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Disjoint Topological Transitivity for Cosine Operator Functions on Groups

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Abstract. In this paper, we study finite sequences of operators, generated by the powers of weighted translations on discrete groups, and give sufficient conditions for such sequences to be disjoint topologically transitive and mixing in terms of the group elements and weights. The sequences of operators are cosine operator functions. Moreover, we also obtain necessary conditions for cosine operator functions to be disjoint transitive and mixing.

1. Introduction

Recently, Bernal-González, Bès and Peris studied new notions of linear dynamics, namely, disjoint transitivity and disjoint hypercyclicity in [2] and [7] respectively. Since then, disjoint transitivity and disjoint hypercyclicity were investigated by many authors [3–6, 13, 14, 19, 24–36]. For instance, the existence of disjoint hypercyclic operators on separable, infinite-dimensional topological vector spaces was studied in [33, 34]. In [3, 4], Bès, Martin and Peris investigated disjoint hypercyclic composition operators on spaces of holomorphic functions. Disjoint mixing composition operators on the Hardy space in the unit ball were also characterized by Liang and Zhou in [28]. In addition, Salas considered dual disjoint hypercyclicity in [31]. The characterizations for weighted shifts and powers of weighted shifts on $\ell^p(\mathbb{Z})$ and weighted sequence spaces to be disjoint hypercyclic and supercyclic were obtained in [6, 7, 27, 29] respectively. In [36], Zhang and Zhou studied disjointness in supercyclicity on the algebra of Hilbert-Schmidt operators. Also, Puig [30] introduced the study of disjoint hypercyclicity along filter, and extended some results of Bès, Martin, Peris and Shkarin [5]. The necessary and sufficient condition for operators, which map a holomorphic function to a partial sum of the Taylor expansion, to be disjoint hypercyclic was given by Vlachou in [35]. Kostić studied disjoint hypercyclicity of C-distribution cosine functions and semigroups in [24, 26]. It should be noted that a survey of recent results on hypercyclic and topologically mixing properties of various types of abstract (time-fractional) PDEs in Banach and Fréchet spaces is contained in this monograph [25].

These new notions, *disjoint topological transitivity* and *disjoint hypercyclicity*, are kind of generalizations of transitivity and hypercyclicity respectively. A sequence of operators (T_n) is called *topologically transitive* if given two nonempty open subsets U, V of a Banach space X, there is some $n \in \mathbb{N}$ such that $T_n(U) \cap V \neq \emptyset$. If $T_n(U) \cap V \neq \emptyset$ from some n onwards, then (T_n) is called *topologically mixing*. For (T_n), it is known that

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topological transitivity is equivalent to hypercyclicity with dense set of hypercyclic vectors. A sequence (T_n) is *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\{T_n x : n \geq 0\}$ is dense in X; if this is the case, then we say that x is a *hypercyclic vector* of (T_n). Transitivity and hypercyclicity have been studied intensely in the past three decades. We refer to these three books [1, 18, 25] for a survey.

In this paper, we study finite sequences of operators $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$ on a complex Banach space X for some $N \in \mathbb{N}$, and give a sufficient condition for them to be disjoint topologically transitive, provided that each sequence $(T_{l,n})$ $(1 \leq l \leq N)$ is generated by a weighted translation operator on groups, and $(T_{l,n})$ is a cosine operator function.

First, we recall some definitions of disjointness for further discussions.

Definition 1.1. *([7, Definition 2.1]) Given $N \geq 2$, the sequences of operators $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$ acting on a Banach space X are disjoint topologically transitive or diagonally topologically transitive (in short, d-topologically transitive) if given nonempty open sets $U, V_1, \cdots, V_N \subset X$, there is some $n \in \mathbb{N}$ such that*

$$\emptyset \neq U \cap T_{1,n}^{-1}(V_1) \cap T_{2,n}^{-1}(V_2) \cap \cdots \cap T_{N,n}^{-1}(V_N).$$

If the above condition is satisfied from some n onwards, then $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$ are called d-mixing.

If these sequences of operators are generated by the iterates of one single operator, that is, $T_{l,n} := T_l^n$ $(1 \leq l \leq N)$, then we say these operators $T_1, T_2, \cdots, T_N$ are d-topologically transitive.

Definition 1.2. *([7, Definition 1.1]) Given $N \geq 2$, the sequences of operators $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$ acting on a Banach space X are disjoint hypercyclic, or diagonally hypercyclic (in short, d-hypercyclic) if there is some vector $(x, x, \cdots, x)$ in the diagonal of $X^N = X \times X \times \cdots \times X$ such that*

$$\{(T_{1,n}x, T_{2,n}x, \cdots, T_{N,n}x) : n \in \mathbb{N}_0\}$$

is dense in X^N; if this is the case, then we say that $x \in X$ is a d-hypercyclic vector associated to the sequences of operators $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$.

We say $T_1, T_2, \cdots, T_N$ are d-hypercyclic if there exists a vector $x \in X$ such that $\{(T_1^n x, T_2^n x, \cdots, T_N^n x) : n \in \mathbb{N}_0\}$ is dense in X^N. For $1 \leq r_1 < r_2 < \cdots < r_N$, in [13, 14], we have recently studied the operators $T_1, T_2, \cdots, T_N$ and the powers of operators $T_1^{r_1}, T_2^{r_2}, \cdots, T_N^{r_N}$, and given sufficient and necessary conditions for the operators and the powers of operators to be d-topologically transitive (equivalent to be d-hypercyclic with dense set of d-hypercyclic vectors), where each T_l $(1 \leq l \leq N)$ is a weighted translation operator on the Lebesgue space of a locally compact group. Some results obtained in [13, 14] subsume some works in [6, 7, 29] about d-hypercyclic weighted shifts on $\ell^p(\mathbb{Z})$, which can be viewed as special cases of d-hypercyclic weighted translation operators on a locally compact group. It is interesting to learn that Han and Liang recently also characterized disjoint hypercyclic weighted translations on groups in [19] by using a different approach from ours.

We note that there are more examples of disjoint transitive operators $T_1, T_2, \cdots, T_N$ in the literature than examples of disjoint transitive sequences of operators $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \cdots, (T_{N,n})_{n=1}^{\infty}$. Therefore, inspired by the work in [24–26] and based on our previous study in [9–12], we will continue our investigation and give a class of concrete examples of disjoint topologically transitive sequences of operators on groups.

In what follows, let G be a discrete group with identity e. We denote by $\ell^p(G)$ $(1 \leq p \leq \infty)$ the complex Lebesgue space. A bounded function $w : G \to (0, \infty)$ is called a *weight* on G. Let $a \in G$ and let δ_a be the unit point mass at a. A *weighted translation* on G is a weighted convolution operator $T_{a,w} : \ell^p(G) \longrightarrow \ell^p(G)$ defined by

$$T_{a,w}(f) = wT_a(f) \qquad (f \in \ell^p(G))$$

where w is a weight on G and $T_a(f) = f * \delta_a \in \ell^p(G)$ is the convolution:

$$(f * \delta_a)(x) = \sum_{G} f(xy^{-1})\delta_a(y) = f(xa^{-1}) \qquad (x \in G).$$

If $w^{-1} \in \ell^\infty(G)$, then the inverse of $T_{a,w}$ is $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$, which is also a weighted translation. To simplify notation, we write $S_{a,w}$ for $T_{a^{-1},w^{-1}*\delta_{a^{-1}}}$, that is,

$$S_{a,w}(h) = \frac{h}{w} * \delta_{a^{-1}} \qquad (h \in \ell^p(G))$$

so that

$$T_{a,w}S_{a,w}(h) = h \qquad (h \in \ell^p(G)).$$

We assume $w, w^{-1} \in \ell^\infty(G)$ throughout, and then define a sequence of bounded linear operators $C_n : \ell^p(G) \to \ell^p(G)$ by

$$C_n = \frac{1}{2}(T^n_{a,w} + S^n_{a,w})$$

for all $n \in \mathbb{Z}$ where $T^{-n}_{a,w} := (T^{-1}_{a,w})^n = S^n_{a,w}$. By the fact that $C_n = C_{-n}$ for all $n \in \mathbb{Z}$, we will only need to consider the sequence of operators $(C_n)_{n\in\mathbb{N}_0}$.

In fact, such a sequence of operators $(C_n)_{n\in\mathbb{Z}}$ is a cosine operator function by letting $C_n := C(n)$. A *cosine operator function* on a Banach space X is a mapping C from the real line into the space of continuous linear operators on X satisfying $C(0) = I$, and $2C(t)C(s) = C(t+s) + C(t-s)$ for all $s, t \in \mathbb{R}$. The latter equality is called the d'Alembert functional equation, which implies $C(t) = C(-t)$ for all $t \in \mathbb{R}$. In [8], Bonilla and Miana gave a sufficient condition for a cosine operator function $C(t)$ defined by

$$C(t) = \frac{1}{2}(T(t) + T(-t))$$

to be topologically transitive, where T is a strongly continuous translation group on some weighted Lebesgue space $L^p(\mathbb{R})$. Also, for a Borel measure μ and $\Omega \subset \mathbb{R}^d$, Kalmes characterized in [21] transitive cosine operator functions, generated by second order partial differential operators on $L^p(\Omega, \mu)$. Moreover, Kostić displayed the main structural properties of hypercyclic and chaotic integrated C-cosine functions and C-distribution semigroups in [22–24].

2. A Sufficient Condition for Disjoint Transitivity

Given some $N \in \mathbb{N}$, in this section, we will study N sequences of operators, and give a condition sufficing them to be disjoint transitive. First, we denote these N sequences by $(C_{1,r_1n})^\infty_{n=1}, (C_{2,r_2n})^\infty_{n=1}, \cdots, (C_{N,r_Nn})^\infty_{n=1}$, and define

$$C_{l,r_ln} := \frac{1}{2}(T^{r_ln}_{a,w_l} + S^{r_ln}_{a,w_l}) \qquad (1 \le l \le N)$$

where each $r_l \in \mathbb{N}$ with $1 \le r_1 < r_2 < \cdots < r_N$, and $T^{r_l}_{a,w_l}$ is the power of a weighted translation T_{a,w_l} with inverse S_{a,w_l}, generated by the group element $a \in G$ and the weight w_l for $l = 1, 2, \cdots, N$.

Before discussing the main result, some elements $a \in G$ and weights w should be excluded from our consideration. For example, if $\|w\|_\infty \le 1$, then $\|T_{a,w}\| \le 1$ and $T_{a,w}$ is not transitive. Also, it is showed in [15] that $T_{a,w}$ is not transitive if a is a torsion element of G. Hence these sequences $(C_{1,r_1n})^\infty_{n=1}, (C_{2,r_2n})^\infty_{n=1}, \cdots, (C_{N,r_Nn})^\infty_{n=1}$ cannot be disjoint transitive if they are generated by torsion elements. An element a in a group G is called a *torsion element* if it is of finite order. It is characterized in [15, Lemma 2.1] that an element a of a discrete group G is not torsion (non-torsion or torsion free) if, and only if, for any finite set $K \subset G$, there exists some $N \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \emptyset$ for all $n > N$.

As in [11–13, 15], the result in this paper for the ℓ^p-space of a discrete group can be extended without difficulty to the L^p-space of a locally compact group. For simplicity and exposing the essential idea, we will confine our discussion on discrete groups. We refer to these two classic books [17, 20] and the recent book [16] on this subject of abstract harmonic analysis on locally compact groups.

Now we are ready to give a sufficient condition for these N sequences of operators $(C_{1,r_1n})^\infty_{n=1}, (C_{2,r_2n})^\infty_{n=1}, \cdots, (C_{N,r_Nn})^\infty_{n=1}$ to be disjoint topologically transitive in terms of the non-torsion group element and the weights.

Theorem 2.1. *Let G be a discrete group and let a be a torsion free element in G. Let $1 \leq p < \infty$. Given some $2 \leq N \in \mathbb{N}$, let $T_l = T_{a,w_l}$ be an invertible weighted translation on $\ell^p(G)$, generated by a and a weight w_l for $1 \leq l \leq N$. Let $r_l \in \mathbb{N}$ and $C_{l,r_ln} = \frac{1}{2}(T_l^{r_ln} + S_l^{r_ln})$, where S_l is the inverse of T_l. Then for $1 \leq r_1 < r_2 < \cdots < r_N$, we have* (ii) *implies* (i).

(i) *$(C_{1,r_1n})_{n=1}^\infty, (C_{2,r_2n})_{n=1}^\infty, \cdots, (C_{N,r_Nn})_{n=1}^\infty$ are disjoint topologically transitive.*

(ii) *For each non-empty finite subset $K \subset G$, there are sequences $(E_{l,k}^+)$ and $(E_{l,k}^-)$ of subsets of K, and a sequence (n_k) of positive numbers such that for $K = E_{l,k}^+ \cup E_{l,k}^-$, both sequences*

$$\varphi_{l,r_ln} := \prod_{j=1}^{r_ln} w_l * \delta_{a^{-1}}^j \quad \text{and} \quad \widetilde{\varphi}_{l,r_ln} := \left(\prod_{j=0}^{r_ln-1} w_l * \delta_a^j\right)^{-1}$$

satisfy (for all $1 \leq l \leq N$)

$$\lim_{k\to\infty} \|\varphi_{l,r_ln_k}|_K\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{l,r_ln_k}|_K\|_\infty = 0,$$

$$\lim_{k\to\infty} \|\varphi_{l,2r_ln_k}|_{E_{l,k}^+}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{l,2r_ln_k}|_{E_{l,k}^-}\|_\infty = 0,$$

and (for $1 \leq s < l \leq N$)

$$\lim_{k\to\infty} \left\| \frac{\widetilde{\varphi}_{s,(r_l-r_s)n_k} \cdot \widetilde{\varphi}_{l,r_ln_k}}{\widetilde{\varphi}_{s,r_ln_k}}\Big|_K \right\|_\infty = \lim_{k\to\infty} \left\| \frac{\varphi_{l,(r_l-r_s)n_k} \cdot \widetilde{\varphi}_{s,r_sn_k}}{\widetilde{\varphi}_{l,r_sn_k}}\Big|_K \right\|_\infty = 0,$$

$$\lim_{k\to\infty} \left\| \frac{\varphi_{s,(r_l-r_s)n_k} \cdot \varphi_{l,r_ln_k}}{\varphi_{s,r_ln_k}}\Big|_K \right\|_\infty = \lim_{k\to\infty} \left\| \frac{\widetilde{\varphi}_{l,(r_l-r_s)n_k} \cdot \varphi_{s,r_sn_k}}{\varphi_{l,r_sn_k}}\Big|_K \right\|_\infty = 0,$$

$$\lim_{k\to\infty} \left\| \frac{\varphi_{s,(r_l+r_s)n_k} \cdot \varphi_{l,r_ln_k}}{\varphi_{s,r_ln_k}}\Big|_K \right\|_\infty = \lim_{k\to\infty} \left\| \frac{\varphi_{l,(r_l+r_s)n_k} \cdot \varphi_{s,r_sn_k}}{\varphi_{l,r_sn_k}}\Big|_K \right\|_\infty = 0,$$

$$\lim_{k\to\infty} \left\| \frac{\widetilde{\varphi}_{s,(r_l+r_s)n_k} \cdot \widetilde{\varphi}_{l,r_ln_k}}{\widetilde{\varphi}_{s,r_ln_k}}\Big|_K \right\|_\infty = \lim_{k\to\infty} \left\| \frac{\widetilde{\varphi}_{l,(r_l+r_s)n_k} \cdot \widetilde{\varphi}_{s,r_sn_k}}{\widetilde{\varphi}_{l,r_sn_k}}\Big|_K \right\|_\infty = 0.$$

Proof. We show that $(C_{1,r_1n})_{n=1}^\infty, (C_{2,r_2n})_{n=1}^\infty, \cdots, (C_{N,r_Nn})_{n=1}^\infty$ are disjoint topologically transitive. For $1 \leq l \leq N$, let U and V_l be non-empty open subsets of $\ell^p(G)$. Since the space $C_c(G)$ of continuous functions on G with finite support is dense in $\ell^p(G)$, we can pick $f, g_l \in C_c(G)$ with $f \in U$ and $g_l \in V_l$. Let K be the union of the finite supports of f and all g_l. Let the sequences $(\varphi_{l,r_ln}), (\widetilde{\varphi}_{l,r_ln})$ satisfy condition (ii).

Since a is torsion free, there exists $M \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \emptyset$ for all $n > M$.

First, we show that $\|T_l^{r_ln_k} f\|_p \to 0$, $\|T_l^{r_ln_k}(g_l\chi_K)\|_p \to 0$ and $\|T_l^{2r_ln_k}(g_l\chi_{E_{l,k}^+})\|_p \to 0$ as $k \to \infty$ for $1 \leq l \leq N$. Let $\varepsilon > 0$. There exists $M' \in \mathbb{N}$ such that $n_k > M$, $\varphi_{l,r_ln_k}^p\big|_K < \min(\frac{\varepsilon}{\|f\|_p^p}, \frac{\varepsilon}{\|g_l\|_p^p})$ and $\varphi_{l,2r_ln_k}^p\big|_{E_{l,k}^+} < \frac{\varepsilon}{\|g_l\|_p^p}$ for $k > M'$. Hence

$$\begin{aligned}
\|T_l^{r_ln_k} f\|_p^p &\leq \sum_{Ka^{r_ln_k}} |w_l(x)w_l(xa^{-1}) \cdots w_l(xa^{-(r_ln_k-1)})|^p |f(xa^{-r_ln_k})|^p \\
&= \sum_K |w_l(xa^{r_ln_k})w_l(xa^{r_ln_k-1}) \cdots w_l(xa)|^p |f(x)|^p \\
&= \sum_K \varphi_{l,r_ln_k}^p(x)|f(x)|^p < \varepsilon,
\end{aligned}$$

and $\|T_l^{r_l n_k}(g_l\chi_K)\|_p < \varepsilon$, similarly. Moreover, for $k > M'$, we have

$$\begin{aligned}
\|T_l^{2r_l n_k}(g_l\chi_{E_{l,k}^+})\|_p^p &= \sum_{E_{l,k}^+ a^{2r_l n_k}} |w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(2r_l n_k-1)})|^p |g_l(xa^{-2r_l n_k})|^p \\
&= \sum_{E_{l,k}^+} |w_l(xa^{2r_l n_k})w_l(xa^{2r_l n_k-1})\cdots w_l(xa)|^p |g_l(x)|^p \\
&= \sum_{E_{l,k}^+} \varphi_{l,2r_l n_k}^p(x)|g_l(x)|^p < \varepsilon.
\end{aligned}$$

Applying similar arguments to $S_l^{r_l n_k}$ and $S_l^{2r_l n_k}$, and using the sequences $(\widetilde{\varphi}_{r_l n_k})$ and $(\widetilde{\varphi}_{2r_l n_k})$, we get

$$\lim_{k\to\infty}\|S_l^{r_l n_k}f\|_p^p = \lim_{k\to\infty}\|S_l^{r_l n_k}(g_l\chi_K)\|_p^p = \lim_{k\to\infty}\|S_l^{2r_l n_k}(g_l\chi_{E_{l,k}^-})\|_p^p = 0.$$

Let $1 \le s < l \le N$. In the following, we consider eight types of composition operators, and show

$$\begin{aligned}
\lim_{k\longrightarrow\infty}\|T_l^{r_l n_k}S_s^{r_s n_k}(g_s\chi_K)\|_p^p &= \lim_{k\longrightarrow\infty}\|T_s^{r_s n_k}S_l^{r_l n_k}(g_l\chi_K)\|_p^p = 0,\\
\lim_{k\longrightarrow\infty}\|T_l^{r_l n_k}T_s^{r_s n_k}(g_s\chi_K)\|_p^p &= \lim_{k\longrightarrow\infty}\|T_s^{r_s n_k}T_l^{r_l n_k}(g_l\chi_K)\|_p^p = 0,\\
\lim_{k\longrightarrow\infty}\|S_l^{r_l n_k}T_s^{r_s n_k}(g_s\chi_K)\|_p^p &= \lim_{k\longrightarrow\infty}\|S_s^{r_s n_k}T_l^{r_l n_k}(g_l\chi_K)\|_p^p = 0,\\
\lim_{k\longrightarrow\infty}\|S_l^{r_l n_k}S_s^{r_s n_k}(g_s\chi_K)\|_p^p &= \lim_{k\longrightarrow\infty}\|S_s^{r_s n_k}S_l^{r_l n_k}(g_l\chi_K)\|_p^p = 0.
\end{aligned}$$

First, we have the estimation for the type of $T_l S_s$ below.

$$\begin{aligned}
&\|T_l^{r_l n_k}(S_s^{r_s n_k}(g_s\chi_K))\|_p^p \\
= &\sum_G |w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l n_k-1)})|^p |S_s^{r_s n_k}g_s\chi_K(xa^{-r_l n_k})|^p \\
= &\sum_G \frac{|w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l n_k-1)})|^p}{|w_s(xa^{-r_l n_k+1})w_s(xa^{-r_l n_k+2})\cdots w_s(xa^{-r_l n_k+r_s n_k})|^p}|g_s\chi_K(xa^{-r_l n_k+r_s n_k})|^p \\
= &\sum_K \frac{|w_l(xa^{-(r_s-r_l)n_k})w_l(xa^{-(r_s-r_l)n_k-1})\cdots w_l(xa^{-(r_s n_k-1)})|^p}{|w_s(xa^{-(r_s n_k-1)})w_s(xa^{-(r_s n_k-2)})\cdots w_s(x)|^p}|g_s(x)|^p \\
= &\sum_K \frac{\varphi_{l,(r_l-r_s)n_k}^p(x)\cdot\widetilde{\varphi}_{s,r_s n_k}^p(x)}{\widetilde{\varphi}_{l,r_s n_k}^p(x)}|g_s(x)|^p \longrightarrow 0
\end{aligned}$$

as $k \longrightarrow \infty$. Similarly, we have

$$\begin{aligned}
&\|T_l^{r_l n_k}(T_s^{r_s n_k}(g_s\chi_K))\|_p^p \\
= &\sum_G |w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l n_k-1)})|^p |T_s^{r_s n_k}g_s\chi_K(xa^{-r_l n_k})|^p \\
= &\sum_G |w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l n_k-1)})|^p \\
&\cdot|w_s(xa^{-r_l n_k})w_s(xa^{-r_l n_k-1})\cdots w_s(xa^{-r_l n_k-r_s n_k+1})|^p|g_s\chi_K(xa^{-r_l n_k-r_s n_k})|^p \\
= &\sum_K |w_l(xa^{r_l n_k+r_s n_k})w_l(xa^{r_l n_k+r_s n_k-1})\cdots w_l(xa^{r_s n_k+1})|^p \\
&\cdot|w_s(xa^{r_s n_k})w_s(xa^{r_s n_k-1})\cdots w_s(xa)|^p|g_s(x)|^p \\
= &\sum_K \frac{\varphi_{l,(r_l+r_s)n_k}^p(x)\cdot\varphi_{s,r_s n_k}^p(x)}{\varphi_{l,r_s n_k}^p(x)}|g_s(x)|^p \longrightarrow 0
\end{aligned}$$

as $k \longrightarrow \infty$.

A similar line of reasoning shows that the remaining six terms go to zero as k tends to infinity. Now we are ready to achieve our goal. For each $k \in \mathbb{N}$, we let

$$\begin{aligned} v_k = f &+ 2T_1^{r_1 n_k}(g_1 \chi_{E^+_{1,k}}) + 2T_2^{r_2 n_k}(g_2 \chi_{E^+_{2,k}}) + \cdots + 2T_N^{r_N n_k}(g_N \chi_{E^+_{N,k}}) \\ &+ 2S_1^{r_1 n_k}(g_1 \chi_{E^-_{1,k}}) + 2S_2^{r_2 n_k}(g_2 \chi_{E^-_{2,k}}) + \cdots + 2S_N^{r_N n_k}(g_N \chi_{E^-_{N,k}}). \end{aligned}$$

As $K \cap Ka^{\pm n_k} = \emptyset$, we arrive at

$$\|v_k - f\|_p^p \le 2^p \sum_{s=1}^{N} \|T_s^{r_s n_k}(g_s \chi_{E^+_{s,k}})\|_p^p + 2^p \sum_{s=1}^{N} \|S_s^{r_s n_k}(g_s \chi_{E^-_{s,k}})\|_p^p$$

and

$$\begin{aligned} &\|C_{l,r_l n_k} v_k - g_l\|_p^p \\ \le\ & \frac{1}{2^p}\|T_l^{r_l n_k} f\|_p^p + \sum_{s=1}^{N} \|T_l^{r_l n_k} T_s^{r_s n_k}(g_s \chi_{E^+_{s,k}})\|_p^p + \sum_{s=1, s\neq l}^{N} \|T_l^{r_l n_k} S_s^{r_s n_k}(g_s \chi_{E^-_{s,k}})\|_p^p \\ +\ & \frac{1}{2^p}\|S_l^{r_l n_k} f\|_p^p + \sum_{s=1}^{N} \|S_l^{r_l n_k} S_s^{r_s n_k}(g_s \chi_{E^-_{s,k}})\|_p^p + \sum_{s=1, s\neq l}^{N} \|S_l^{r_l n_k} T_s^{r_s n_k}(g_s \chi_{E^+_{s,k}})\|_p^p. \end{aligned}$$

Hence $\lim_{k\to\infty} v_k = f$ and $\lim_{k\to\infty} C_{l,r_l n_k} v_k = g_l$ which imply that, for each l, we have

$$C_{l,r_l n_k}(U) \cap V_l \neq \emptyset$$

for some k. Therefore $(C_{1,r_1 n})_{n=1}^{\infty}, (C_{2,r_2 n})_{n=1}^{\infty}, \cdots, (C_{N,r_N n})_{n=1}^{\infty}$ are disjoint topologically transitive. $\square$

We note that Theorem 2.1 is closely related with the assertion of [24, Theorem 45] where Kostić investigated disjoint hypercyclicity of a class of cosine operator functions on weighted function spaces. As the weight conditions in the above theorem is quite complex and is not easy to be verified, we assume all the operators T_l are generated by the same weight w and the same group element $a \in G$, that is, $T_l := T_{a,w}$ $(1 \le l \le N)$, in order to simplify the weight conditions.

Corollary 2.2. *Let G be a discrete group and let a be a torsion free element in G. Let $1 \le p < \infty$. Given some $2 \le N \in \mathbb{N}$, let $T_{a,w}$ be an invertible weighted translation on $\ell^p(G)$ with inverse $S_{a,w}$. Let $r_l \in \mathbb{N}$ and $C_{r_l n} = \frac{1}{2}(T_{a,w}^{r_l n} + S_{a,w}^{r_l n})$ for $1 \le l \le N$. Then for $1 \le r_1 < r_2 < \cdots < r_N$, we have* (ii) *implies* (i).

(i) *$(C_{r_1 n})_{n=1}^{\infty}, (C_{r_2 n})_{n=1}^{\infty}, \cdots, (C_{r_N n})_{n=1}^{\infty}$ are disjoint topologically transitive.*

(ii) *For each non-empty finite subset $K \subset G$, there are sequences $(E^+_{l,k})$ and $(E^-_{l,k})$ of subsets of K, and a sequence (n_k) of positive numbers such that for $K = E^+_{l,k} \cup E^-_{l,k}$, both sequences*

$$\varphi_{r_l n} := \prod_{j=1}^{r_l n} w * \delta^j_{a^{-1}} \quad \text{and} \quad \widetilde{\varphi}_{r_l n} := \left(\prod_{j=0}^{r_l n - 1} w * \delta^j_a\right)^{-1}$$

satisfy (for all $1 \le l \le N$)

$$\lim_{k\to\infty} \|\varphi_{r_l n_k}|_K\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{r_l n_k}|_K\|_\infty = 0,$$

$$\lim_{k\to\infty} \|\varphi_{2r_l n_k}|_{E^+_{l,k}}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{2r_l n_k}|_{E^-_{l,k}}\|_\infty = 0,$$

and (for $1 \le s < l \le N$)

$$\lim_{k\to\infty} \|\widetilde{\varphi}_{(r_l - r_s)n_k}|_K\|_\infty = \lim_{k\to\infty} \|\varphi_{(r_l - r_s)n_k}|_K\|_\infty = 0,$$

$$\lim_{k\to\infty} \|\widetilde{\varphi}_{(r_l + r_s)n_k}|_K\|_\infty = \lim_{k\to\infty} \|\varphi_{(r_l + r_s)n_k}|_K\|_\infty = 0.$$

Example 2.3. Let $G = \mathbb{Z}$ and $a = 1$. Let w be a weight on $\mathbb{Z}$ with $w^{-1} \in \ell^\infty(\mathbb{Z})$. Then the weighted translation $T_{1,w*\delta_1}$ on $\ell^2(\mathbb{Z})$ is the bilateral weighted forward shift T with a weight sequence (w_j), defined by $Te_j = w_j e_{j+1}$ for $j \in \mathbb{Z}$ where $w_j = w(j)$ and (e_j) is the canonical basis of $\ell^2(\mathbb{Z})$. Let $S = T^{-1}$ and let $C_{r_l n} = \frac{1}{2}(T^{r_l n} + S^{r_l n})$ for $1 \le l \le N$. Choose $r_1 = 1, r_2 = 2, \cdots, r_N = N$. Then, by Theorem 2.1, $(C_{1n})_{n=1}^\infty, (C_{2n})_{n=1}^\infty, \cdots, (C_{Nn})_{n=1}^\infty$ are disjoint transitive if given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists an arbitrarily large n such that, for all $|j| < q$ and $1 \le l \le 2N$, we have

$$\varphi_{ln}(j) = \prod_{i=0}^{ln-1} w(j+i) < \varepsilon, \qquad \widetilde{\varphi}_{ln}(j) = \frac{1}{\prod_{i=1}^{ln} w(j-i)} < \varepsilon.$$

If we define $w : \mathbb{Z} \to (0, \infty)$ by

$$w(j) = \begin{cases} 2 & \text{if } j < 0 \\ \frac{1}{2} & \text{if } j \ge 0 \end{cases},$$

then w satisfies the weight condition above.

Strengthening the weight condition (ii) in Theorem 2.1, we obtain a sufficient condition for the sequences $(C_{1,r_1 n})_{n=1}^\infty, (C_{2,r_2 n})_{n=1}^\infty, \cdots, (C_{N,r_N n})_{n=1}^\infty$ to be disjoint mixing.

Corollary 2.4. *Let G be a discrete group and let a be a torsion free element in G. Let $1 \le p < \infty$. Given some $2 \le N \in \mathbb{N}$, let $T_l = T_{a,w_l}$ be an invertible weighted translation on $\ell^p(G)$, generated by a and a weight w_l for $1 \le l \le N$. Let $r_l \in \mathbb{N}$ and $C_{l,r_l n} = \frac{1}{2}(T_l^{r_l n} + S_l^{r_l n})$ where S_l is the inverse of T_l. Then for $1 \le r_1 < r_2 < \cdots < r_N$, we have* (ii) *implies* (i).

(i) $(C_{1,r_1 n})_{n=1}^\infty, (C_{2,r_2 n})_{n=1}^\infty, \cdots, (C_{N,r_N n})_{n=1}^\infty$ *are disjoint mixing.*

(ii) *For each non-empty finite subset $K \subset G$, both sequences*

$$\varphi_{l,r_l n} := \prod_{j=1}^{r_l n} w_l * \delta_{a^{-1}}^j \quad \text{and} \quad \widetilde{\varphi}_{l,r_l n} := \left(\prod_{j=0}^{r_l n-1} w_l * \delta_a^j\right)^{-1}$$

satisfy (for all $1 \le l \le N$)

$$\lim_{n\to\infty} \|\varphi_{l,r_l n}|_K\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_{l,r_l n}|_K\|_\infty = 0,$$

and (for $1 \le s < l \le N$)

$$\lim_{n\to\infty} \left\| \frac{\widetilde{\varphi}_{s,(r_l-r_s)n} \cdot \widetilde{\varphi}_{l,r_l n}}{\widetilde{\varphi}_{s,r_l n}}\Big|_K \right\|_\infty = \lim_{n\to\infty} \left\| \frac{\varphi_{l,(r_l-r_s)n} \cdot \widetilde{\varphi}_{s,r_s n}}{\widetilde{\varphi}_{l,r_s n}}\Big|_K \right\|_\infty = 0,$$

$$\lim_{n\to\infty} \left\| \frac{\varphi_{s,(r_l-r_s)n} \cdot \varphi_{l,r_l n}}{\varphi_{s,r_l n}}\Big|_K \right\|_\infty = \lim_{n\to\infty} \left\| \frac{\widetilde{\varphi}_{l,(r_l-r_s)n} \cdot \varphi_{s,r_s n}}{\varphi_{l,r_s n}}\Big|_K \right\|_\infty = 0,$$

$$\lim_{n\to\infty} \left\| \frac{\varphi_{s,(r_l+r_s)n} \cdot \varphi_{l,r_l n}}{\varphi_{s,r_l n}}\Big|_K \right\|_\infty = \lim_{n\to\infty} \left\| \frac{\varphi_{l,(r_l+r_s)n} \cdot \varphi_{s,r_s n}}{\varphi_{l,r_s n}}\Big|_K \right\|_\infty = 0,$$

$$\lim_{n\to\infty} \left\| \frac{\widetilde{\varphi}_{s,(r_l+r_s)n} \cdot \widetilde{\varphi}_{l,r_l n}}{\widetilde{\varphi}_{s,r_l n}}\Big|_K \right\|_\infty = \lim_{n\to\infty} \left\| \frac{\widetilde{\varphi}_{l,(r_l+r_s)n} \cdot \widetilde{\varphi}_{s,r_s n}}{\widetilde{\varphi}_{l,r_s n}}\Big|_K \right\|_\infty = 0.$$

Proof. Using the full sequences $(\varphi_{l,r_l n}), (\widetilde{\varphi}_{l,r_l n})$ instead of the subsequences $(\varphi_{l,r_l n_k}), (\widetilde{\varphi}_{l,r_l n_k})$ in the proof of Theorem 2.1, and letting

$$v_n = f + 2T_1^{r_1 n} g_1 + 2T_2^{r_2 n} g_2 + \cdots + 2T_N^{r_N n} g_N + 2S_1^{r_1 n} g_1 + 2S_2^{r_2 n} g_2 + \cdots + 2S_N^{r_N n} g_N,$$

one can deduce that for each l,

$$C_{l,r_l n}(U) \cap V_l \neq \emptyset$$

from some n onwards. Therefore $(C_{1,r_1 n})_{n=1}^\infty, (C_{2,r_2 n})_{n=1}^\infty, \cdots, (C_{N,r_N n})_{n=1}^\infty$ are disjoint mixing. $\square$

Next, we turn our attention to some special case of T_l. We assume $T_l = T_{a,w}$ $(1 \leq l \leq N)$, generated by the same weight w and the same group element $a \in G$.

Corollary 2.5. *Let G be a discrete group and let a be a torsion free element in G. Let $1 \leq p < \infty$. Given some $2 \leq N \in \mathbb{N}$, let $T_{a,w}$ be an invertible weighted translation on $\ell^p(G)$ with inverse $S_{a,w}$. Let $r_l \in \mathbb{N}$ and $C_{r_l n} = \frac{1}{2}(T_{a,w}^{r_l n} + S_{a,w}^{r_l n})$ for $1 \leq l \leq N$. Then for $1 \leq r_1 < r_2 < \cdots < r_N$, we have* (ii) *implies* (i).

(i) $(C_{r_1 n})_{n=1}^{\infty}, (C_{r_2 n})_{n=1}^{\infty}, \cdots, (C_{r_N n})_{n=1}^{\infty}$ *are disjoint mixing.*

(ii) *For each non-empty finite subset $K \subset G$, both sequences*

$$\varphi_{r_l n} := \prod_{j=1}^{r_l n} w * \delta_{a^{-1}}^{j} \quad \text{and} \quad \widetilde{\varphi}_{r_l n} := \left(\prod_{j=0}^{r_l n-1} w * \delta_a^j \right)^{-1}$$

satisfy (for all $1 \leq l \leq N$)

$$\lim_{n\to\infty} \|\varphi_{r_l n}|_K\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_{r_l n}|_K\|_\infty = 0,$$

and (for $1 \leq s < l \leq N$)

$$\lim_{n\to\infty} \|\widetilde{\varphi}_{(r_l - r_s)n}|_K\|_\infty = \lim_{n\to\infty} \|\varphi_{(r_l - r_s)n}|_K\|_\infty = 0,$$

$$\lim_{n\to\infty} \|\widetilde{\varphi}_{(r_l + r_s)n_k}|_K\|_\infty = \lim_{n\to\infty} \|\varphi_{(r_l + r_s)n}|_K\|_\infty = 0.$$

Example 2.6. Let $(C_{r_l n})$ be defined as in Example 2.3 for $l = 1, 2$ with $r_1 = 3, r_2 = 8$. Then, by Corollary 2.5 above, $(C_{3n})_{n=1}^{\infty}, (C_{8n})_{n=1}^{\infty}$ are disjoint mixing if given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists some $M \in \mathbb{N}$ such that

$$\varphi_{3n}(j) = \prod_{i=0}^{3n-1} w(j+i) < \varepsilon, \qquad \widetilde{\varphi}_{3n}(j) = \frac{1}{\prod_{i=1}^{3n} w(j-i)} < \varepsilon;$$

$$\varphi_{8n}(j) = \prod_{i=0}^{8n-1} w(j+i) < \varepsilon, \qquad \widetilde{\varphi}_{8n}(j) = \frac{1}{\prod_{i=1}^{8n} w(j-i)} < \varepsilon;$$

$$\varphi_{5n}(j) = \prod_{i=0}^{5n-1} w(j+i) < \varepsilon, \qquad \widetilde{\varphi}_{5n}(j) = \frac{1}{\prod_{i=1}^{5n} w(j-i)} < \varepsilon;$$

$$\varphi_{11n}(j) = \prod_{i=0}^{11n-1} w(j+i) < \varepsilon, \qquad \widetilde{\varphi}_{11n}(j) = \frac{1}{\prod_{i=1}^{11n} w(j-i)} < \varepsilon,$$

for all $|j| < q$ and all $n > M$.

3. A Necessary Condition for Disjoint Transitivity

In this section, we will give a necessary condition for disjoint topological transitivity of the sequences $(C_{1,r_1 n})_{n=1}^{\infty}, (C_{2,r_2 n})_{n=1}^{\infty}, \cdots, (C_{N,r_N n})_{n=1}^{\infty}$.

Theorem 3.1. *Let G be a discrete group and let a be a torsion free element in G. Let $1 \leq p < \infty$. Given some $2 \leq N \in \mathbb{N}$, let $T_l = T_{a,w_l}$ be an invertible weighted translation on $\ell^p(G)$, generated by a and a weight w_l for $1 \leq l \leq N$. Let $r_l \in \mathbb{N}$ and $C_{l,r_l n} = \frac{1}{2}(T_l^{r_l n} + S_l^{r_l n})$ where S_l is the inverse of T_l. Then for $1 \leq r_1 < r_2 < \cdots < r_N$, we have* (i) *implies* (ii).

(i) $(C_{1,r_1 n})_{n=1}^{\infty}, (C_{2,r_2 n})_{n=1}^{\infty}, \cdots, (C_{N,r_N n})_{n=1}^{\infty}$ *are disjoint topologically transitive.*

(ii) *For each non-empty finite subset $K \subset G$, there are sequences $(E^{+}_{l,k})$ and $(E^{-}_{l,k})$ of subsets of K, and a sequence (n_k) of positive numbers such that for $K = E^{+}_{l,k} \cup E^{-}_{l,k}$, both sequences*

$$\varphi_{l,r_l n} := \prod_{j=1}^{r_l n} w_l * \delta^{j}_{a^{-1}} \quad \text{and} \quad \widetilde{\varphi}_{l,r_l n} := \left(\prod_{j=0}^{r_l n-1} w_l * \delta^{j}_{a} \right)^{-1}$$

satisfy (for all $1 \le l \le N$)

$$\lim_{k\to\infty} \|\varphi_{l,r_l n_k}|_K\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{l,r_l n_k}|_K\|_\infty = 0,$$

and

$$\lim_{k\to\infty} \|\varphi_{l,2r_l n_k}|_{E^{+}_{l,k}}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{l,2r_l n_k}|_{E^{-}_{l,k}}\|_\infty = 0.$$

Proof. Let $(C_{1,r_1 n})_{n=1}^{\infty}, (C_{2,r_2 n})_{n=1}^{\infty}, \cdots, (C_{N,r_N n})_{n=1}^{\infty}$ be disjoint topologically transitive. Let $K \subset G$ be a non-empty finite set. Let χ_K be the characteristic function, and let $\varepsilon \in (0,1)$. By the assumption on disjoint transitivity and the fact that $a \in G$ is non-torsion, there exist a vector $f \in \ell^p(G)$ and some $m \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \emptyset$ for all $n \ge m$,

$$\|f - \chi_K\|_p < \varepsilon \quad \text{and} \quad \|C_{l,r_l m} f - \chi_K\|_p < \varepsilon$$

for $1 \le l \le N$. By the continuity of the mapping $h \in \ell^p(G,\mathbb{C}) \mapsto Re\ h \in \ell^p(G,\mathbb{R})$ and the fact T_l and S_l commute with it, we can assume without loss of generality that f is real-valued. Then we have the following estimates

$$f(x) > 1-\varepsilon > 0 \qquad \text{and} \qquad C_{l,r_l m} f(x) > 1-\varepsilon > 0 \qquad (1 \le l \le N, x \in K).$$

Combing all above, we have

$$\begin{aligned}
2^p \varepsilon^p &> \|2C_{l,r_l m} f - 2\chi_K\|_p^p \ge \|T_l^{r_l m}(f\chi_K) + S_l^{r_l m}(f\chi_K) - 2\chi_K\|_p^p \\
&\ge \sum_{Ka^{r_l m}} |T_l^{r_l m} f\chi_K(x) + S_l^{r_l m} f\chi_K(x)|^p \\
&= \sum_{K} |T_l^{r_l m} f\chi_K(xa^{r_l m}) + S_l^{r_l m} f\chi_K(xa^{r_l m})|^p \\
&= \sum_{K} |\varphi_{l,r_l m}(x) f\chi_K(x) + \varphi^{-1}_{l,r_l m}(xa^{r_l m}) f\chi_K(xa^{2r_l m})|^p \\
&= \sum_{K} |\varphi_{l,r_l m}(x) f\chi_K(x)|^p \\
&> (1-\varepsilon)^p \sum_{K} \varphi^{p}_{l,r_l m}(x).
\end{aligned}$$

For a subset F of G, we have $\|(C_{l,r_l n} h)\chi_F\|_p \le \|C_{l,r_l n} h\|_p$ for arbitrary n and $h \in \ell^p(G,\mathbb{R})$. Let $h^- := \max\{0,-h\}$ and $h^+ := \max\{0,h\}$. Obviously the mapping $h \in \ell^p(G,\mathbb{R}) \mapsto h^- \in \ell^p(G,\mathbb{R})$ satisfies $\|(h+g)^-\|_p \le \|h^- + g^-\|_p$ and commutes with T_l and S_l so that we have

$$\begin{aligned}
\|(C_{l,r_l m} f^-)\chi_F\|_p &\le \|(C_{l,r_l m} f)^-\|_p = \|(C_{l,r_l m} f - \chi_K + \chi_K)^-\|_p \\
&\le \|(C_{l,r_l m} f - \chi_K)^-\|_p + \|\chi_K^-\|_p \\
&= \|(C_{l,r_l m} f - \chi_K)^-\|_p \le \|C_{l,r_l m} f - \chi_K\|_p < \varepsilon.
\end{aligned}$$

Now define $E^{-}_{l} = \{x \in K : T_l^{r_l m} f(x) > 1-\varepsilon\}$ and $E^{+}_{l} = K \setminus E^{-}_{l}$. Then for $x \in E^{+}_{l}$, one has $S_l^{r_l m} f(x) > 1-\varepsilon$ which follows from the fact

$$1-\varepsilon < C_{l,r_l m} f(x) = \frac{1}{2} T_l^{r_l m} f(x) + \frac{1}{2} S_l^{r_l m} f(x) \le \frac{1}{2}(1-\varepsilon) + \frac{1}{2} S_l^{r_l m} f(x).$$

By $K \cap Ka^{\pm m} = \emptyset$, the inequality $\|f+g\|_p^p \le 2^p\|f\|_p^p + 2^p\|g\|_p^p$, and the positivity of $T_l^{r_lm} f^+$ and $S_l^{r_lm} f^+$, we have

$$
\begin{aligned}
& (1-\varepsilon)^p \sum_{E_l^+} \varphi_{l,2r_lm}^p(x) \\
< & \sum_{E_l^+} |w_l(xa^{2r_lm}) w_l(xa^{2r_lm-1}) \cdots w_l(xa)|^p |S_l^{r_lm} f^+(x)|^p \\
= & \sum_{E_l^+ a^{2r_lm}} |w_l(x) w_l(xa^{-1}) \cdots w_l(xa^{-(2r_lm-1)})|^p |S_l^{r_lm} f^+(xa^{-2r_lm})|^p \\
= & \sum_{E_l^+ a^{2r_lm}} |T_l^{2r_lm} S_l^{r_lm} f^+(x)|^p \\
= & \sum_{E_l^+ a^{2r_lm}} |T_l^{r_lm} f^+(x)|^p \\
\le & 2^p \sum_{E_l^+ a^{2r_lm}} |C_{l,r_lm} f^+(x)|^p \\
= & 2^p \|(C_{l,r_lm} f^+)\chi_{E_l^+ a^{2r_lm}}\|_p^p \\
= & 2^p \|(C_{l,r_lm}(f^- + f))\chi_{E_l^+ a^{2r_lm}}\|_p^p \\
= & 2^p \|(C_{l,r_lm} f^-)\chi_{E_l^+ a^{2r_lm}} + (C_{l,r_lm} f - \chi_K)\chi_{E_l^+ a^{2r_lm}} + \chi_{K\cap E_l^+ a^{2r_lm}}\|_p^p \\
\le & 2^p (2^p \|C_{l,r_lm} f^- \chi_{E_l^+ a^{2r_lm}}\|_p^p + 2^p \|C_{l,r_lm} f - \chi_K\|_p^p) \\
\le & 2^{2p}(\varepsilon^p + \varepsilon^p).
\end{aligned}
$$

The final conclusion of theorem follows by plugging $n_k = m(\varepsilon) = m(1/k)$. □

As in Section 2, one has the necessary condition for disjoint mixing.

Corollary 3.2. *Let G be a discrete group and let a be a torsion free element in G. Let $1 \le p < \infty$. Given some $2 \le N \in \mathbb{N}$, let $T_l = T_{a,w_l}$ be an invertible weighted translation on $\ell^p(G)$, generated by a and a weight w_l for $1 \le l \le N$. Let $r_l \in \mathbb{N}$ and $C_{l,r_ln} = \frac{1}{2}(T_l^{r_ln} + S_l^{r_ln})$ where S_l is the inverse of T_l. Then for $1 \le r_1 < r_2 < \cdots < r_N$, we have* (i) *implies* (ii).

(i) *$(C_{1,r_1n})_{n=1}^\infty, (C_{2,r_2n})_{n=1}^\infty, \cdots, (C_{N,r_Nn})_{n=1}^\infty$ are disjoint mixing.*
(ii) *For each non-empty finite subset $K \subset G$, both sequences*

$$\varphi_{l,r_ln} := \prod_{j=1}^{r_ln} w_l * \delta_{a^{-1}}^j \quad \text{and} \quad \widetilde{\varphi}_{l,r_ln} := \left(\prod_{j=0}^{r_ln-1} w_l * \delta_a^j\right)^{-1}$$

satisfy (for all $1 \le l \le N$)

$$\lim_{n\to\infty} \|\varphi_{l,r_ln}|_K\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_{l,r_ln}|_K\|_\infty = 0.$$

Remark 3.3. *We end up this paper with an observation that $T_{a,w}$ is mixing if, and only if, $(C_{1n})_{n=1}^\infty, (C_{2n})_{n=1}^\infty, \cdots, (C_{Nn})_{n=1}^\infty$ are disjoint mixing where $C_{ln} = \frac{1}{2}(T_{a,w}^{ln} + S_{a,w}^{ln})$. Indeed, with $r_l = l$ and $w_l = w$ for each l and the weight w in Corollary 2.5 and Corollary 3.2, we can deduce that*

$$\lim_{n\to\infty} \|\varphi_n|_K\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_n|_K\|_\infty = 0$$

is a sufficient and necessary condition for $(C_{1n})_{n=1}^\infty, (C_{2n})_{n=1}^\infty, \cdots, (C_{Nn})_{n=1}^\infty$ to be disjoint mixing. The above condition also characterizes the mixing weighted translation operator $T_{a,w}$ in [15, Corollary 2.6].

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