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L^p – estimates of Solutions of Backward Doubly Stochastic Differential Equations

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Abstract. This paper deals with a large class of nonhomogeneous backward doubly stochastic differential equations which have a more general form of the forward Itô integrals. Terms under which the solutions of these equations are bounded in the L^p-sense, $p \geq 2$, under both the Lipschitz and non-Lipschitz conditions, are given, i.e. L^p – stability for this general type of backward doubly stochastic differential equations is established.

1. Introduction

It is well-known that the theory of backward stochastic differential equations (BSDEs for short) was introduced and developed by Pardoux and Peng [22–24] in the 90s. They defined the notation of nonlinear BSDE and proved the existence and uniqueness of adapted solutions in their fundamental paper [22]. Furthermore, it is shown in various papers that BSDEs give the probabilistic representation of solutions (at least in the viscosity sense) for a large class of systems of semi-linear parabolic partial differential equations (PDEs) (see [20],[24]). After that, BSDEs are widely used to describe numerous mathematical problems in finance (see [8], [9], [25]), stochastic control and stochastic games (see [6], [7], [10], [11]), stochastic partial differential equations (for short SPDEs, see [4], [12], [26]) etc. Consequently, all these applications incited to introduce various types of BSDEs.

In paper [21], Pardoux and Peng introduced a new class of backward stochastic differential equations – *backward doubly stochastic differential equations*, (BDSDEs for short), and, under certain conditions, they provided a probabilistic interpretation for the solutions of a special class of quasilinear partial differential equations. Precisely, they studied BDSDE

$$y(t) = \xi + \int_t^T f(s, y(s), z(s))\, ds + \int_t^T g(s, y(s), z(s))\, dB(s) - \int_t^T z(s)\, dW(s), \tag{1}$$

where $t \in [0, T]$ and the integral with respect to the Brownian motion $B(t)$ is the "backward" Itô integral, while the integral with respect to the Brownian motion $W(t)$ is the standard forward Itô integral, and both the integrals are particular cases of the Itô-Skorohod integral (see [17]). The solution is a pair $(y(t), z(t))$

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of processes adapted to the past of the Brownian motions. Pardoux and Peng [21] gave the existence and uniqueness result for Eq. (1) and produced a probabilistic representation of certain quasilinear stochastic partial differential equations, under the Lipschitz conditions for the coefficients. Since then, many authors tried to weaken the conditions for the functions f and g and to give more general results. For example, Lin [14], Aman [1], Aman and Owo [2], N'zi and Owo [18, 19], Boufoussi, Casteren and Mrhardy [4], Ren, Lin and Hu [26], Hu and Ren [12].

All previous papers deal with BDSDEs which have only the process z in the forward integral, i.e. the forward integral has the form $\int_t^T z(s)\,dW(s)$. In their paper [13], Janković, Djordjević and Jovanović extended this type of equations by adding the function dependent on the process y in the forward integral. More precisely, for the equation

$$Y(t) = \xi + \int_t^T f(s, Y(s), Z(s))\,ds + \int_t^T g(s, Y(s), Z(s))\,dB(s) - \int_t^T [h(s, Y(s)) + Z(s)]\,dW(s), \quad t \in [0, T], \tag{2}$$

they proved the existence and uniqueness results under both the Lipschitz and non-lipschitz conditions, comparison theorems and relations between the solutions of these equations and between the solutions of the appropriate SPDEs. Even more, they studied the relation between the solution to Eq. (2) and the following equation,

$$y(t) = \xi + \int_t^T f(s, y(s), z(s) - h(s, y(s)))\,ds + \int_t^T g(s, y(s), z(s) - h(s, y(s)))\,dB(s) - \int_t^T z(s)\,dW(s), \quad t \in [0, T], \tag{3}$$

which is a type of Eq. (1). Eq. (2) can be treated as an additively perturbed Eq. (1), where all perturbations except the one in the forward integral, are zero. The function h in the forward integral can be treated as the perturbation, so that Eq. (2) is called the *nonhomogeneous* BDSDE, with respect to the appropriate *homogenous* Eq. (3).

Recently, in paper [5], Djordjević studied the existence, comparison problems, existence of the maximal solution and the structure of the solution, i.e. the Kneser problem for Eq. (2), when the coefficients are continuous.

In present paper, we extend the previous results of stability to nonhomogeneous BDSDE (2). The main idea of this paper is to study the L^p-stability, $p \geq 2$, for Eq. (2), under both the Lipschitz and non-Lipschitz conditions. The paper is organized as follows: In Section 2, we introduce some notations and notions about BDSDE (2). In Section 3, we give the L^p-stability for Eq. (2) under the Lipschitz conditions, while Section 4 is dedicated to the main result of the paper – the L^p-stability under non-Lipschitz conditions for the coefficients of the equation. Last two sections refer to conclusion marks and appendix about application of known inequality.

2. Notation and Preliminary Results

First, we usually denote that $|\cdot|$ is the Euclidean norm in $\mathbb{R}^k$ and $\|A\| = \sqrt{trace(A^\top A)}$ is the Frobenius trace-norm for a matrix A in $\mathbb{R}^{k\times d}$, where $A^\top$ is the transpose of A. We also generally assume that all random variables and processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and that $[0, T]$ is an arbitrarily large fixed time duration. We assume that $\{W_t, t \in [0, T]\}$ and $\{B_t, t \in [0, T]\}$ are two mutually independent standard Brownian motions with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively. Denote that $\mathcal{N}$ is a class of $\mathbf{P}$-null sets of $\mathcal{F}$. For every $t \in [0, T]$, let us define

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where, for any process η_t, $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s, r \in [s, t]\} \vee \mathcal{N}$ and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$. Since $\{\mathcal{F}_t^W, t \in [0, T]\}$ and $\{\mathcal{F}_{t,T}^B, t \in [0, T]\}$ are increasing and decreasing filtrations, respectively, the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing and, therefore, it does not constitute a filtration.

As usual, for any $n \in \mathbb{N}$, let $\mathcal{M}^2([0, T]; \mathbb{R}^n)$ be the set of (class of $d\mathbf{P} \times dt$ a.e. equal) $\mathbb{R}^n$-valued jointly measurable stochastic processes $\{\varphi_t, t \in [0, T]\}$ satisfying:

(i) $\|\varphi\|_{\mathcal{M}^2} = \mathbf{E}\int_0^T |\varphi_t|^2\, dt < \infty$;
(ii) φ_t is $\mathcal{F}_t$-measurable for a.e. $t \in [0,T]$.

Similarly, let $\mathcal{S}^2([0,T];\mathbb{R}^n)$ be the set of continuous $\mathbb{R}^n$-valued stochastic processes satisfying:
(i) $\|\varphi\|_{\mathcal{S}^2} = \mathbf{E}\sup_{t\in[0,T]} |\varphi_t|^2 < \infty$;
(ii) φ_t is $\mathcal{F}_t$-measurable for any $t \in [0,T]$.

Throughout the paper, the following basic assumption holds:

($\mathbf{H_0}$) The terminal value is a random variable $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbb{R}^k)$ and random functions $f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^k$, $g : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \to \mathbb{R}^{k\times l}$, $h : \Omega \times [0,T] \times \mathbb{R}^k \to \mathbb{R}^{k\times d}$ are jointly measurable and such that for any $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$
\begin{aligned}
&f(\cdot, y, z) \in \mathcal{M}^2([0,T];\mathbb{R}^k),\\
&g(\cdot, y, z) \in \mathcal{M}^2([0,T];\mathbb{R}^{k\times l}),\\
&h(\cdot, y) \in \mathcal{M}^2([0,T];\mathbb{R}^{k\times d}).
\end{aligned}
$$

Definition 2.1. *A solution of Eq. (2) is a pair of $\mathbb{R}^k \times \mathbb{R}^{k\times d}$-valued processes $(Y,Z) = \{(Y_t, Z_t), t \in [0,T]\} \in \mathcal{S}^2([0,T];\mathbb{R}^k) \times \mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$ which satisfies Eq. (2).*

Definition 2.2. *A solution $\{(Y_t, Z_t), t \in [0,T]\}$ of Eq. (2) is said to be unique, if for any other solution $\{(\bar{Y}_t, \bar{Z}_t), t \in [0,T]\}$ it follows that $\mathbf{P}\{Y_t = \bar{Y}_t,\ t \in [0,T]\} = 1$ and $\mathbf{E}\int_0^T \|Z_t - \bar{Z}_t\|^2\, dt = 0$.*

We need the so-called extension of Itô's formula in our investigation.

Lemma 2.3 (Pardoux, Peng [21]). *Let functions $\alpha \in \mathcal{S}^2([0,T];\mathbb{R}^k)$, $\beta \in \mathcal{M}^2([0,T];\mathbb{R}^k)$, $\gamma \in \mathcal{M}^2([0,T];\mathbb{R}^{k\times l})$ and $\delta \in \mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$ be such that*

$$
\alpha_t = \alpha_0 + \int_0^t \beta_s\, ds + \int_0^t \gamma_s\, dB_s + \int_0^t \delta_s\, dW_s, \quad t \in [0,T].
$$

Then,

$$
|\alpha_t|^2 = |\alpha_0|^2 + 2\int_0^t \alpha_s^\top \beta_s\, ds + 2\int_0^t \alpha_s^\top \gamma_s\, dB_s + 2\int_0^t \alpha_s^\top \delta_s\, dW_s - \int_0^t \|\gamma_s\|^2\, ds + \int_0^t \|\delta_s\|^2\, ds.
$$

More generally, for $\Phi \in C^2(\mathbb{R}^k)$,

$$
\begin{aligned}
\Phi(\alpha_t) = \Phi(\alpha_0) &+ \int_0^t \Phi'^T(\alpha_s)\beta_s\, ds + \int_0^t \Phi'^T(\alpha_s)\gamma_s\, dB_s + \int_0^t \Phi'^T(\alpha_s)\delta_s\, dW_s\\
&- \frac{1}{2}\int_0^t trace[\Phi''(\alpha_s)\gamma_s\gamma_s^T]\, ds + \frac{1}{2}\int_0^t trace[\Phi''(\alpha_s)\delta_s\delta_s^T]\, ds.
\end{aligned}
$$

3. L^p-stability under Lipschitz Coefficients

Beside the problem of the existence and uniqueness of the solution to Eq. (2), the important topic is the boundness of the L^p-moments of its solution. Before we present the main theorem, we will introduce some additional assumptions and recall theorems of the existence and uniqueness of solutions of homogenous and nonhomogeneous BDSDEs.

The following *Lipschitz conditions* are assumed in our study:

($\mathbf{H_1}$) For f, g and h satisfying (H_0), there exist constants $K > 0$ and $0 < \alpha < 1$ such that for every $(\omega, t) \in \Omega \times [0,T]$ and $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$
\begin{aligned}
|f(t, y_1, z_1) - f(t, y_2, z_2)|^2 &\le K\Big(|y_1 - y_2|^2 + \|z_1 - z_2\|^2\Big),\\
\|g(t, y_1, z_1) - g(t, y_2, z_2)\|^2 &\le K|y_1 - y_2|^2 + \alpha\|z_1 - z_2\|^2,\\
\|h(t, y_1) - h(t, y_2)\|^2 &\le K|y_1 - y_2|^2.
\end{aligned}
$$

The following propositions give the existence and uniqueness results for the homogenous and nonhomogeneous Eqs. (1) and (2).

Proposition 3.1 (Pardoux, Peng [21]). *Let (H_0) and (H_1) hold for ξ, f and g. Then, Eq. (1) has a unique solution $(y,z) \in \mathcal{S}^2([0,T];\mathbb{R}^k) \times \mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$.*

Proposition 3.2 (Janković, Djordjević, Jovanović [13]). *Let (H_0) and (H_1) hold for ξ, f, g and h and let $\{(y_t, z_t), t \in [0,T]\}$ be the solution of Eq. (3). Then,*

$$\{(y_t, z_t - h(t, y_t)), t \in [0,T]\} \in \mathcal{S}^2([0,T];\mathbb{R}^k) \times \mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$$

is the unique solution of Eq. (2).

Pardoux and Peng in [21] proved the boundness of the L^p-moments of Eq. (1) if the functions f and g satisfy the Lipschitz condition from $(\mathbf{H_1})$ and g satisfies the following additional condition:

A1. There exists constant K, such that for every $(t,y,z) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$g^T(t,y,z)g(t,y,z) \le z^T z + K(\|g(t,0,0)\|^2 + |y|^2)I,$$

where I is the unit matrix of order l.

Theorem 3.3 (Pardoux, Peng, [21]). *Let all the conditions from Proposition 3.1 hold and assumption **A1** be satisfied. If $p \ge 2$, $\xi \in L^p(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbb{R}^d)$ and*

$$E\int_0^T (|f(t,0,0)|^p + \|g(t,0,0)\|^p)\,dt < \infty,$$

then for the solution $\{(y(t), z(t)), t \in [0,T]\}$ of Eq. (1) the following holds,

$$E\sup_{0\le t\le T}|y(t)|^p < \infty, \quad E\left(\int_0^T \|z(t)\|^2\,dt\right)^{\frac{p}{2}} < \infty. \tag{4}$$

Using the previous result, it can be shown that L^p-moments of the solution of Eq. (2) are finite, if f, g and h satisfy Lipschitz conditions $(\mathbf{H_1})$.

Proposition 3.4. *Let ξ, f, g and h satisfy $(\mathbf{H_0})$, $(\mathbf{H_1})$ and let assumption **A1** holds. If $\xi \in L^p(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbb{R}^d)$ for $p \ge 2$ and*

$$E\int_0^T (|f(t,0,0)|^p + \|g(t,0,0)\|^p + \|h(t,0)\|^p)\,dt < \infty, \tag{5}$$

for the solution $\{(Y(t), Z(t)), t \in [0,T]\}$ of Eq. (2) the following holds,

$$E\sup_{0\le t\le T}|Y(t)|^p < \infty, \quad E\left(\int_0^T \|Z(t)\|^2\,dt\right)^{\frac{p}{2}} < \infty. \tag{6}$$

Proof. Since all the conditions of Proposition 3.2 are satisfied, then there exist unique solutions $\{(y(t), z(t)), t \in [0,T]\}$ and $\{(y(t), z(t) - h(t, y(t))), t \in [0,T]\} \in \mathcal{S}^2([0,T];\mathbb{R}^k) \times \mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$ of Eqs. (1) and (2), respectively, where

$$Y(t) = y(t) \text{ a.s. for every } t \in [0,T], \quad \mathbf{E}\int_0^T \|h(t, y(t)) + Z(t) - z(t)\|^2\,dt = 0. \tag{7}$$

From (4) and (7), it follows that

$$E(\sup_{0\le t\le T}|Y(t)|^p) = E(\sup_{0\le t\le T}|y(t)|^p) < \infty.$$

By applying the elementary inequality $\left(\sum_{i=1}^m a_i\right)^k \le (m^{k-1}\vee 1)\sum_{i=1}^m a_i^k,\ a_i \ge 0,\ k \ge 0$, Hölder's inequality ($E|XY| \le (E|X|^p)^{1/p}(E|Y|^q)^{1/q}, \frac{1}{p}+\frac{1}{q}=1,\ p,q>0$) and (7), we have

$$\begin{aligned}
&E\left(\int_0^T \|Z(t)\|^2\,dt\right)^{\frac{p}{2}} \qquad (8)\\
&\le 4^{\frac{p}{2}}E\left(\int_0^T \|Z(t)-z(t)+h(t,y(t))\|^2\,dt + \int_0^T \|z(t)\|^2\,dt + K\int_0^T |y(t)|^2\,dt + \int_0^T \|h(t,0)\|^2\,dt\right)^{\frac{p}{2}}\\
&\le 4^{\frac{p}{2}}3^{\frac{p}{2}-1}E\left[\left(\int_0^T \|z(t)\|^2\,dt\right)^{\frac{p}{2}} + (KT)^{\frac{p}{2}}\sup_{0\le t\le T}|y(t)|^p + T^{\frac{p-2}{p}}\left(\int_0^T \|h(t,0)\|^p\right)^{\frac{2}{p}}\right].
\end{aligned}$$

From (4), (5) and (8), it follows that

$$E\left(\int_0^T \|Z(t)\|^2\,dt\right)^{\frac{p}{2}} < \infty,$$

which completes the proof. □

4. L^p-stability under Non-Lipschitz Coefficients

In the previous section, we concluded under Lipschitz conditions (H_1) that the solution of Eq. (2) can be represented with the help of the solution of Eq. (1). In the sequel, we recall that the same conclusions holds under more general conditions, under the so-called non-Lipschitz coefficients. Analogously, we introduce the *non-Lipschitz conditions* for the coefficients of Eq. (2):

(H₂) For f, g and h satisfying (H_0), there exist constants $C > 0$ and $0 < \alpha < 1$ such that for every $(\omega, t) \in \Omega\times[0,T]$ and $(y_1,z_1),(y_2,z_2) \in \mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$\begin{aligned}
|f(t,y_1,z_1)-f(t,y_2,z_2)|^2 &\le \rho(t,|y_1-y_2|^2) + C\|z_1-z_2\|^2,\\
\|g(t,y_1,z_1)-g(t,y_2,z_2)\|^2 &\le \rho(t,|y_1-y_2|^2) + \alpha\|z_1-z_2\|^2,\\
\|h(t,y_1)-h(t,y_2)\|^2 &\le \rho(t,|y_1-y_2|^2).
\end{aligned}$$

Here $\rho : [0,T]\times R^+ \to R^+$ satisfies: For fixed $t \in [0,T]$, $\rho(t,\cdot)$ is a concave and non-decreasing function with $\rho(t,0)\equiv 0$; for fixed u, $\int_0^T \rho(t,u)\,dt < \infty$; for any $M > 0$, the ODE

$$u' = -M\,\rho(t,u), \quad u(T) = 0$$

has a unique solution $u(t)\equiv 0$, $t\in[0,T]$.

Obviously, if $\rho(t,u)\equiv Cu$, the non-Lipschitz conditions (H_2) are reduced to the Lipschitz conditions (H_1).

The following propositions give the results of the existence and uniqueness of the homogenous and nonhomogeneous BDSDEs under non-Lipschitz condition for the coefficients.

Proposition 4.1 (N'zi,Owo [18]). *Let (H_0) and (H_2) hold for ξ, f and g. Then, Eq. (1) has a unique solution $(y,z)\in\mathcal{S}^2([0,T];\mathbb{R}^k)\times\mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$.*

Proposition 4.2 (Janković, Djordjević, Jovanović [13]). *Let (H_0) and (H_2) hold for ξ, f, g and h and let $\{(y_t,z_t), t\in[0,T]\}$ be the solution of Eq. (3). Then,*

$$\{(y_t, z_t - h(t,y_t)), t\in[0,T]\}\in\mathcal{S}^2([0,T];\mathbb{R}^k)\times\mathcal{M}^2([0,T];\mathbb{R}^{k\times d})$$

is the unique solution of Eq. (2).

The estimate of the L^p-moments of the solution of Eq. (2) can be extended under weaker conditions for f, g and h. In that goal, we introduce the following hypothesis.

(H$_3$) For the functions f, g and h satisfying (**H$_0$**), there exist constants $C > 0$ and $0 < \alpha < 1$ such that for $p \geq 2$, $(\omega, t) \in \Omega \times [0, T]$ and $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

$$\begin{aligned}
|f(t, y_1, z_1) - f(t, y_2, z_2)|^2 &\leq \min\left\{\rho(|y_1 - y_2|^2), \rho^{\frac{2}{p}}(|y_1 - y_2|^p)\right\} + C\|z_1 - z_2\|^2,\\
\|g(t, y_1, z_1) - g(t, y_2, z_2)\|^2 &\leq \min\left\{\rho(|y_1 - y_2|^2), \rho^{\frac{2}{p}}(|y_1 - y_2|^p)\right\} + \alpha\|z_1 - z_2\|^2,\\
\|h(t, y_1) - h(t, y_2)\|^2 &\leq \min\left\{\rho(|y_1 - y_2|^2), \rho^{\frac{2}{p}}(|y_1 - y_2|^p)\right\},
\end{aligned}$$

where the function $\rho : R^+ \to R^+$ satisfies the conditions:
– ρ is continuous, nondecreasing and concave;
– $\rho(0) = 0$ and $\rho(u) > 0$ for every $u > 0$;
– $\int_{0^+} \frac{du}{\rho(u)} = \infty$.

It should be noted that the hypothesis (**H$_2$**) follows from the hypothesis (**H$_3$**). Indeed, because of the continuity of the function ρ, the solving of the Cauchy problem

$$u' = -M\,\rho(u), \quad u(T) = 0$$

is equivalent with the solving of the integral equation

$$u(t) = M \int_t^T \rho(u(s))\,ds, \quad t \in [0, T].$$

If $\int_{0^+}^{\infty} \frac{du}{\rho(u)} = \infty$, the equation $u' = -M\,\rho(u)$ has a unique solution $u(t) \equiv 0$, $t \in [0, T]$. This implies that the class of the functions satisfying the hypothesis (**H$_2$**) is more general than the the class of the functions satisfying the hypothesis (**H$_3$**). Even more, it should be noted that in the more general assumption (**H$_2$**), function ρ depends also on a time parameter t, (ρ is dependent of two parameters t and u, $\rho : [0, T] \times R^+ \to R^+$) which is not the case in the assumption (**H$_3$**) where ρ depends only of u ($\rho : R^+ \to R^+$). By Proposition 4.1, it follows that BDSDE (2) has a unique solution when ξ, f, g and h satisfy (**H$_0$**) and (**H$_3$**).

Following theorem gives the estimate of the L^p-moments of the solution of Eq. (2) under assumption that f, g and h satisfy the non-Lipschitz conditions (**H$_3$**) and under the additional condition for α, i.e. $\alpha \in \left(0, 1/2(p-1)\right)$.

Lemma 4.3. *Let ξ, f, g, h and $p \geq 2$ satisfy (**H$_3$**) for $\alpha \in \left(0, 1/2(p-1)\right)$. If $\xi \in L^p(\Omega, \mathcal{F}_T, \mathbf{P}; \mathbb{R}^d)$ and*

$$E \int_0^T \left(|f(t, 0, 0)|^p + \|g(t, 0, 0)\|^p + \|h(t, 0)\|^p\right) dt < \infty, \tag{9}$$

than the solution $\{(Y(t), Z(t)), t \in [0, T]\}$ of Eq. (2) satisfies

$$\sup_{0 \leq t \leq T} E|Y(t)|^p + E \int_0^T |Y(t)|^{p-2}\|Z(t)\|^2\,dt < \infty.$$

Proof. By applying Itô's formula on $|Y(t)|^p$, we have

$$\begin{aligned}|Y(t)|^p &= |\xi|^p + p\int_t^T |Y(s)|^{p-2}(Y(s))^T f(s,Y(s),Z(s))\,ds + \frac{p}{2}\int_t^T |Y(s)|^{p-2}\|g(s,Y(s),Z(s))\|^2\,ds \\ &+ \frac{p(p-2)}{2}\int_t^T |Y(s)|^{p-4}|(Y(s))^T g(s,Y(s),Z(s))|^2\,ds + p\int_t^T |Y(s)|^{p-2}(Y(s))^T g(s,Y(s),Z(s))\,dB(s) \\ &- \frac{p}{2}\int_t^T |Y(s)|^{p-2}\|h(s,Y(s))+Z(s)\|^2\,ds - \frac{p(p-2)}{2}\int_t^T |Y(s)|^{p-4}|(Y(s))^T[h(s,Y(s))+Z(s)]|^2\,ds \\ &- p\int_t^T |Y(s)|^{p-2}(Y(s))^T[h(s,Y(s))+Z(s)]\,dW(s).\end{aligned} \tag{10}$$

Taking expectation, we find that

$$\begin{aligned}E|Y(t)|^p &\le E|\xi|^p + pE\int_t^T |Y(s)|^{p-2}(Y(s))^T f(s,Y(s),Z(s))\,ds + \frac{p(p-1)}{2}E\int_t^T |Y(s)|^{p-2}\|g(s,Y(s),Z(s))\|^2\,ds \\ &\quad - \frac{p}{2}E\int_t^T |Y(s)|^{p-2}\|h(s,Y(s))+Z(s)\|^2\,ds, \\ &\equiv E|\xi|^p + pI_1 + \frac{p(p-1)}{2}I_2 + \frac{p}{2}I_3.\end{aligned} \tag{11}$$

where I_1, I_2 and I_3 are the appropriate integrals which must be estimated. By applying $(\mathbf{H_3})$ and elementary inequalities: $\pm 2ab \le \frac{a^2}{\epsilon} + b^2\epsilon$, $(a+b)^2 \le 2a^2 + 2b^2$, $a^{p-2}b^2 \le \frac{p-2}{p}a^p + \frac{2}{p}b^p$, for $\epsilon_1 > 0$, we have

$$\begin{aligned}I_1 &= E\int_t^T |Y(s)|^{p-2}(Y(s))^T f(s,Y(s),Z(s))\,ds \\ &\le \frac{1}{2\epsilon_1}E\int_t^T |Y(s)|^p ds + \epsilon_1 E\int_t^T |Y(s)|^{p-2}|f(s,Y(s),Z(s)) - f(s,0,0)|^2\,ds + \epsilon_1 E\int_t^T |Y(s)|^{p-2}|f(s,0,0)|^2\,ds \\ &\le \frac{1}{2\epsilon_1}E\int_t^T |Y(s)|^p ds + \epsilon_1 E\int_t^T |Y(s)|^{p-2}\rho^{\frac{2}{p}}(|Y(s)|^p)\,ds \\ &+ \epsilon_1 CE\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\,ds + \epsilon_1 E\int_t^T |Y(s)|^{p-2}|f(s,0,0)|^2\,ds \\ &\le \left(\frac{1}{2\epsilon_1} + \frac{2(p-2)\epsilon_1}{p}\right)E\int_t^T |Y(s)|^p ds + \frac{2\epsilon_1}{p}E\int_t^T \rho(|Y(s)|^p)\,ds \\ &+ \frac{2\epsilon_1}{p}E\int_t^T |f(s,0,0)|^p\,ds + \epsilon_1 CE\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\,ds.\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}I_2 &= E\int_t^T |Y(s)|^{p-2}\|g(s,Y(s),Z(s))\|^2\,ds \\ &\le 2E\int_t^T |Y(s)|^{p-2}\rho^{\frac{2}{p}}(|Y(s)|^p)\,ds + 2\alpha E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\,ds + 2E\int_t^T |Y(s)|^{p-2}\|g(s,0,0)\|^2\,ds \\ &\le \frac{4(p-2)}{p}E\int_t^T |Y(s)|^p\,ds + \frac{4}{p}E\int_t^T \rho(|Y(s)|^p)\,ds + 2\alpha E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\,ds + \frac{4}{p}E\int_t^T \|g(s,0,0)\|^p\,ds.\end{aligned} \tag{13}$$

Analogously, for $\epsilon_2 > 0$,

$$
\begin{aligned}
I_3 &\le -2E\int_t^T trace[|Y(s)|^{p-2}(Z(s))^T h(s,Y(s))]\ ds - E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\ ds\\
&\le \frac{1}{\epsilon_2}E\int_t^T |Y(s)|^{p-2}\|h(s,Y(s)) - h(s,0) + h(s,0)\|^2\ ds\\
&\quad + \epsilon_2 E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\,ds - E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\ ds\\
&\le (\epsilon_2-1)E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\,ds + \frac{2}{\epsilon_2}E\int_t^T |Y(s)|^{p-2}\rho^{\frac{2}{p}}(|Y(s)|^p)\ ds + \frac{2}{\epsilon_2}E\int_t^T |Y(s)|^{p-2}\|h(s,0)\|^2\,ds\\
&\le (\epsilon_2-1)E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\,ds + \frac{4(p-2)}{p\epsilon_2}E\int_t^T |Y(s)|^p\ ds\\
&\quad + \frac{4}{p\epsilon_2}E\int_t^T \rho(|Y(s)|^p)\ ds + \frac{4}{p\epsilon_2}E\int_t^T \|h(s,0)\|^p\,ds.
\end{aligned}
\tag{14}
$$

In view of the previous estimates, we find from the relations (12), (13) and (14) that

$$
\begin{aligned}
E|Y(t)|^p \le E|\xi|^p &+ \left[\frac{p}{2\epsilon_1} + 2(p-2)\epsilon_1 + 2(p-1)(p-2) + \frac{2(p-2)}{\epsilon_2}\right]E\int_t^T |Y(s)|^p ds\\
&+ \left[2\epsilon_1 + 2(p-1) + \frac{2}{\epsilon_2}\right]E\int_t^T \rho(|Y(s)|^p)\,ds\\
&+ \left[\frac{p}{2}(\epsilon_2-1) + p\epsilon_1 C + p(p-1)\alpha\right]E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\ ds\\
&+ E\int_t^T [2\epsilon_1|f(s,0,0)|^p + 2(p-1)\|g(s,0,0)\|^p + \frac{2}{\epsilon_2}\|h(s,0)\|^p]\,ds.
\end{aligned}
\tag{15}
$$

Let us denote that

$$
\begin{aligned}
c_1 &= \frac{p}{2\epsilon_1} + 2(p-2)\epsilon_1 + 2(p-1)(p-2) + \frac{2(p-2)}{\epsilon_2},\\
c_2 &= 2\epsilon_1 + 2(p-1) + \frac{2}{\epsilon_2},\\
c_3 &= p\left[\frac{1}{2} - (p-1)\alpha - \epsilon_1 C - \frac{\epsilon_2}{2}\right],\\
c_4 &= E\int_0^T \left[2\epsilon_1|f(t,0,0)|^p + 2(p-1)\|g(t,0,0)\|^p + \frac{2}{\epsilon_2}\|h(t,0)\|^p\right]dt.
\end{aligned}
$$

From (9), we have that c_4 is a finite constant.

Applying previous notations for the constants, it follows from the inequality (15) that

$$
\begin{aligned}
&E|Y(t)|^p + c_3 E\int_t^T |Y(s)|^{p-2}\|Z(s)\|^2\ ds\\
&\qquad \le E|\xi|^p + c_1 E\int_t^T |Y(s)|^p ds + c_2 E\int_t^T \rho(|Y(s)|^p) + c_4.
\end{aligned}
\tag{16}
$$

Regarding that $C > 0$ and $\alpha \in \big(0, 1/2(p-1)\big)$ are the known constants, we can determine positive numbers ϵ_1 and ϵ_2 such that $c_3 > 0$. Applying Jensen's inequality [16] on the concave function ρ, we find from (16) that

$$
E|Y(t)|^p \le E|\xi|^p + \max\{c_1, c_2\}\int_t^T [E|Y(s)|^p ds + \rho(E|Y(s)|^p)]\,ds + c_4. \tag{17}
$$

Applying Bihari's inequality on (17) (see more about the application od Bihari's inequality in 6. Appendix), it follows that

$$E|Y(t)|^p \leq G^{-1}[G\,(E|\xi|^p + c_4) + \max\{c_1, c_2\}(T-t)] < \infty,\ t \in [0,T], \tag{18}$$

and from (16) and (18),

$$E\int_0^T |Y(s)|^{p-2}\|Z(s)\|^2\, ds \leq \frac{1}{c_3}G^{-1}[G\,(E(|\xi|^p) + c_4) + \max\{c_1, c_2\}T]. \tag{19}$$

From the last two estimates, it follows that

$$\sup_{0\leq t\leq T} E|Y(t)|^p + E\int_0^T |Y(t)|^{p-2}\|Z(t)\|^2\, dt < \infty,$$

which completes the proof. □

Following two theorems give a very important results for the solution of the Eq.(2), ie L^p estimate for $p \geq 2$.

Theorem 4.4. *Let all assumptions of Lemma 4.3 hold and let* $\{(Y(t), Z(t)), t \in [0,T]\}$ *be the solution of Eq. (2), than process* $Y(t)$ *satisfies*

$$E\sup_{0\leq t\leq T}|Y(t)|^p < \infty.$$

Proof. For chosen $p \geq 2$ and each integer $n \geq 1$, let us introduce stopping time

$$\tau_n = \inf\left\{t \in [0,T], E\sup_{0\leq t\leq \tau_n}|Y(t)|^p \geq n\right\} \wedge T.$$

Applying the same estimates, we find from (10) that

$$\begin{aligned} E\sup_{0\leq t\leq \tau_n}|Y(t)|^p \leq C_1 &+ pE\sup_{0\leq t\leq \tau_n}\left|\int_t^{\tau_n}|Y(s)|^{p-2}(Y(s))^T g(s, Y(s), Z(s))\, dB(s)\right| \\ &+ pE\sup_{0\leq t\leq \tau_n}\left|\int_t^{\tau_n}|Y(s)|^{p-2}(Y(s))^{\tau_n}[h(s, Y(s)) + Z(s)]\, dW(s)\right|, \end{aligned} \tag{20}$$

where C_1 is a positive constant. By applying Burkholder-Davis-Gundy inequality [15], hypothesis ($\mathbf{H_3}$) and elementary inequalities, we will obtain the estimates for the integrals from (20). For arbitrary $\epsilon_3 > 0$,

$$\begin{aligned} &E\sup_{0\leq t\leq \tau_n}\left|\int_t^{\tau_n}|Y(s)|^{p-2}(Y(s))^T g(s, Y(s), Z(s))\, dB(s)\right| \\ &\quad\leq 4E\left(\int_0^{\tau_n}|Y(t)|^{2p-2}\|g(t, Y(t), Z(t)\|^2\, dt\right)^{\frac{1}{2}} \\ &\quad\leq 4E\left\{\left((\sup_{0\leq t\leq \tau_n}|Y(t)|^p)\right)^{\frac{1}{2}}\left(\int_0^{\tau_n}|Y(t)|^{p-2}\|g(t, Y(t), Z(t))\|^2\, dt\right)^{\frac{1}{2}}\right\} \\ &\quad\leq 2\epsilon_3 E\sup_{0\leq t\leq \tau_n}|Y(t)|^p + \frac{4}{\epsilon_3}E\int_0^{\tau_n}|Y(t)|^{p-2}\rho^{\frac{2}{p}}(|Y(t)|^p)\, dt \\ &\qquad+\frac{4\alpha}{\epsilon_3}E\int_0^{\tau_n}|Y(t)|^{p-2}\|Z(t)\|^2\, dt + \frac{4}{\epsilon_3}E\int_0^{\tau_n}|Y(t)|^{p-2}\|g(t, 0, 0)\|^2\, dt. \end{aligned} \tag{21}$$

Similarly, for arbitrary $\epsilon_4 > 0$,

$$
\begin{aligned}
E \sup_{0\le t\le\tau_n} &\left| \int_t^{\tau_n} |Y(s)|^{p-2}(Y(s))^T[h(s,Y(s)) + Z(s)]\,dW(s) \right| \qquad (22)\\
&\le 4E\left(\int_0^{\tau_n} |Y(t)|^{2p-2}\|h(t,Y(t)) + Z(t)\|^2\,ds \right)^{\frac{1}{2}}\\
&\le 4E\left\{ \left(\sup_{0\le t\le\tau_n} |Y(t)|^p \right)^{\frac{1}{2}} \left(\int_0^{\tau_n} |Y(t)|^{p-2}\|h(s,Y(t)) + Z(t)\|^2\,dt \right)^{\frac{1}{2}} \right\}\\
&\le 2\epsilon_4 E\left(\sup_{0\le t\le\tau_n} |Y(t)|^p \right) + \frac{6}{\epsilon_4}E\int_0^{\tau_n} |Y(t)|^{p-2}\rho^{\frac{2}{p}}(|Y(t)|^p)\,dt\\
&\quad + \frac{6}{\epsilon_4}E\int_0^{\tau_n} |Y(t)|^{p-2}\|Z(t)\|^2\,dt + \frac{6}{\epsilon_4}E\int_0^{\tau_n} |Y(t)|^{p-2}\|h(t,0)\|^2\,dt.
\end{aligned}
$$

Substituting (21) and (22) in (20), applying Hölder and Jensen's inequalities, we find that

$$
\begin{aligned}
E \sup_{0\le t\le\tau_n} |Y(t)|^p &\le C_2 + 2p(\epsilon_3 + \epsilon_4)E \sup_{0\le t\le\tau_n} |Y(t)|^p + 8(p-2)\left(\frac{2}{\epsilon_3} + \frac{3}{\epsilon_4} \right)\int_0^{\tau_n} E|Y(t)|^p\,dt \qquad (23)\\
&\quad + 4\left(\frac{2}{\epsilon_3} + \frac{3}{\epsilon_4} \right)\int_0^{\tau_n} \rho(E|Y(t)|^p)\,dt + 2p\left(\frac{2\alpha}{\epsilon_3} + \frac{3}{\epsilon_4} \right)E\int_0^{\tau_n} |Y(t)|^{p-2}\|Z(t)\|^2\,dt,
\end{aligned}
$$

where

$$
C_2 = C_1 + \frac{8}{\epsilon_3}E\int_0^{\tau_n} \|g(t,0,0)\|^p\,dt + \frac{12}{\epsilon_4}E\int_0^{\tau_n} \|h(t,0)\|^p\,dt.
$$

Constants ϵ_3 and ϵ_4 are arbitrary and can be chosen such that $1 - 2p(\epsilon_3 + \epsilon_4) > 0$. Therefore, we conclude from (9), (18), (19), (23) and Lemma 4.3 that

$$
E(\sup_{0\le t\le\tau_n} |Y(t)|^p) < \infty.
$$

If we let $n \to \infty$, by applying Fatou's lemma we have that

$$
E(\sup_{0\le t\le T} |Y(t)|^p) < \infty,
$$

which completes the proof. □

Theorem 4.5. *Let all assumptions of Lemma 4.3 hold and let* $\{(Y(t), Z(t)), t \in [0,T]\}$ *be the solution of Eq. (2), than process* $Z(t)$ *satisfies*

$$
E\left(\int_0^T \|Z(t)\|^2\,dt \right)^{\frac{p}{2}} < \infty.
$$

Proof. For each integer $n \ge 1$, let us introduce the stopping time

$$
\tau_n = \left\{ t \in [0,T], \int_0^t \|Z(t)\|^2 dt \ge n \right\} \wedge n.
$$

From (10), for $t = 0$ and $p = 2$,

$$
\begin{aligned}
|Y(0)|^2 \le |\xi|^2 &+ 2\int_t^{\tau_n} Y^T(s)f(s,Y(s),Z(s))\,ds - \int_t^{\tau_n} \|Z(s)\|^2\,ds\\
&+ \int_t^{\tau_n} \|g(s,Y(s),Z(s))\|^2\,ds + 2\int_t^{\tau_n} (Y(s))^T g(s,Y(s),Z(s))\,dB(s)\\
&- 2\int_t^{\tau_n} trace[(Z(s))^T h(s,Y(s))]\,ds - 2\int_t^{\tau_n} (Y(s))^T[h(s,Y(s)) + Z(s)]\,dW(s).
\end{aligned}
$$

Hence,

$$\begin{aligned}\int_t^{\tau_n} \|Z(s)\|^2\, ds \le |\xi|^2 - |Y(0)|^2 + 2\int_t^{\tau_n} (Y(s))^T f(s,Y(s),Z(s))\, ds \\ + \int_t^{\tau_n} \|g(s,Y(s),Z(s))\|^2\, ds + 2\int_t^{\tau_n} (Y(s))^T g(s,Y(s),Z(s))\, dB(s) \\ - 2\int_t^{\tau_n} trace[(Z(s))^T h(s,Y(s))]\, ds - 2\int_t^{\tau_n} (Y(s))^T [h(s,Y(s)) + Z(s)]\, dW(s).\end{aligned}$$

By applying the elementary inequality $|a+b|^p \le (1+\delta)|a|^p + \phi(\delta)|b|^p$, where $\delta > 0$ and $\phi(\delta)$ is a generic constant (see [16]), we find that

$$\begin{aligned}&\left(\int_t^{\tau_n} \|Z(s)\|^2\, ds\right)^{\frac{p}{2}} \\ &\le (1+\delta)\left|\int_t^{\tau_n} \|g(s,Y(s),Z(s))\|^2\, ds\right|^{\frac{p}{2}} + q(\delta,p)\left[|\xi|^p + |Y_0|^p + 2^{\frac{p}{2}}\left|\int_t^{\tau_n} (Y(s))^T f(s,Y(s),Z(s))\, ds\right|^{\frac{p}{2}}\right. \\ &\quad + 2^{\frac{p}{2}}\left|\int_t^{\tau_n} (Y(s))^T g(s,Y(s),Z(s))\, dB(s)\right|^{\frac{p}{2}} + 2^{\frac{p}{2}}\left|\int_t^{\tau_n} trace[(Z(s))^T h(s,Y(s))]\, ds\right|^{\frac{p}{2}} \\ &\quad \left. + 2^{\frac{p}{2}}\left|\int_t^{\tau_n} (Y(s))^T [h(s,Y(s)) + Z(s)]\, dW(s)\right|^{\frac{p}{2}}\right],\end{aligned}$$

for an arbitrary constant $\delta > 0$ and generic constant $q(\delta,p)$.

Previous three estimates resembles to some standard methods for BSDEs in general used first by Pardoux and Peng (for example see [21]). A proper credit is given to the authors for this part of the proof.

Therefore,

$$E\left(\int_t^{\tau_n} \|Z(s)\|^2\, ds\right)^{\frac{p}{2}} \le (1+\delta)J_2 + q(\delta,p)\left[2E\left(\sup_{0\le t\le T} |Y(s)|^p\right) + J_1 + J_3 + J_4 + J_5\right], \tag{24}$$

where J_1, J_2, J_3 and J_4 are the appropriate integrals. By applying hypothesis (**H$_3$**), for an arbitrary $\varepsilon_5 > 0$,

$$\begin{aligned}J_1 &\le 2E\left[\int_t^{\tau_n} \left(\frac{|Y(s)|^2}{\epsilon_5} + \epsilon_5 |f(s,Y(s),Z(s))|^2\right) ds\right]^{\frac{p}{2}} \qquad (25)\\ &\le 2^{\frac{p}{2}-1}\left[\frac{T^{\frac{p}{2}-1}}{\epsilon_5^{\frac{p}{2}}} E\int_t^{\tau_n} |Y(s)|^p\, ds + \epsilon_5^{\frac{p}{2}} 2^{\frac{p}{2}-1} E\left(\int_t^{\tau_n} |f(s,Y(s),Z(s)) - f(s,0,0)|^2\, ds + \int_t^{\tau_n} |f(s,0,0)|^2\, ds\right)^{\frac{p}{2}}\right] \\ &\le 2^{\frac{p}{2}-1}\left[\frac{T^{\frac{p}{2}}}{\epsilon_5^{\frac{p}{2}}} \sup_{0\le t\le T} E|Y(t)|^p + \epsilon_5^{\frac{p}{2}} 2^{p-1}\left(E\int_t^{\tau_n} [\rho^{\frac{2}{p}}(|Y(s)|^p) + C\|Z(s)\|^2]\, ds\right)^{\frac{p}{2}}\right. \\ &\qquad \left. + \epsilon_5^{\frac{p}{2}} 2^{p-1} T^{\frac{p}{2}-1} E\int_t^{\tau_n} \|f(s,0,0)\|^p\, ds\right] \\ &\le 2^{\frac{p}{2}-1}\left[\frac{T^{\frac{p}{2}}}{\epsilon_5^{\frac{p}{2}}} \sup_{0\le t\le T} E|Y(t)|^p + \epsilon_5^{\frac{p}{2}} 2^{\frac{3p}{2}-2}\left(T^{\frac{p}{2}} \sup_{0\le t\le T} \rho(E|Y(t)|^p) + C^{\frac{p}{2}} E\left(\int_t^{\tau_n} \|Z(s)\|^2\, ds\right)^{\frac{p}{2}}\right)\right. \\ &\qquad \left. + \epsilon_5^{\frac{p}{2}} 2^{p-1} T^{\frac{p}{2}-1} E\int_t^{\tau_n} |f(s,0,0)|^p\, ds\right] \\ &\le (C\epsilon_5)^{\frac{p}{2}} 2^{2p-3} E\left(\int_t^{\tau_n} \|Z(s)\|^2\, ds\right)^{\frac{p}{2}} + r_1,\end{aligned}$$

where r_1 is a generic constant.

In order to estimate J_2, it should be noted that from hypothesis $(\mathbf{H}_3)$ it follows that there exists a constant $\varepsilon_0 > 0$ such that

$$\|g(t,y,z)\|^2 \le (1+\varepsilon_0)\rho^{\frac{2}{p}}(|y|^2) + \alpha(1+\varepsilon_0)\|z\|^2 + \left(1+\frac{1}{\varepsilon_0}\right)\|g(t,0,0)\|^2.$$

Substituting the last inequality in integral J_2 and applying the previous elementary inequalities, we derive that

$$\begin{aligned} J_2 &\le (1+\delta)\alpha(1+\varepsilon_0)\left(\int_t^{\tau_n}\|Z(s)\|^2\,ds\right)^{\frac{p}{2}} + q'(\delta,p)\left|\int_t^{\tau_n}[(1+\varepsilon_0)\rho^{\frac{2}{p}}(|Y(s)|^2) + (1+\frac{1}{\varepsilon_0})\|g(s,0,0)\|^2]\,ds\right|^{\frac{p}{2}} \qquad (26)\\ &\le (1+\delta)\alpha(1+\varepsilon_0)\left(\int_t^{\tau_n}\|Z(s)\|^2\,ds\right)^{\frac{p}{2}} + r_2, \end{aligned}$$

where r_2 and $q'(\delta,p)$ are generic constants.

Analogously, for $\varepsilon_i > 0$, $i = 6, ..., 11$,

$$\begin{aligned} J_3 &\le 2^{\frac{p}{2}} E\left(\int_t^{\tau_n}|Y(s)|^2\|g(s,Y(s),Z(s))\|^2\,ds\right)^{\frac{p}{4}} \qquad (27)\\ &\le 2^p E\left[\sup_{0\le t\le T}|Y(t)|^{\frac{p}{2}}\left(\int_t^{\tau_n}[\rho^{\frac{2}{p}}(|Y(s)|^p) + \alpha\|Z(s)\|^2 + \|g(s,0,0)\|^2]\,ds\right)^{\frac{p}{4}}\right]\\ &\le 2^{p-1}(3T)^{\frac{p}{4}-1}E\left(\frac{1}{\epsilon_6}(\sup_{0\le t\le T}|Y(t)|^{\frac{p}{2}})^2 + \epsilon_6 T\rho(|Y(s)|^p)\right)\\ &\quad + 2^{p-1}(3T)^{\frac{p}{4}-1}\alpha^{\frac{p}{4}}E\left[\frac{1}{\epsilon_7}(\sup_{0\le t\le T}|Y(t)|^{\frac{p}{2}})^2 + \epsilon_7\left(\int_t^{\tau_n}\|Z(s)\|^2\,ds\right)^{\frac{p}{2}}\right]\\ &\quad + 2^{p-1}(3T)^{\frac{p}{4}-1}E\left[\frac{1}{\epsilon_8}(\sup_{0\le t\le T}|Y(t)|^{\frac{p}{2}})^2 + \epsilon_8 T\int_t^{\tau_n}\|g(s,0,0)\|^p\,ds\right]\\ &\le 2^{p-1}(3T)^{\frac{p}{4}-1}\alpha^{\frac{p}{4}}\epsilon_7 E\left(\int_t^{\tau_n}\|Z(s)\|^2\,ds\right)^{\frac{p}{2}} + r_3,\\ J_4 &\le E\left(\int_t^{\tau_n}[\epsilon_9\|Z(s)\|^2 + \frac{1}{\epsilon_9}\rho^{\frac{2}{p}}(|Y_s|^p)]\,ds\right)^{\frac{p}{2}} \qquad (28)\\ &\le 2^{\frac{p}{2}-1}\left\{\epsilon_9^{\frac{p}{2}}E\left(\int_t^{\tau_n}\|Z(s)\|^2\,ds\right)^{\frac{p}{2}} + \frac{1}{\epsilon_9^{\frac{p}{2}}}T^{\frac{p}{2}}\sup_{0\le t\le T}\rho(E|Y_s|^p)\right\}\\ &\le 2^{\frac{p}{2}-1}\epsilon_9^{\frac{p}{2}}E\left(\int_t^{\tau_n}\|Z(s)\|^2\,ds\right)^{\frac{p}{2}} + r_4, \end{aligned}$$

and

$$J_5 \leq 2^{\frac{p}{2}} E\left(\int_t^{\tau_n} |Y(s)|^2 \|h(s,Y(s)) + Z(s)\|^2\,ds\right)^{\frac{p}{4}} \tag{29}$$

$$\leq 2^{\frac{p}{2}} E \sup_{0\leq t\leq T} |Y(t)|^{\frac{p}{2}} \left(\int_t^{\tau_n} [2\rho^{\frac{2}{p}}(|Y(s)|^p) + 2\|Z(s)\|^2]\,ds\right)^{\frac{p}{4}}$$

$$\leq 2^{\frac{p}{2}} E \sup_{0\leq t\leq T} |Y(t)|^{\frac{p}{2}} 2^{\frac{p}{4}-1}\left(2^{\frac{p}{4}} T^{\frac{p}{4}-1} \int_t^{\tau_n} \rho^{\frac{1}{2}}(|Y(s)|^p)\,ds + \left(\int_t^{\tau_n} \|Z(s)\|^2\,ds\right)^{\frac{p}{4}}\right)$$

$$\leq 2^{p-1} T^{\frac{p}{4}-1} E \sup_{0\leq t\leq T} |Y(t)|^{\frac{p}{2}} \int_t^{\tau_n} \rho^{\frac{1}{2}}(|Y(s)|^p)\,ds + 2^{\frac{3p}{4}-1} T^{\frac{p}{4}-1} E \sup_{0\leq t\leq T} |Y(t)|^{\frac{p}{2}} \left(\int_t^{\tau_n} \|Z(s)\|^2\,ds\right)^{\frac{p}{4}}$$

$$\leq 2^{2p-4} T^{\frac{p}{2}-2} \epsilon_{10} E \sup_{0\leq t\leq T} |Y(t)|^p + \frac{T}{\epsilon_{10}} \sup_{0\leq t\leq T} \rho(E|Y(s)|^p)\,ds + \frac{2^{\frac{3p}{2}-2} T^{\frac{p}{2}-2}}{\epsilon_{11}} E \sup_{0\leq t\leq T} |Y(t)|^p + \epsilon_{11} \left(\int_t^{\tau_n} \|Z(s)\|^2\,ds\right)^{\frac{p}{2}}$$

$$\leq \epsilon_{11} \left(\int_t^{\tau_n} \|Z(s)\|^2\,ds\right)^{\frac{p}{2}} + r_5,$$

where r_3, r_4, r_5 are generic constants.

Since we have from Theorem 4.4, Lemma 4.3 and property of function ρ that $E\sup_{0\leq t\leq T}|Y(t)|^p < \infty$, $\sup_{0\leq t\leq T} \rho(E|Y(t)|^p) < \infty$, and as assumption (9) holds, substituting (25)–(29) in (24), we obtain that

$$E\left(\int_t^{\tau_n} \|Z(s)\|^2 ds\right)^{\frac{p}{2}} \leq R + \left[q(\delta,p)\varepsilon + (1+\delta)^2\alpha(1+\varepsilon_0)\right] E\left(\int_t^{\tau_n} \|Z(s)\|^2\,ds\right)^{\frac{p}{2}}, \tag{30}$$

where R is a generic constant and $\varepsilon = (C\epsilon_5)^{\frac{p}{2}} 2^{2p-3} + 2^{p-1}(3T)^{\frac{p}{4}-1}\alpha^{\frac{p}{4}}\epsilon_7 + 2^{\frac{p}{2}-1}\epsilon_9^{\frac{p}{2}} + \epsilon_{11}$. For a given $\alpha \in (0,1)$, we can choose constants $\varepsilon_0, \epsilon_5, \epsilon_7, \epsilon_9, \epsilon_{11}$ and δ such that

$$(1+\delta)^2\alpha(1+\varepsilon_0) + q(\delta,p)\varepsilon < 1.$$

Since R is finite, from (30) we find that

$$E\left(\int_t^{\tau_n} \|Z(s)\|^2\,ds\right)^{\frac{p}{2}} < \infty.$$

If we let $n \to \infty$, by applying Fatou's lemma we have that

$$E\left(\int_t^{T} \|Z(s)\|^2\,ds\right)^{\frac{p}{2}} < \infty.$$

which completes the proof. □

Remark. An important remark should be added for the particular case when $p = 2$. As it is already known, and it is given here by Proposition 4.2 (Janković, Djordjević, Jovanović [13]), under assumptions (H_0) and (H_2) for the functions f, g i h, where $\alpha \in (0,1)$, there exists a solution of Eq. (2) such that

$$E\sup_{0\leq t\leq T}|Y(t)|^2 < \infty, \quad E\left(\int_0^T \|Z(t)\|^2\,dt\right) < \infty.$$

It is proven in Theorem 4.4 and Theorem 4.5 that under assumption (H_3) for functions f, g i h, more flexible condition than (H_0), ie condition (9), but with restriction of α, $\alpha \in (0, 1/2)$, the same result holds for the solution of the Eq. (2). So, when conditions of Proposition 4.2 are not satisfied, but solution of the Eq. (2) exists, Theorems 4.4 and 4.5 could be applied to establish L^2-estimate of the solution.

5. Conclusion

The difference between Eq. (1) and Eq. (2) is a function h in the forward integral which can be treated as a perturbation. It is interesting to analyze the equation which is obtained by perturbing not only the function h in the forward integral, but also by perturbing the final condition and the functions f and g. In order to analyze this type of completely perturbed equation, it is necessary to study the L^p-stability of the solution of the nonhomogeneous BDSDE. Some interesting problems arise from the problem of perturbations, such as the closeness between the solutions of the perturbed and unperturbed equations, time intervals on which the difference of these solutions stay close enough etc., which could be the topic of the study of forthcoming papers.

6. Appendix

Let function satisfies condition (H_3). Since $\rho(x)$ is the concave function with $\rho(0) = 0$, it follows for the $0 \leq x \leq 1$ that $\frac{\rho(x)}{x} \geq \frac{\rho(1)}{1}$, i.e.

$$\rho(x) \geq \rho(1)x, \ \ 0 \leq x \leq 1.$$

Let us define the function $\rho_1(x) := x + \rho(x)$, which has the same properties as the function ρ: $\rho_1 : R^+ \to R^+$, it is continuous, nondecreasing and concave, such that $\rho_1(0) = 0$ and

$$\int_{0+} \frac{dx}{x + \rho(x)} \geq \frac{\rho(1)}{1 + \rho(1)} \int_{0+} \frac{dx}{\rho(x)} = \infty.$$

We can apply Bihari's inequality [3] adapted to (17): Let g be monotone continuous function, strictly positive on an interval I, containing a point u_0, which vanishes nowhere in I. Let u and k be continuous functions on an interval $J = (\alpha, \beta]$ such that $u(J) \subset I$, and suppose that k has a fixed sign in J. Suppose that

$$u(t) \leq a + \int_t^\beta k(s)g(u(s))\,ds, \ \ t \in J.$$

If either

(i) g is nondecreasing and k is nonnegative,

or

(ii) g is nonincreasing and k is nonpositive,

then

$$u(t) \leq G^{-1}\left(G(a) + \int_t^\beta k(s)\,ds\right), \ \ \alpha_1 < t \leq \beta, \tag{31}$$

where $G(u) = \int_{u_0}^u \frac{dx}{g(x)}$, $u \in I$ and $\alpha_1 = \max\{\mu_1, \mu_2\}$, with

$$\mu_1 = \inf\left\{\mu \in J : a + \int_t^\beta k(s)g(u(s))\,ds \in I, \ \mu \leq t \leq \beta\right\},$$

$$\mu_2 = \inf\left\{\mu \in J : G(a) + \int_t^\beta k(s)\,ds \in G(I), \ \mu \leq t \leq \beta\right\}.$$

In order to apply Bihari's inequality adapted to (17), it is necessary to justify its conditions. The intervals I and J are $I = [0, \infty)$ and $J = [0, T]$, and $u(s) = E|Y(s)|^p$, $\ k(s) = \max\{c_1, c_2\}$. Further, for every $s \in [0, T]$, we have that $E|Y(s)|^p \in I$, which implies $u(J) \subset I$. A constant a from Bihari's theorem is $E|\xi|^p + c_4$, the function $G(u) = \int_{u_0}^u \frac{dx}{x+\rho(x)}$ and

$$\mu_1 = \inf\left\{\mu \in J : E|\xi|^p + c_4 + \max\{c_1, c_2\} \int_t^{\tau_n} [E|Y(s)|^p + \rho(E|Y(s)|^p)]\,ds \in I, \mu \leq t \leq T\right\} = 0,$$

$$\mu_2 = \inf\left\{\mu \in J : G(E|\xi|^p + c_4 C) + \max\{c_1, c_2\} \int_t^{\tau_n} [E|Y(s)|^p + \rho(E|Y(s)|^p)]\,ds \in G(I), \mu \leq t \leq T\right\} = 0,$$

while $\alpha_1 = \max\{\mu_1, \mu_2\} = 0$.

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