frac{\psi(z_1) + \psi(z_n)}{2} \sum_{i=1}^{n} b_i. \tag{26}$$

Proof. We apply Theorem 3.1 and obtain

$$\sum_{j=1}^{n} b_j \psi(z_j) \le a_1 \psi(z_1) + a_2 \psi(z_n) \tag{27}$$

with a_1 and a_2 given by (20)-(21).

By using the symmetry of $\mathbf{b} = (b_1, \dots, b_n)$ about $\frac{n+1}{2}$, we find that formulas (20)-(21) become (25).

In fact, we have

$$\begin{aligned} a_1 &= \frac{1}{n-1}[(n-1)b_1 + (n-2)b_2 + \ldots + 1 \cdot b_{n-1} + 0 \cdot b_n] \\ &= \frac{1}{n-1}[(n-1)b_1 + \ldots + (n - N_1 + 1)b_{N_1-1} + (n - N_1)\epsilon_n b_{N_1} \\ &\quad + (n - N_2)\epsilon_n b_{N_2} + (n - N_2 - 1)b_{N_2+1} + \ldots + 0 \cdot b_n], \end{aligned}$$

because

$$(n - N_1)\epsilon_n b_{N_1} + (n - N_2)\epsilon_n b_{N_2} = \begin{cases} (n - \frac{n}{2})b_{\frac{n}{2}} + (n - \frac{n}{2} - 1)b_{\frac{n}{2}+1} & \text{for even } n \\ (n - \frac{n+1}{2})\frac{1}{2}b_{\frac{n+1}{2}} + (n - \frac{n+1}{2})\frac{1}{2}b_{\frac{n+1}{2}} & \text{for odd } n \end{cases}$$

$$= \begin{cases} \frac{n}{2} b_{\frac{n}{2}} + (\frac{n}{2} - 1) b_{\frac{n}{2}+1} & \text{for even } n \\ \frac{n-1}{2} b_{\frac{n+1}{2}} & \text{for odd } n. \end{cases}$$

Therefore we derive

$$a_1 = \frac{1}{n-1}[(n-1+0)b_1 + \ldots + (n - N_1 + 1 + n - N_2 - 1)b_{N_1-1} + (n - N_1 + n - N_2)\epsilon_n b_{N_1}]$$

$$= \frac{1}{n-1} \cdot (n-1)(b_1 + \ldots + b_{N_1-1} + \epsilon_n b_{N_1})$$

$$= \frac{1}{2}(b_1 + b_2 + \ldots + b_{n-1} + b_n) = \frac{1}{2}\sum_{i=1}^{n} b_i,$$

as desired.

Likewise, we deduce from the symmetry of $\mathbf{b} = (b_1, \ldots, b_n)$ about $\frac{n+1}{2}$ that

$$a_2 = \frac{1}{n-1}[0 \cdot b_1 + 1 \cdot b_2 + \ldots + (n-2) \cdot b_{n-1} + (n-1) \cdot b_n]$$

$$= \frac{1}{n-1}[0 \cdot b_n + 1 \cdot b_{n-1} + \ldots + (n-2) \cdot b_2 + (n-1) \cdot b_1]$$

$$= a_1 = \frac{1}{2}\sum_{i=1}^{n} b_i,$$

as claimed.

So, in the context of the proved statement (25), we conclude that inequality (27) implies (26), completing the proof of Theorem 3.2. □

By applying Theorem 3.2 to the identity function $\psi(t) = t$ for $t \in I = \mathbb{R}$, we obtain the following right-hand side of Fejér type inequality for convex sequences due to Latreuch and Belaïdi [8, Theorem 1.4] (cf. Theorem G in Section 1):

$$\sum_{j=1}^{n} b_j z_j \leq \frac{z_1 + z_n}{2} \sum_{i=1}^{n} b_i.$$

A further specification of Theorem 3.2 is to establish a Hermite–Hadamard type inequality for a convex sequence.

Corollary 3.3. *Let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in I^n$ be a convex sequence.*

Then the following Hermite-Hadamard type inequality holds

$$\frac{1}{n}\sum_{j=1}^{n} \psi(z_j) \leq \frac{\psi(z_1) + \psi(z_n)}{2}. \tag{28}$$

Proof. We introduce

$$\mathbf{b} = (b_1, \ldots, b_n) = \frac{1}{n}(1, \ldots, 1) \in \mathbb{R}^n_+.$$

Clearly, $\mathbf{b}$ is symmetric about $\frac{n+1}{2}$.

By virtue of Theorem 3.2 we can write

$$\sum_{j=1}^{n} b_j \psi(z_j) \leq \frac{\psi(z_1) + \psi(z_n)}{2} \sum_{i=1}^{n} b_i. \tag{29}$$

Moreover, we have

$$\sum_{i=1}^{n} b_i = n \cdot \frac{1}{n} = 1.$$

So, inequality (29) can be restated as (28), as required. □

By putting $\psi(t) = t$ for $t \in I = \mathbb{R}$ in Corollary 3.3 we obtain the following right-hand side of Hermite-Hadamard type inequality due to Latreuch and Belaïdi [8, Section 4] (cf. Theorem G in Section 1):

$$\frac{1}{n}\sum_{j=1}^{n} z_j \le \frac{z_1 + z_n}{2}.$$

Now we turn our attention to inequalities with $k = 2$ terms on their left-hand sides. Below we give further interpretation of Theorem 2.1 to establish such results.

Theorem 3.4. *Let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in I^n$ be a convex sequence.*

Let $b_1, b_2 \ge 0$ and $p_1, p_2 \in \{1, 2, \dots, n\}$, where

$$p_1 = 1 \cdot \lambda_1 + 2 \cdot \lambda_2 + \ldots + n \cdot \lambda_n, \tag{30}$$

$$p_2 = 1 \cdot \mu_1 + 2 \cdot \mu_2 + \ldots + n \cdot \mu_n \tag{31}$$

for some $\lambda_i \ge 0$, $\mu_i \ge 0$, $i = 1, 2, \dots, n$, $\sum\limits_{i=1}^{n} \lambda_i = 1 = \sum\limits_{i=1}^{n} \mu_i$.

Then the following inequality holds

$$b_1\psi(z_{p_1}) + b_2\psi(z_{p_2}) \le \sum_{i=1}^{n} a_i\psi(z_i), \tag{32}$$

where

$$a_i = b_1 \cdot \lambda_i + b_2 \cdot \mu_i \quad \text{for} \quad i = 1, 2, \dots, n. \tag{33}$$

Proof. We denote

$$\mathbf{p} = (p_1, p_2) \quad \text{and} \quad \mathbf{q} = (q_1, q_2, \dots, q_n) = (1, 2, \dots, n).$$

Then it follows from (30)-(31) that

$$\mathbf{p} = \mathbf{q}\,\mathbf{S},$$

where $\mathbf{S}$ is the n-by-2 column stochastic matrix given by

$$\mathbf{S} = \begin{pmatrix} \lambda_1, & \mu_1 \\ \lambda_2, & \mu_2 \\ \vdots & \vdots \\ \lambda_n, & \mu_n \end{pmatrix}.$$

On the other hand, from (33) we also have

$$\mathbf{a} = \mathbf{b}\,\mathbf{S}^T \quad \text{with} \quad \mathbf{a} = (a_1, a_2, \dots, a_n) \quad \text{and} \quad \mathbf{b} = (b_1, b_2).$$

In conclusion, on account of Theorem 2.1 for $m = n$ and $k = 2$, we infer that inequality (32) holds true, whenever (33) is fulfilled. □

We now consider the case when the majorization $(p_1, p_2) \prec (1, n)$ is met.

Theorem 3.5. *Let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in I^n$ be a convex sequence.*

Let $b_1, b_2 \geq 0$ and $p_1, p_2 \in \{1, 2, \dots, n\}$, where

$$p_1 = 1 \cdot \lambda_1 + 2 \cdot \lambda_2 + \ldots + n \cdot \lambda_n \tag{34}$$

for some $0 \leq \lambda_i \leq \frac{2}{n}$, $i = 1, 2, \dots, n$, $\sum\limits_{i=1}^{n} \lambda_i = 1$.

Assume that the pair (p_1, p_2) is majorized by $(1, n)$.

Then the following inequality holds

$$b_1 \psi(z_{p_1}) + b_2 \psi(z_{p_2}) \leq \sum_{i=1}^{n} a_i \psi(z_i), \tag{35}$$

where

$$a_i = b_1 \cdot \lambda_i + b_2 \cdot \left(\frac{2}{n} - \lambda_i \right) \quad \text{for} \quad i = 1, 2, \dots, n. \tag{36}$$

Proof. We put

$$\mathbf{p} = (p_1, p_2) \quad \text{and} \quad \mathbf{q} = (q_1, q_2, \dots, q_n) = (1, 2, \dots, n).$$

Since the pair (p_1, p_2) is majorized by the pair $(1, n)$, we obtain

$$p_1 + p_2 = n + 1. \tag{37}$$

It is obvious that

$$n + 1 = \frac{2}{n}(1 + 2 + \ldots + n). \tag{38}$$

By combining (34), (37) and (38), we have

$$p_2 = 1 \cdot \left(\frac{2}{n} - \lambda_1 \right) + 2 \cdot \left(\frac{2}{n} - \lambda_2 \right) + \ldots + n \cdot \left(\frac{2}{n} - \lambda_n \right). \tag{39}$$

Furthermore, $\frac{2}{n} - \lambda_i \geq 0$ for $i = 1, 2, \dots, n$, and

$$\sum_{i=1}^{n} \left(\frac{2}{n} - \lambda_i \right) = n \cdot \frac{2}{n} - \sum_{i=1}^{n} \lambda_i = 2 - 1 = 1.$$

By defining

$$\mu_i = \frac{2}{n} - \lambda_i \quad \text{for } i = 1, 2, \dots, n,$$

we see that the statement (39) implies (31), and (36) implies (33).

Therefore we are allowed to use Theorem 3.4. So, we deduce that inequality (32) is satisfied. In consequence, inequality (35) holds provided that condition (36) is fulfilled. This completes the proof of Theorem 3.5. □

Corollary 3.6. *Let $\psi : I \to \mathbb{R}$ be a nondecreasing convex function defined on an interval $I \subset \mathbb{R}$. Let $\mathbf{z} = (z_1, z_2, \dots, z_n) \in I^n$ be a convex sequence.*

Let $p_1, p_2 \in \{1, 2, \dots, n\}$, where

$$p_1 = 1 \cdot \lambda_1 + 2 \cdot \lambda_2 + \ldots + n \cdot \lambda_n$$

for some $0 \leq \lambda_i \leq \frac{2}{n}$, $i = 1, 2, \dots, n$, $\sum\limits_{i=1}^{n} \lambda_i = 1$.

Assume that the pair (p_1, p_2) is majorized by $(1, n)$.
Then the following inequality holds

$$\frac{1}{2}\psi(z_{p_1}) + \frac{1}{2}\psi(z_{p_2}) \le \frac{1}{n}\sum_{i=1}^{n}\psi(z_i). \tag{40}$$

In particular, if $p_1 = p_2 = \frac{n+1}{2}$ (with odd n) then

$$\psi\left(z_{\frac{n+1}{2}}\right) \le \frac{1}{n}\sum_{i=1}^{n}\psi(z_i). \tag{41}$$

If $p_1 = \frac{n}{2}$ and $p_2 = \frac{n}{2} + 1$ (with even n) then

$$\frac{1}{2}\psi\left(z_{\frac{n}{2}}\right) + \frac{1}{2}\psi\left(z_{\frac{n}{2}+1}\right) \le \frac{1}{n}\sum_{i=1}^{n}\psi(z_i). \tag{42}$$

Proof. Substituting $b_1 = b_2 = \frac{1}{2}$ in Theorem 3.5 gives

$$a_i = \frac{1}{2}\cdot\lambda_i + \frac{1}{2}\cdot\left(\frac{2}{n} - \lambda_i\right) = \frac{1}{n} \quad \text{for} \quad i = 1, 2, \ldots, n$$

by (36).

For this reason, inequality (35) reduces to (40), as wanted.

Inequalities (41)-(42) are simple consequences of (40). □

Remark 3.7. By writting $p_1 = N_1 - c$ and $p_2 = N_2 + c$, where $N_1 = N_2 = \frac{n+1}{2}$ for odd n, and $N_1 = \frac{n}{2}$ and $N_2 = \frac{n}{2} + 1$ for even n, we get the majorization $(p_1, p_2) \prec (1, n)$.

So, we can restate inequality (40) in the form

$$\frac{1}{2}\psi(z_{N_1-c}) + \frac{1}{2}\psi(z_{N_2+c}) \le \frac{1}{n}\sum_{i=1}^{n}\psi(z_i). \tag{43}$$

This shows that (43) is a discrete version of a continuous counterpart (see [12, [p. 56]).

Finally, we emphasize that for the identity function $\psi(t) = t$ with $t \in I = \mathbb{R}$, the inequalities (41)-(42) read as

$$z_{\frac{n+1}{2}} \le \frac{1}{n}\sum_{i=1}^{n} z_i \quad \text{when } n \text{ is odd},$$

$$\frac{1}{2}z_{\frac{n}{2}} + \frac{1}{2}z_{\frac{n}{2}+1} \le \frac{1}{n}\sum_{i=1}^{n} z_i \quad \text{when } n \text{ is even}$$

(see Theorem G). This can be viewed as the left-hand side of Hermite-Hadamard inequality for a convex sequence $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{R}^n$ (cf. (5)).

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References

[1] S. Abramovich, J. Barić, J. Pečarić, Fejer and Hermite-Hadamard type inequalities for superquadratic functions, Journal of Mathematical Analysis and Applications, 344 (2008) 1048–1056.

[2] J. Borcea, Equilibrium points of logarithmic potentials, Transactions of the American Mathematical Society, 359 (2007) 3209–3237.

[3] L. Bougoffa, New inequalities about convex functions, Journal of Inequalities in Pure and Applied Mathematics, 7 (4) (2006) Art. 148.

[4] W. W. Breckner, T. Trif, Convex Functions and Related Functional Equations: Selected Topics, Cluj University Press, Cluj, 2008.

[5] A.-M. Burtea, Two examples of weighted majorization, Annals of the University of Craiova, Mathematics and Computer Science Series, 32 (2) (2010) 92–99.

[6] S. S. Dragomir, Some majorisation type discrete inequalities for convex functions, Mathematical Inequalities and Applications, 7 (2) (2004) 207–216.

[7] M. Klaričić-Bakula, J. Pečarić, J. Perić, Extensions of the Hermite-Hadamard inequality with applications, Mathematical Inequalities and Applications, 12 (4) (2012) 899–921.

[8] Z. Latreuch, B. Belaïdi, New inequalities for convex sequences with applications, International Journal of Open Problems in Computer Science and Mathematics, 5 (3) (2012) 15–27.

[9] A. W. Marshall, I. Olkin, B. C. Arnold, Inequalities: Theory of Majorization and Its Applications, Second Edition, Springer, New York, 2011.

[10] N. Minculete, F.-C. Mitroi, Fejér-type inequalities, The Australian Journal of Mathematical Analysis and Applications, 9 (1) (2012) 1–8.

[11] C. P. Niculescu, L.-E. Persson, Old and new on the Hermite-Hadamard inequality, Real Analysis Exchange, 29 (2) (2003/2004) 663–685.

[12] C. P. Niculescu, L.-E. Persson, Convex functions and their applications: A contemporary approach, Springer, Berlin/New York, 2004.

[13] M. Niezgoda, An extension of results of A. McD. Mercer and I. Gavrea, Journal of Inequalities in Pure and Applied Mathematics, 6 (4) (2005) Art. 107

[14] M. Niezgoda, Remarks on Sherman like inequalities for (α, β)-convex functions, Mathematical Inequalities and Applications, 17 (4) (2014) 1579–1590.

[15] M. Niezgoda, Vector joint majorization and generalization of Csiszár-Körner's inequality for f-divergence, Discrete Applied Mathematics, 198 (2016) 195–205.

[16] J. Pečarić, J. Perić, Improvements of the Giaccardi and the Petrović inequality and related Stolarsky type means, Annals of the University of Craiova, Mathematics and Computer Science Series, 39 (1) (2012) 65–75.

[17] R. Pereira, S. Plosker, Dirichlet polynomials, majorization, and trumping, Journal of Physics A: Mathematical and Theoretical, 46 (2013) 225302.

[18] S. Sherman, On a theorem of Hardy, Littlewood, Pólya, and Blackwell, Proceedings of the National Academy of Sciences, 37 (1) (1957) 826–831.

[19] S. Wu, L. Debnath, Inequalities for convex sequences and their applications, Computers and Mathematics with Applications, 54 (2007) 525–534.