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A Completion-Invariant Extension of the Concept of Quasi C-continuous Lattices

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Abstract. In this paper, the concepts of C-precontinuous posets, quasi C-precontinuous posets and meet C-precontinuous posets are introduced. The main results are: (1) A complete semilattice P is C-precontinuous (resp., quasi C-precontinuous) if and only if its normal completion is a C-continuous lattice (resp., quasi C-continuous lattice); (2) A poset is both quasi C-precontinuous and Frink quasicontinuous if and only if it is generalized completely continuous; (3) A complete semilattice is meet C-precontinuous if and only if its normal completion is meet C-continuous; (4) A poset is both quasi C-precontinuous and meet C-precontinuous if and only if it is C-precontinuous.

1. Introduction

Generalizing Dedekind's pioneer construction of the real line by cuts of rational numbers, MacNeille invented his famous *normal completion* for arbitrary partially ordered sets. Unfortunately, there are many algebraic properties that are not preserved by normal completions. For instance, it is well known that the normal completion of a distributive lattice (resp., continuous lattice) need not be distributive (resp., continuous) (see [1, 2]). There are, however, interesting posets that are known to be closed under normal completions. These include Boolean algebras, Heyting algebras, precontinuous posets, completely precontinuous posets, generalized completely continuous posets and hypercontinuous posets (see [2–4, 8, 9]). The concept of quasicontinuous domains was introduced by Gierz, Lawson and Stralka (see [12]) as a common generalization of both continuous domains (see [13]) and generalized continuous lattices (see [14]). The basic idea is to generalize the *way-below relation* between points and points to the one between sets and sets. This is done, however, only for dcpos, which is rather restrictive. Since many models may not be dcpos, there are more and more demand to study posets which miss join of directed sets (see [15, 19, 20]). Furthermore, the quasicontinuous domains are not completion-invariant (see [2]), that is, the normal completion of a quasicontinuous domains is not a quasicontinuous lattice (see [11]). In [5], using Frink ideals (see [6])

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instead of directed lower sets, M. Erné introduced a new way-below relation and the concept of *precontinuous posets* for arbitrary posets. The precontinuous posets have good completion-invariant properties, namely, a poset is precontinuous iff its normal completion is a continuous lattice. Recently, many good completion-invariant properties were investigated. For example, Zhang and Xu introduced the concept of *Frink quasicontinuous posets* (resp., *meet precontinuous*) by Frink ideals and proved that a poset is a Frink quasicontinuous (resp., meet precontinuous) poset if and only if its normal completion is a quasicontinuous lattice (resp., meet continuous lattice) (see [10, 11]). It is well known that L is a completely distributive lattice if and only if it is isomorphic to $\sigma(P)$ for a continuous domain P. Since a completely distributive lattice is self-dual, it follows that the Scott-closed set lattices of continuous domains are, up to isomorphism, exactly the completely distributive lattices. In [16], Ho and Zhao studied the order-theoretic properties of lattices of Scott-closed sets and introduced the concept of *C-continuous poset*, which is similar to the well studied notion of continuity. C-continuity is not completion-invariant: a simple example will be given in Section 3. In [17], the concept of C-continuity was generalized by the notion of *quasi C-continuity*. In terms of quasi C-continuity, it was proved that quasicontinuity of a dcpo is equivalent to the quasicontinuity of its Scott-closed lattice.

Now the following questions naturally arise: What is the counterpart of the notion of a precontinuous poset in the ream of C-continuous posets (resp., quasi C-precontinuous) and to what extent does the counterpart possess the completion-invariant property? In this paper, we solve these questions. First, we introduce a kind of new way below relation for arbitrary posets by means of weak Scott closed sets. Then we introduce the concepts of quasi C-precontinuous posets, C-precontinuous posets and C-meet precontinuous posets. It is proved that a complete semilattice is C-precontinuous if and only if its normal completion is a C-continuous lattice. In general, a complete semilattice P is quasi C-precontinuous if and only if its normal completion is a quasi C-continuous lattice. It is also proved that a complete semilattice P is meet C-precontinuous if and only if its normal completion is a meet C-continuous lattice. Meanwhile, we also discuss the relationships among quasi C-precontinuity, Frink quasicontinuity and generalized completely continuity, and show that a poset is both quasi C-precontinuous and Frink quasicontinuous if and only if it is generalized completely continuous. In particular, we derive that a poset is both C-precontinuous and precontinuous if and only if it is completely precontinuous. Finally, we further characterize the relationships among meet C-precontinuity, quasi C-precontinuity and C-precontinuity.

2. Preliminaries

Let P be a partially ordered set (in short, poset). For $x \in P, A \subseteq P$, we define $\downarrow A = \{x \in P : x \leq a$ for some $a \in A\}$ and $\uparrow A$ is defined dually. A subset D of P is directed provided it is nonempty, and every finite subset of D has an upper bound in D. If the join (supremum) of any directed subset D of P exists, then P is called a directed complete poset (in short, dcpo). Let $A^{\uparrow}$ and $A^{\downarrow}$ denote the sets of all upper and lower bounds of A, respectively. A cut operator δ is defined by $A^{\delta} = (A^{\uparrow})^{\downarrow}$. Notice that whenever A has a join then $x \in A^{\delta}$ means $x \leq \vee A$. Let $\delta(P) = \{A^{\delta} : A \subseteq P\}$. $(\delta(P), \subseteq)$ is called the normal completion, or the Dedekind-MacNeille completion of P and it is a complete lattice. By a *completion-invariant property* we mean a property that holds for P if and only if it holds for any normal completion of P.

For a poset P, a subset U of P is called Scott open provided that $U = \uparrow U$ and $D \cap U \neq \emptyset$ for all directed sets $D \subseteq P$ with $\vee D \in U$ whenever $\vee D$ exists. The topology formed by all the Scott open sets of P is called the Scott topology on P, written as $\sigma(P)$. The complement of a Scott open set is called *Scott closed*. The topology on P generated by the collection of sets $P \setminus \downarrow x$ (as a subbase) is called the upper topology and is denoted by $v(P)$. For $x, y \in P$, define a relation $\prec_v$ on P by $x \prec_v y \Leftrightarrow y \in int_{v(P)} \uparrow x$. A complete lattice L is called hypercontinuous if $x = \vee\{y \in L : y \prec_v x\}$ for all $x \in L$.

Definition 2.1. ([5, 7]) *Let P be a poset. A subset $U \subseteq P$ is called weak Scott open if it satisfies*

(1) $U = \uparrow U$;
(2) *For all directed sets $D \subseteq P$, $D^{\delta} \cap U \neq \emptyset$ implies $D \cap U \neq \emptyset$.*

The collection of all weak Scott open subsets of P forms a topology which will be called the weak Scott topology on P and will be denoted by $\sigma_2(P)$. The complement of a weak Scott open set is called weak Scott closed. Let $C(P) = \{P \setminus U : U \in \sigma_2(P)\}$. Then $F \in C(P)$ if and only if (i) $F = \downarrow F$, and (ii) for all directed subsets $D \subseteq F$, we have $D^\delta \subseteq F$.

Remark 2.2. *In general, $\sigma_2(P)$ is coarser than $\sigma(P)$, and both topologies coincide on dcpos.*

Example 2.3. *(Example 2.5 in* [5]*) Consider three disjoint countable sets $A = \{a_n : n \in \mathbf{N_0}\}, B = \{b_n : n \in \mathbf{N_0}\}, C = \{c_n : n \in \mathbf{N}\}$, and the order $\leq$ on $P = A \cup B \cup C$ is defined as follows:*

$\downarrow a_0 = \{a_0\} \cup B$,

$\downarrow a_n = \{b_m : m < n\}(n \in \mathbf{N}, n \neq 2)$,

$\downarrow a_2 = \{b_0, b_1\} \cup C$,

$\downarrow b_n = \{b_n\}(n \in \mathbf{N_0})$,

$\downarrow c_n = \{c_m : m \leq n\}(n \in \mathbf{N})$,

$x \leq y \Leftrightarrow x \in \downarrow y$.

Then $\uparrow b_0$ is open in $\sigma(P)$ but not in $\sigma_2(P)$ since $C = \{c_n : n \in \mathbf{N}\}$ is a directed lower set with $b_0 \in C^\delta \cap \uparrow b_0 \neq \emptyset$ while $C \cap \uparrow b_0 = \emptyset$. Hence in this example, we have that $\sigma_2(P)$ is contained properly in $\sigma(P)$.

Definition 2.4. ([13]) *Let L be a complete lattice and $A, B \subseteq L$.*

(1) *We say that A is way-below B, in symbols $A \ll B$, if for each directed subset $D \subseteq L$, $\vee D \in \uparrow B$ implies $d \in \uparrow A$ for some $d \in D$.*
(2) *We say that A is strongly way-below B, in symbols $A \blacktriangleleft B$, if for any subset $S \subseteq L$, $\vee S \in \uparrow B$ implies $s \in \uparrow A$ for some $s \in S$.*
(3) *L is called quasicontinuous if for all $x \in L$, $\uparrow x = \bigcap\{\uparrow F : F \in P^{(<\omega)}, F \ll x\}$, where $P^{(<\omega)} = \{F \subseteq P : F$ is finite$\}$.*
(4) *L is called continuous if for all $x \in L$, $x = \vee\{a \in P : a \ll x\}$.*
(5) *L is called generalized completely distributive if for all $x \in L$, $\uparrow x = \bigcap\{\uparrow F : F \in P^{(<\omega)}, F \blacktriangleleft x\}$.*
(6) *L is called completely distributive if for all $x \in L$, $x = \vee\{a \in P : a \blacktriangleleft x\}$.*

Definition 2.5. ([6]) *A subset I of a poset P is called a Frink ideal in P if $Z^\delta \subseteq I$ for all finite subsets $Z \subseteq I$. Let $Fid(P)$ denote the set of all Frink ideals.*

Definition 2.6. ([5, 11]) *Let P be a poset and $A, B \subseteq P$.*

(1) *We say that $A \ll_e B$, if for all $I \in Fid(P)$, $\uparrow B \cap I^\delta \neq \emptyset$ implies $\uparrow A \cap I \neq \emptyset$. $F \ll_e \{x\}$ is shortly written as $F \ll_e x$. Let $v(x) = \{F \in P^{(<\omega)} : F \ll_e x\}$.*
(2) *P is called Frink quasicontinuous if for all $x \in P$, $\uparrow x = \bigcap\{\uparrow F : F \in v(x)\}$.*
(3) *P is called precontinuous if for all $x \in P$, $x \in \{a \in P : a \ll_e x\}^\delta$.*

Definition 2.7. ([8, 9]) *Let P be a poset and $A, B \subseteq P$.*

(1) *We say that $A \blacktriangleleft B$, if for any subset $S \subseteq P$, $\uparrow B \cap S^\delta \neq \emptyset$ implies $\uparrow A \cap S \neq \emptyset$.*
(2) *P is called generalized completely continuous if for all $x \in P$, $\uparrow x = \bigcap\{\uparrow F : F \in P^{(<\omega)}, F \blacktriangleleft x\}$.*
(3) *P is called completely precontinuous if for all $x \in P$, $x \in \{a \in P : a \lhd x\}^\delta$.*

Definition 2.8. ([7]) *A poset P is called meet continuous if for any $x \in P$ and any directed set D with $x \leq \vee D$ whenever $\vee D$ exists, we have that $x \in cl_{\sigma(P)}(\downarrow D \cap \downarrow x)$.*

Lemma 2.9. ([21]) *Let L be a complete lattice. Then L is a generalized completely distributive lattice if and only if L^{op} is a hypercontinuous lattice.*

Remark 2.10. *Let P be a poset and $A, B \subseteq P$.*

(1) *If $A \subseteq B$, then $A^\uparrow \supseteq B^\uparrow$ and $A^\downarrow \supseteq B^\downarrow$.*
(2) *$A \subseteq A^\delta$, $A^{\delta\delta} = A^\delta$, $A^{\uparrow\downarrow\uparrow} = A^\uparrow$.*
(3) *For all $\{{A_j}^\delta : j \in J\} \subseteq \delta(P)$, $\bigwedge_{\delta(P)}\{{A_j}^\delta : j \in J\} = \bigcap\{{A_j}^\delta : j \in J\}$, $\bigvee_{\delta(P)}\{{A_j}^\delta : j \in J\} = (\bigcup\{{A_j}^\delta : j \in J\})^\delta = (\bigcup\limits_{j \in J} A_j)^\delta$.*

3. Quasi *C*-precontinuous Posets

In this section, the concepts of *C*-precontinuous and quasi *C*-precontinuous posets are introduced, and it is proved that a complete semilattice is *C*-precontinuous(resp. quasi *C*-precontinuous) if and only if its normal completion is *C*-continuous(resp. quasi *C*-precontinuous), that is, *C*-precontinuity and quasi *C*-precontinuity are completion−invariant for a complete semilattice.

Definition 3.1. ([17]) *Let P be a poset and $\emptyset \neq A, B \subseteq P$.*

(1) *A is beneath B, denoted by $A \prec B$, if for any nonempty Scott-closed set $C \subseteq P$ for which $\vee C$ exists, the relation $\vee C \in \uparrow B$ implies that $C \cap \uparrow A \neq \emptyset$. $F \prec \{x\}$ is shortly written as $F \prec x$.*
(2) *P is called a quasi C-continuous poset, if for all $x \in P, \uparrow x = \bigcap\{\uparrow F : F \in P^{(<\omega)}, F \prec x\}$. A quasi C-continuous poset which is also a complete lattice is called a quasi C-continuous lattice.*

Definition 3.2. ([16]) *A poset P is called C-continuous if for all $x \in P$, $x = \vee\{a \in P : a \prec x\}$. A C-continuous poset which is also a complete lattice is called a C-continuous lattice.*

Lemma 3.3. ([16]) *If L is a C-continuous lattice, then L is distributive.*

It is easy to see that for a finite lattice L, both L and L^{op} are continuous, and $v(L) = \sigma(L)$. It follows that L and L^{op} are hypercontinuous lattices, and hence by Lemma 2.9, L and L^{op} are generalized completely distributive lattices. Clearly a generalized completely distributive lattice is a quasi C-continuous lattice. By this observation, we see that every finite lattice is a quasi C-continuous lattice. Thus quasi C-continuous lattice may not be distributive. By Lemma 3.3, we have that C-continuous lattice must be quasi C-continuous, but the converse may not be true.

Example 3.4. *Let $P = \{a, b, c\}$ with discrete order. Obviously, P is C-continuous. But $\delta(P) = \{\emptyset, \{a\}, \{b\}, \{c\}, P\}$ is not distributive. By Lemma 3.3, $\delta(P)$ is not a C-continuous lattice.*

The above example shows that C-continuity is not completion-invariant. However, our intention is not to generalize the concept of C-continuity but to give a completion-invariant definition of C-(pre)continuity for arbitrary posets.

Definition 3.5. *Let P be a poset and let $x \in P, \emptyset \neq A, B \subseteq P$.*

(1) *A is called way below B, denoted by $A \preceq B$, if for any nonempty $S \in C(P), S^{\delta} \cap \uparrow B \neq \emptyset$ implies $S \cap \uparrow A \neq \emptyset$. $F \preceq \{x\}$ is shortly written as $F \preceq x$. Let $w(x) = \{F \subseteq P : F \in P^{(<\omega)}, F \preceq x\}$, $\Uparrow A = \{x \in P : A \preceq x\}$, $\Downarrow x = \{y \in P : y \preceq x\}$.*
(2) *P is called a C-precontinuous poset, if for all $x \in P$, $x \in (\Downarrow x)^{\delta}$.*
(3) *P is called a quasi C-precontinuous poset, if for all $x \in P, \uparrow x = \bigcap\{\uparrow F : F \in w(x)\}$.*

Obviously, a C-precontinuous poset is quasi C-precontinuous.

Remark 3.6. *For complete lattices the above definition of the way below relation is equivalent to Definition 3.1 (as in a complete lattice, $S^{\delta} \cap \uparrow B \neq \emptyset$ means $\vee S \in \uparrow B$). Thus a quasi C-precontinuous poset which is also a complete lattice must be a quasi C-continuous lattice.*

The following proposition is basic and the proof is omitted.

Proposition 3.7. *Let P be a poset and G, H,K, $M \subseteq P$. Then*

(1) *$G \preceq H \Leftrightarrow G \preceq h$ for all $h \in H$;*
(2) *$G \preceq H \Leftrightarrow \uparrow G \preceq \uparrow H$;*
(3) *$G \preceq H \Rightarrow G \leq H$, where $G \leq H \Leftrightarrow \uparrow H \subseteq \uparrow G$;*
(4) *$G \leq H \preceq K \leq M \Rightarrow G \preceq M$.*

The following theorem characterizes the relationships among quasi C-precontinuity, Frink quasicontinuity and generalized complete continuity.

Theorem 3.8. *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is both quasi C-precontinuous and Frink quasicontinuous;*
(2) *P is generalized completely continuous.*

Proof. (1) $\Rightarrow$ (2) Suppose that P is both quasi C-precontinuous and Frink quasicontinuous. Then for all $x \in P$, $\uparrow x = \bigcap\{\uparrow F : F \in v(x)\}$ by the Frink quasicontinuity of P. Now we show that $\uparrow F = \bigcap\{\uparrow G : G \in P^{(<\omega)}, G \leq F\}$ for all $F \in v(x)$. It suffices to show that $\bigcap\{\uparrow G : G \in P^{(<\omega)}, G \leq F\} \subseteq \uparrow F$. Suppose that there is $z \in \bigcap\{\uparrow G : G \in P^{(<\omega)}, G \leq F\}$ such that $z \notin \uparrow F$. Then $y \nleq z$ for all $y \in F$. By the quasi C-precontinuity of P, there is $F_y \in w(y)$ such that $F_y \leq y$ and $z \notin \uparrow F_y$. Take $F' = \bigcup_{y \in F} F_y$. Then F' is finite and $F' \leq F$ but $z \notin F'$, a contradiction. Thus $\uparrow x = \bigcap\{\uparrow F : F \in v(x)\} = \bigcap\{\uparrow G : G \in P^{(<\omega)}$, there is $F \in v(x)$ such that $G \leq F\}$. Next we show that $G \lhd x$ whenever there exists $F \subseteq P$ such that $G \leq F \ll_e x$. Indeed, let $S \subseteq P$ and suppose that $S^\delta \cap \uparrow x \neq \emptyset$. Then $x \in S^\delta$. Set $H = \bigcup\{E^\delta : E \in S^{(<\omega)}\}$. The following assertions (i) and (ii) hold.

(i) $H \in Fid(P)$. Indeed, let $Z \in H^{(<\omega)}$. Then there are $E_1^\delta, E_2^\delta, \ldots, E_n^\delta \subseteq H$ such that $Z \subseteq \bigcup\limits_{i=1}^{n} E_i^\delta \subseteq (\bigcup\limits_{i=1}^{n} E_i)^\delta$. So $Z^\delta \subseteq (\bigcup\limits_{i=1}^{n} E_i)^{\delta\delta} = (\bigcup\limits_{i=1}^{n} E_i)^\delta \subseteq H$.

(ii) $x \in H^\delta$. If $x \notin H^\delta$, then there is $y \in P$ such that $H \subseteq \downarrow y$ but $x \nleq y$. Then for any $s \in S$, we have $\{s\}^\delta = \downarrow s \subseteq \downarrow y$. Thus $S \subseteq \downarrow y$. Since $x \in S^\delta$, we obtain that $x \leq y$, a contradiction.

Since $F \ll_e x, H \in Fid(P)$ and $x \in H^\delta$, we have $\uparrow F \cap H \neq \emptyset$, and hence there is some $h \in H$such that $h \in \uparrow F$. Thus there is $E \in S^{(<\omega)}$ such that $h \in E^\delta = (\downarrow E)^\delta$. Note that $\downarrow E \in C(P)$. Since $G \leq F$ and $h \in (\downarrow E)^\delta \cap \uparrow F$, we get that $\downarrow E \cap \uparrow G \neq \emptyset$. Thus $S \cap \uparrow G \neq \emptyset$. This shows that $G \lhd x$. Now we obtain that

$$\uparrow x \subseteq \bigcap\{\uparrow W : W \in P^{(<\omega)}, W \lhd x\} \subseteq \bigcap\{\uparrow G : G \in P^{(<\omega)}, \text{there is } F \in v(x) \text{ such that } G \leq F\} = \uparrow x.$$

Therefore $\uparrow x = \bigcap\{\uparrow W : W \in P^{(<\omega)}, W \lhd x\}$, and hence P is generalized completely continuous.

(2) $\Rightarrow$ (1) Suppose that P is generalized completely continuous. Then for any $x \in P$, we have $\uparrow x = \bigcap\{\uparrow F : F \in P^{(<\omega)}, F \lhd x\}$. Note that $F \lhd x$ implies $F \ll_e x$ and $F \leq x$. Then it is easy to show that $\uparrow x = \bigcap\{\uparrow F : F \in v(x)\}$ and $\uparrow x = \bigcap\{\uparrow F : F \in w(x)\}$. Thus P is quasi C-precontinuous and Frink quasicontinuous. □

Similarly, we have the following theorem.

Theorem 3.9. *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is both C-precontinuous and precontinuous;*
(2) *P is completely precontinuous.*

By Theorem 3.8 and Theorem 3.9, we have the following two corollaries.

Corollary 3.10. ([16]) *Let L be a complete lattice. Then the following conditions are equivalent:*

(1) *L is both C-continuous and continuous;*
(2) *L is completely distributive.*

Corollary 3.11. ([17]) *Let L be a complete lattice. Then the following conditions are equivalent:*

(1) *L is both quasi C-continuous and quasicontinuous;*
(2) *L is generalized completely distributive.*

Lemma 3.12. *Let P be poset and $\mathcal{A} = \{A_i^\delta : i \in I\} \in C(\delta(P))$. Then $\bigcup\limits_{i \in I} A_i^\delta \in C(P)$.*

Proof. Clearly, $\bigcup\limits_{i \in I} A_i^\delta$ is a lower set. Next we show that $D^\delta \subseteq \bigcup\limits_{i \in I} A_i^\delta$ for all directed subsets D of $\bigcup\limits_{i \in I} A_i^\delta$. Indeed, since $\mathcal{A} \in C(\delta(P))$ and $\{\downarrow d : d \in D\}$ is a directed subset of $\mathcal{A}$, we have that $\bigvee_{\delta(P)}\{\downarrow d : d \in D\} \in \mathcal{A}$. Notice that $D^\delta = (\bigcup\limits_{d \in D} \downarrow d)^\delta$. By Remark 2.10, we get that $D^\delta = \bigvee_{\delta(P)}\{\downarrow d : d \in D\}$. Now we obtain that $D^\delta \subseteq \bigcup \mathcal{A}$. □

Recall that a poset is a complete semilattice if and only if every nonempty subset has an inf and every directed subset has a sup. It is well known that a dcpo is a complete semilattice if and only if every subset that is bounded above has a sup(see [13]).

Lemma 3.13. *Let P be a complete semilattice and $Y \in C(P)$. Then $\mathcal{A} = \{A^\delta \in \delta(P) : A^\delta \subseteq \downarrow y$ for some $y \in Y\} \in C(\delta(P))$ and $\bigvee_{\delta(P)} \mathcal{A} = Y^\delta$.*

Proof. Clearly, $\mathcal{A}$ is a lower set. Let $\mathcal{E} = \{A_i^\delta : i \in I\} \in \mathcal{D}(\delta(P))$ and $\mathcal{E} \subseteq \mathcal{A}$, where $\mathcal{D}(\delta(P))$ denotes all directed subsets of $\delta(P)$. By Remark 2.10, we get that $\bigvee_{\delta(P)} \mathcal{E} = (\bigcup_{i\in I} A_i^\delta)^\delta$. Now we show that $(\bigcup_{i\in I} A_i^\delta)^\delta \in \mathcal{A}$. Indeed, for any $E^\delta \in \mathcal{E}$, by the condition, there exists $y \in Y$ such that $E^\delta \subseteq \downarrow y$. Since P is a complete semilattice, $\vee E^\delta$ exists. It is easy to see that the set $\{\vee E^\delta : E^\delta \in \mathcal{E}\}$ is a directed subset of P. Set $e = \vee\{\vee E^\delta : E^\delta \in \mathcal{E}\}$. Since $Y \in C(P)$, $Y = \downarrow Y$. Then we get that $\vee E^\delta \in Y$ for all $E^\delta \in \mathcal{E}$, and hence $e \in Y$. Obviously $\vee E^\delta \leq e$ for all $E^\delta \in \mathcal{E}$. This implies that $E^\delta \subseteq \downarrow \vee E^\delta \subseteq \downarrow e$ for all $E^\delta \in \mathcal{E}$. Thus $(\bigcup_{i\in I} A_i^\delta)^\delta \subseteq \downarrow e$. This shows that $\bigvee_{\delta(P)} \mathcal{E} = (\bigcup_{i\in I} A_i^\delta)^\delta \in \mathcal{A}$. So $\mathcal{A} \in C(\delta(P))$.

Next, we show that the second assertion of our lemma holds as well. Indeed, it is easy to see that Y^δ is an upper bound of $\mathcal{A}$ in $\delta(P)$. If X^δ is also an upper bound of $\mathcal{A}$ in $\delta(P)$, then for all $y \in Y$, $\downarrow y \subseteq X^\delta$. Notice that $Y^\delta = (\bigcup_{y\in Y} \downarrow y)^\delta$. This implies that $Y^\delta \subseteq X^\delta$, and hence we obtain that $\bigvee_{\delta(P)} \mathcal{A} = Y^\delta$. □

The following theorem shows that quasi C-precontinuity is completion−invariant for a complete semilattice.

Theorem 3.14. *Let P be a poset and consider the following conditions:*

(1) *P is a quasi C-precontinuous poset;*
(2) *The normal completion $\delta(P)$ is a quasi C-continuous lattice.*

Then (1) ⇒ (2). Moreover, if P is a complete semilattice, then the two conditions are equivalent.

Proof. (1) ⇒ (2) We need to show that $\uparrow_{\delta(P)} \{A^\delta\} = \bigcap\{\uparrow_{\delta(P)} \mathcal{F} : \mathcal{F} \in w(A^\delta)\}$ for all $A^\delta \in \delta(P)$. Indeed, it is easy to show that $\uparrow_{\delta(P)} \{A^\delta\} \subseteq \bigcap\{\uparrow_{\delta(P)} \mathcal{F} : \mathcal{F} \in w(A^\delta)\}$. On the other hand, let $B^\delta \in \delta(P)$ with $A^\delta \not\subseteq B^\delta$. Then $A \not\subseteq B^\delta$, and hence there is $x \in A$ such that $x \notin B^\delta$. Thus there is $y \in P$ with $B \subseteq \downarrow y$ such that $x \not\leq y$. By (1), there is $F \in P^{(<\omega)}$ with $F \leq x$ such that $y \notin \uparrow F$. Let $\mathcal{F} = \{\downarrow u : u \in F\}$. Then $\mathcal{F} \in \delta(P)^{(<\omega)}$. We conclude that $\mathcal{F} \leq A^\delta$ and $B^\delta \notin \uparrow_{\delta(P)} \mathcal{F}$. If $B^\delta \in \uparrow_{\delta(P)} \mathcal{F}$, then there is $u \in F$ such that $\downarrow u \subseteq B^\delta$. Thus $u \in B^\delta \subseteq \downarrow y$, a contradiction to $y \notin \uparrow F$. Now, by Lemma 3.12, we show that $\mathcal{F} \leq A^\delta$. Indeed, by Remark 2.10, for all $\{A_j^\delta : j \in J\} \in C(\delta(P))$ with $A^\delta \subseteq \bigvee_{j\in J} A_j^\delta$, we have $A^\delta \subseteq (\bigcup_{j\in J} A_j^\delta)^\delta$. Now we have that $\bigcup_{j\in J} A_j^\delta \in C(P)$. Since $x \in A \subseteq A^\delta \subseteq \bigvee_{j\in J} A_j^\delta$, and $F \leq x$, we get that $F \cap \bigcup_{j\in J} A_j^\delta \neq \emptyset$. Thus there are $u \in F$ and $j \in J$ such that $u \in A_j^\delta$. Notice that $\downarrow x \subseteq A_j^\delta$. Hence $\uparrow_{\delta(P)} \mathcal{F} \bigcap\{A_j^\delta : j \in J\} \neq \emptyset$. This shows that $\delta(P)$ is a quasi C-continuous lattice.

(2) ⇒ (1) Obviously, $\uparrow x \subseteq \bigcap\{\uparrow F : F \in w(x)\}$. If $x \not\leq y$, then $\downarrow x \not\subseteq \downarrow y$. By (2), there is $\mathcal{F} = \{A_1, A_2, \ldots, A_n\} \in \delta(P)^{(<\omega)}$ with $\mathcal{F} \leq \downarrow x$ such that $\downarrow y \notin \uparrow_{\delta(P)} \mathcal{F}$. This shows that $A_i \not\subseteq \downarrow y$ for all $i \in \{1, 2, \ldots, n\}$. Thus there is $y_j \in A_j$ with $y_i \not\leq y$ for all $j \in \{1, 2, \ldots, n\}$. Let $F = \{y_1, y_2, \ldots, y_n\}$. Clearly, $y \notin F$. Next we show that $F \leq x$. Indeed, let $S \in C(P)$ and $x \in S^\delta$. Then, by Remark 2.10, $\downarrow x \subseteq S^\delta \subseteq (\bigcup\{G^\delta : G^\delta \subseteq \downarrow y$ for some $y \in S\})^\delta = \bigvee_{\delta(P)}\{G^\delta : G^\delta \subseteq \downarrow y$ for some $y \in S\}$. Now, by Lemma 3.13, we obtain that $\{G^\delta : G^\delta \subseteq \downarrow y$ for some $y \in S\}$ is a Scott closed subset of $\delta(P)$. Since $\mathcal{F} \leq \downarrow x$, we obtain that $\uparrow_{\delta(P)} \mathcal{F} \bigcap\{G^\delta : G^\delta \subseteq \downarrow y$ for some $y \in S\} \neq \emptyset$. Since $S \in C(P)$, there exist $j \in \{1, 2, \ldots, n\}$ and some $y \in S$ with $G^\delta \subseteq \downarrow y$ such that $y_j \in A_j \subseteq G^\delta \subseteq S$. Hence $\uparrow F \cap S \neq \emptyset$. This shows that $F \leq x$. Since $y \notin F$, we get that $y \notin \bigcap\{\uparrow F : F \in w(x)\}$. Therefore, $\uparrow x = \bigcap\{\uparrow F : F \in w(x)\}$. □

Using the proof of Theorem 3.14 we similarly obtain the following theorem.

Theorem 3.15. *Let P be a poset and consider the following conditions:*

(1) *P is C-precontinuous;*

(2) *The normal completion* $\delta(P)$ *is a C-precontinuous lattice.*

Then (1) $\Rightarrow$ (2)*. Moreover, if P is a complete semilattice, then the two conditions are equivalent.*

Definition 3.16. ([2]) *We call a map f between posets P and Q cut-stable if* $f(A^{\uparrow})^{\downarrow} = f(A)^{\uparrow\downarrow}$ *and* $f(A^{\downarrow})^{\uparrow} = f(A)^{\downarrow\uparrow}$ *for all* $A \subseteq P$.

Proposition 3.17. ([2]) *A map f from a poset P to a complete lattice L is cut-stable if and only if there is a unique complete homomorphism g from* $\delta(P)$ *into L such that* $f = g \circ e_P$.

A subcategory **A** *of category* **C** *is called a reflective subcategory (see* [18]*) of* **C***, if for each* **C***-object C, there is an* **A***-object* A_0 *and a* **C***-morphism* $r : C \to A_0$ *such that for each* **A***-object A and* **C***-morphism* $f : C \to A$ *there is a unique* **A***-morphism* $g : A_0 \to A$ *such that* $f = g \circ r$.

By Proposition 3.17 and Theorem 3.14, we have the following theorem.

Theorem 3.18. *The category of quasi C-continuous lattices with complete homomorphisms is a full reflective subcategory of the category of quasi C-precontinuous complete semilattices with cut-stable maps.*

By Proposition 3.17 and Theorem 3.15, similarly we have the following theorem.

Theorem 3.19. *The category of C-continuous lattices with complete homomorphisms is a full reflective subcategory of the category of C-precontinuous complete semilattices with cut-stable maps.*

4. Meet *C*-precontinuous Posets

In this section, the concept of meet *C*-precontinuous is introduced. We show that a complete semilattice is meet *C*-precontinuous if and only if its normal completion is a meet *C*-continuous lattice, that is, meet *C*-precontinuity is completion−invariant for a complete semilattice. Meanwhile, relationships between meet *C*-precontinuity, quasi *C*-precontinuity and *C*-precontinuity are revealed.

Definition 4.1. *A poset P is called meet C-precontinuous if for all* $x \in P$ *and all* $F \in C(P)$ *with* $x \in F^{\delta}$*, we have that* $x \in (\downarrow x \cap F)^{\delta}$*. A meet C-precontinuous poset which is also a complete lattice is called a meet C-continuous lattice.*

The following examples show that the notions of meet *C*-precontinuous poset and meet continuous poset are different.

Example 4.2. *Let* $P = \{0, a, b, c, 1\}$ *with the order* $0 < a < 1, 0 < b < 1, 0 < c < 1$*. Obviously, P is a continuous lattice, thus a meet continuous lattice. Take* $F = \{0, b, c\} \in C(P)$*. Now* $a \in F^{\delta} = \downarrow \vee F = P$*, but* $a \notin (\downarrow a \cap F)^{\delta} = \{0\}$ *and thus P is not meet C-precontinuous.*

Example 4.3. *Let* $P = [0, 1] \cup \{a\}$ *with* $0 < a < 1$ *and* $[0, 1]$ *endowed with usual order. Consider* $D = \{1 - 1/n : n = 1, 2, 3, \ldots\}$ *and a, then we have* $a \wedge \bigvee D = a$*, but* $\bigvee\{a \wedge d : d \in D\} = 0 \neq a$*. Thus P is not meet continuous but it is easy to check that P is meet C-precontinuous.*

Example 4.4. (1) *The unit interval* $P = [0, 1]$ *with usual order is meet C-precontinuous;*

(2) *Let* $S = \{0, 1, 2, \ldots\}$ *with the order* $\leq$ *on S such that* $0 < 1 < 2 < \cdots$*. If a and b are two incomparable upper bounds of S, then the resulting poset* $P = S \cup \{a, b\}$ *is a meet C-precontinuous poset.*

(3) *Let* $P = \{0, 1, 2, \ldots\} \cup \{a\}$*. The order* $\leq$ *on P is given by:* $0 < 1 < 2 < \cdots$ *and* $0 < a$*. Clearly, P is both meet-continuous and meet C-precontinuous.*

Proposition 4.5. *Let P be a C-precontinuous poset. Then P is meet C-precontinuous.*

Proof. Let *P* be *C*-precontinuous. Assume that $x \in F^{\delta}$, where $F \in C(P)$. To show that $x \in (\downarrow x \cap F)^{\delta}$, it suffices to show $\Downarrow x \subseteq \downarrow x \cap F$. Indeed, if $y \leq x$, then not only is $y \leq x$ but $y \in F$. So $y \in \downarrow x \cap F$, as desired. □

The following proposition gives a characterization of meet *C*-precontinuous posets.

Proposition 4.6. *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is meet C-precontinuous;*

(2) *$\delta : C(P) \to \delta(P)$ preserves finite intersections, i.e., for all $\{I_j : 1 \leq j \leq n\} \subseteq C(P)$, $\bigcap_{j=1}^{n} I_j^\delta = (\bigcap_{j=1}^{n} I_j)^\delta$;*

(3) *For all $I_1, I_2 \in C(P)$, $I_1^\delta \cap I_2^\delta = (I_1 \cap I_2)^\delta$;*

(4) *For all $x \in P, I \in C(P)$, $\downarrow x \cap I^\delta = (\downarrow x \cap I)^\delta$.*

Proof. (2) $\Leftrightarrow$ (3) Trivial.

(1) $\Rightarrow$ (3) For all $I_1, I_2 \in C(P)$, it is clear that $(I_1 \cap I_2)^\delta \subseteq I_1^\delta \cap I_2^\delta$. Let $x \in I_1^\delta \cap I_2^\delta$. By (1), $x \in \downarrow x = (\downarrow x \cap I_1)^\delta = (\downarrow x \cap I_2)^\delta$. Set $A = \downarrow x \cap I_1$, $B = \downarrow x \cap I_2$. Then $b \in B$ implies that $b \in \downarrow x = A^\delta$, which in turn yields $b \in \downarrow b = (\downarrow b \cap A)^\delta$. Hence, by Remark 2.10, we obtain that $x \in B^\delta = (\bigcup_{b \in B} \downarrow b)^\delta = (\bigcup_{b \in B} (\downarrow b \cap A)^\delta)^\delta \subseteq ((A \cap B)^\delta)^\delta = (A \cap B)^\delta \subseteq (I_1 \cap I_2)^\delta$.

(3) $\Rightarrow$ (4) Trivial.

(4) $\Rightarrow$ (1) Let $x \in P$ and $I \in C(P)$. If $x \in I^\delta$ then, by (4), $x \in \downarrow x \cap I^\delta = (\downarrow x \cap I)^\delta$. Thus P is meet C-precontinuous. □

The following shows that meet C-precontinuity is completion−invariant for a complete semilattice.

Theorem 4.7. *Let P be a poset and consider the following conditions:*

(1) *P is a meet C-precontinuous poset;*

(2) *$(\delta(P), \subseteq)$ is a meet C-continuous lattice.*

Then (1) $\Rightarrow$ (2). *Moreover, if P is a complete semilattice, then the two conditions are equivalent.*

Proof. (1) $\Rightarrow$ (2) For all $A^\delta \in \delta(P)$ and $\{A_j^\delta : j \in J\} \in C(\delta(P))$, it suffices to show that the distributive law $A^\delta \wedge \bigvee_{j \in J} A_j^\delta = \bigvee_{j \in J} (A^\delta \wedge A_j^\delta)$ holds in $\delta(P)$. Indeed, it is not difficult to show that $A^\delta \wedge \bigvee_{j \in J} A_j^\delta \supseteq \bigvee_{j \in J} (A^\delta \wedge A_j^\delta)$. Conversely, let $x \in A^\delta \wedge \bigvee_{j \in J} A_j^\delta$. By Remark 2.10, we have $A^\delta \wedge \bigvee_{j \in J} A_j^\delta = A^\delta \cap (\bigcup_{j \in J} A_j^\delta)^\delta$. Note that $x \in A^\delta$. Thus we get that $x \in (\bigcup_{j \in J} A_j^\delta)^\delta$. By Lemma 3.12, we have $(\bigcup_{j \in J} A_j^\delta)^\delta \in C(P)$. By meet C-precontinuity of P, $x \in (\downarrow x \cap (\bigcup_{j \in J} A_j^\delta))^\delta = \bigvee_{j \in J} (A^\delta \cap A_j^\delta)$. Now we obtain that $(\delta(P), \subseteq)$ is a meet C-continuous lattice.

(2) $\Rightarrow$ (1) For $x \in P$ and $D \in C(P)$ with $x \in D^\delta$, we have that $\downarrow x \cap D^\delta = (\downarrow x \cap D)^\delta$. Indeed, it is easy to show that $\downarrow x \cap D^\delta \supseteq (\downarrow x \cap D)^\delta$. On the other hand, by the meet C-precontinuity of $\delta(P)$ and Lemma 3.13, we have that

$$\begin{aligned}\downarrow x \cap D^\delta &= \downarrow x \wedge D^\delta = \downarrow x \wedge \bigvee_{\delta(P)} \{G^\delta : G^\delta \subseteq \downarrow d \text{ for some } d \in D\} \\ &= \bigvee_{\delta(P)} \{\downarrow x \cap G^\delta : G^\delta \subseteq \downarrow d \text{ for some } d \in D\} \subseteq (\downarrow x \cap D)^\delta.\end{aligned}$$

Now we obtain that $x \in \downarrow x \cap D^\delta \subseteq (\downarrow x \cap D)^\delta$. Thus P is meet C-precontinuous. □

By Proposition 3.17 and Theorem 4.7, we have

Theorem 4.8. *The category of meet C-continuous lattices with complete homomorphisms is a full reflective subcategory of the category of meet C-precontinuous complete semilattices with cut-stable maps.*

The following theorem characterizes the relationships among meet C-precontinuity, quasi C-precontinuity and C-precontinuity.

Theorem 4.9. *Let P be a poset. Then the following conditions are equivalent:*

(1) *P is C-precontinuous;*

(2) *P is both meet C-precontinuous and quasi C-precontinuous.*

Proof. (1) ⇒ (2) Clearly, every C-precontinuous poset is quasi C-precontinuous. By Proposition 4.5, every C-precontinuous poset is also meet C-precontinuous.

(2) ⇒ (1) Suppose that P is both meet C-precontinuous and quasi C-precontinuous. Now we show $x \in \{y \in P : y \leq x\}^\delta$ for all $x \in P$. Indeed, suppose that there is $z \in P$ with $\{y \in P : y \leq x\} \subseteq \downarrow z$ such that $x \not\leq z$. Then, by quasi C-precontinuity of P, there is $F = \{y_1, y_2, \ldots, y_n\} \in P^{(<\omega)}$ with $F \leq x$ such that $z \notin \uparrow F$. We claim that there is some $y_j \in F$ such that $y_j \leq x$. Indeed, if $y_j \not\leq x$ for all $y_j \in F$, then there exists $I_j \in C(P)$ with $x \in I_j^\delta$ such that $y_j \notin I_j$. Now, by Proposition 4.6, we obtain that $x \in \bigcap_{j=1}^{n} I_j^\delta = (\bigcap_{j=1}^{n} I_j)^\delta$. Note that $\bigcap_{j=1}^{n} I_j \in C(P)$. Since $F \leq x$, we get that $\uparrow F \cap \bigcap_{j=1}^{n} I_j \neq \emptyset$. Hence there is $m \in \bigcap_{j=1}^{n} I_j$ and $m \in \uparrow F$, which in turn yields $y_j \leq m$ for some $y_j \in F$. Now we obtain that $y_j \in I_j$, a contradiction. Therefore $y_j \leq x$ for some $y_j \in F$. This implies that $z \geq y_j$ which in turn yields $z \in \uparrow F$, a contradiction. □

By Theorem 4.9, we immediately have the following corollary.

Corollary 4.10. *Let L be a complete lattice. Then the following conditions are equivalent:*

(1) *L is C-continuous;*

(2) *L is both meet C-continuous and quasi C-continuous.*

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