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Bessel Wavelet Transform on the Spaces with Exponential Growth

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(In the memory of Prof. R. S. Pathak.)

Abstract. The Bessel wavelet transform on χ_μ and Q_μ type spaces of exponential growth are investigated and their properties discussed by using the theory of the Hankel transform. Using this said theory, the integral equation of Fredholm type is defined and some examples associated with this integral equation are given.

1. Introduction

From Zemanian [6, 7] and Betancor - Mesa [1], the Hankel integral transformation is defined by

$$(h_\mu f)(y) = \int_0^\infty (xy)^{1/2} J_\mu(xy) f(x)dx, \quad y \in (0,\infty), \mu \geq -\frac{1}{2}, \tag{1}$$

where J_μ is the Bessel function of the first kind.
If $f \in L^1(0,\infty)$ and $h_\mu f \in L^1(0,\infty)$, then inverse Hankel transform is defined by

$$f(x) = \int_0^\infty (xy)^{1/2} J_\mu(xy)(h_\mu f)(y)dy. \tag{2}$$

Let $f \in L^1(0,\infty)$, $g \in L^1(0,\infty)$ then Hankel convolution of f and g is

$$(f\#g)(x) = \int_0^\infty f(y)(\tau_x g)(y)dy, \tag{3}$$

where

$$(\tau_x g)(y) = g(x,y) = \int_0^\infty g(z)\, D_\mu(x,y,z)dz \tag{4}$$

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and $D_\mu(x,y,z) = \int_0^\infty t^{-\mu-\frac{1}{2}}(xt)^{1/2}J_\mu(xt)(yt)^{1/2}J_\mu(yt)(zt)^{1/2}J_\mu(zt)dt,\ x,y,z \in (0,\infty)$, provided that the above integrals exist.
If f and g in $L^1(0,\infty)$, then

$$h_\mu(f\#g)(x) = x^{-\mu-\frac{1}{2}}(h_\mu f)(x)(h_\mu g)(x). \tag{5}$$

J.J.Betancor and Mesa [1] defined the space χ_μ which consists of all smooth complex-valued function $\phi(x)$, $x \in (0,\infty)$ satisfies the following norm

$$\gamma^\mu_{k,m}(\phi) = \sup_{x\in(0,\infty)} \left| e^{kx}\left(x^{-1}\frac{d}{dx}\right)^m (x^{-\mu-\frac{1}{2}}\phi(x))\right| < \infty, \tag{6}$$

for every $k, m \in \mathbb{N}_0$.
The semi norm for $\phi \in \chi_\mu$ is given by

$$\eta^\mu_{k,m}(\phi) = \sup_{x\in(0,\infty)} \left| e^{kx}x^{-\mu-\frac{1}{2}}S^m_\mu\phi(x)\right|, \qquad k,m \in \mathbb{N}_0, \tag{7}$$

where $S_\mu = x^{-\mu-\frac{1}{2}}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx}x^{-\mu-\frac{1}{2}}$, induces on χ_μ the same topology as defined by $\left\{\gamma^\mu_{k,m}\right\}_{k,m\in\mathbb{N}_0}$.
From [1], Q_μ as the space of all complex-valued functions Φ which is like [4] and satisfy the following two conditions

1. $z^{-\mu-\frac{1}{2}}\Phi(z)$ is an even entire function.
2. For every $k, m \in \mathbb{N}_0$, the following norm is given by

$$\omega^\mu_{k,m}(\Phi) = \sup_{|Imz|\le k}(1+|z|^2)^m|z^{-\mu-\frac{1}{2}}\phi(z)| < \infty. \tag{8}$$

The boundedness properties of Bessel functions are given below:

$$\text{(i). } \left|z^{-\mu}J_\mu(z)\right| \le Ce^{|Imz|}, \quad z \in \mathbb{C} \tag{9}$$
$$\text{(ii). } |z^{1/2}H^{(1)}_\mu(z)| \le Ce^{-Imz}, \quad z \in \mathbb{C}, |z| \ge 1 \tag{10}$$

where $H^{(1)}_\mu$ denotes the Hankel function of the first kind of order μ and C is a positive constant depending on μ in (9) and (10).
From [5], we recall the Bessel wavelet transform of a function $f \in L^2(0,\infty)$ with respect to a Bessel wavelet $\psi \in L^2(0,\infty)$ is

$$(B_\psi f)(b,a) = \int_0^\infty f(t)\tau_b\psi_a(x) = a^{\mu-\frac{1}{2}}\int_0^\infty f(t)\psi\left(\frac{t}{a},\frac{b}{a}\right)dt. \tag{11}$$

If $\psi \in L^2(0,\infty)$ and $f \in L^2(0,\infty)$, then using the techniques of [5], we have

$$(B_\psi f)(b,a) = \int_0^\infty (bx)^{1/2}J_\mu(bx)x^{-\mu-\frac{1}{2}}(h_\mu f)(x)(h_\mu\psi)(ax)dx. \tag{12}$$

In this paper, the Bessel wavelet transform on $\chi_\mu(I)$ and $Q_\mu(I)$ spaces are investigated and it is shown that Bessel wavelet transform $B_\psi : \chi_\mu(I) \longrightarrow \chi_\mu(I\times I)$, $B_\psi : Q_\mu(I) \longrightarrow Q_\mu(I\times I)$ are linear and continuous. Applying the continuity property of the Bessel wavelet transform, some properties of Hankel convolution are studied. Various examples of Fredholm integral equation associated with Hankel convolution and the Bessel wavelet transform are discussed.

2. The Bessel Wavelet Transform on the Spaces χ_μ and Q_μ

In this section the properties of Bessel wavelet transform on χ_μ and Q_μ type spaces are studied.

Lemma 2.1. *If $\psi \in \chi_\mu(I), I = (0, \infty)$ then we have the following estimate*

$$\left|\left(a^{-1}\frac{d}{da}\right)^q (h_\mu\psi)(a(\xi + i(k+1))\right| \leq \sum_{r=0}^{q}\binom{q}{r}C_\mu a^{\mu+\frac{1}{2}-2r}|\xi + i(k+1)|^{\mu-2r+2q+\frac{1}{2}} \gamma^\mu_{k+1,0}(\psi)\Gamma(2\mu - 2r + 2q + 1), \tag{13}$$

where $a > 0$, $\mu > r - q - \frac{1}{2}$ and $C_\mu = C \times A_{\mu,r}$ with arbitrary constant C and $A_{\mu,r} = (\mu + \frac{1}{2})(\mu + \frac{1}{2} - 2)...(\mu + \frac{1}{2} - 2(r-1))$.

Proof. We have

$$\begin{aligned}\left(v^{-1}\frac{d}{dv}\right)^q (h_\mu\psi)(v) &= \left(v^{-1}\frac{d}{dv}\right)^q \int_0^\infty (vy)^{1/2}J_\mu(vy)\psi(y)dy\\ &= \int_0^\infty \left(v^{-1}\frac{d}{dv}\right)^q (vy)^{-\mu}J_\mu(vy)(vy)^{\mu+1/2}\psi(y)dy\\ &= \int_0^\infty \sum_{r=0}^{q}\binom{q}{r}\left(v^{-1}\frac{d}{dv}\right)^{q-r}\left(v^{-\mu}J_\mu(vy)\right)\left(v^{-1}\frac{d}{dv}\right)^r v^{\mu+1/2}\, y^{1/2}\,\psi(y)dy. \end{aligned}\tag{14}$$

Therefore, we obtain

$$\left(v^{-1}\frac{d}{dv}\right)^q (h_\mu\psi)(v) = \sum_{r=0}^{q}\binom{q}{r}(-1)^{q-r}A_{\mu,r}\, v^{\mu+\frac{1}{2}-2r}\int_0^\infty (vy)^{-(\mu+q-r)}J_{\mu+q-r}(vy)\, y^{\mu+2q-2r+\frac{1}{2}}\,\psi(y)\,dy.$$

The following estimate can be obtained from (9),

$$\begin{aligned}\left|\left(v^{-1}\frac{d}{dv}\right)^q (h_\mu\psi)(v)\right| &\leq \sum_{r=0}^{q}\binom{q}{r}C_\mu\,|v|^{\mu+\frac{1}{2}-2r}\sup_{y\in(0,\infty)}\left|e^{(k+1)y}y^{-\mu-\frac{1}{2}}\psi(y)\right|\int_0^\infty e^{-y}y^{2\mu+2q-2r+1}dy, \quad if\ |Im\, v| \leq k\\ &\leq \sum_{r=0}^{q}\binom{q}{r}C_\mu\,|v|^{\mu+\frac{1}{2}-2r}\,\gamma^\mu_{k+1,0}(\psi)\,\Gamma(2\mu - 2r + 2q + 1).\end{aligned}$$

Putting $v = a\,(\xi + i(k+1))$, we get

$$\left|\left(a^{-1}\frac{d}{da}\right)^q (h_\mu\psi)\,(a(\xi + i(k+1)))\right| \leq \sum_{r=0}^{q}\binom{q}{r}C_\mu\, a^{\mu-2r+\frac{1}{2}}\,|\xi + i(k+1)|^{\mu-2r+2q+\frac{1}{2}}\,\gamma^\mu_{k+1,0}(\psi)\Gamma(2\mu - 2r + 2q + 1).$$

□

Theorem 2.2. *Bessel wavelet transform B_ψ is a continuous linear map from $\chi_\mu(I)$ to $\chi_\mu(I \times I)$ for $\mu \geq -\frac{1}{2}$.*

Proof. Let $\phi \in \chi_\mu(I)$. Then by using (12) and [1], we have

$$\begin{aligned}(-1)^q\left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi\phi)(b,a) &= \frac{1}{2}\int_{-\infty}^{\infty} b^{-\mu-q}(\xi + i\eta)^{q+\frac{1}{2}}H^{(1)}_{\mu+q}(b(\xi + i\eta))\\ &\quad \times\left((\xi + i\eta)^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(\xi + i\eta)\right)\left(h_\mu\psi\right)(a(\xi + i\eta))d\xi.\end{aligned}$$

Therefore,

$$\left|(-1)^q\left(a^{-1}\frac{d}{da}\right)^l\left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi,\phi)(b,a)\right|$$
$$\leq \int_{-\infty}^{\infty} b^{-\mu-q-\frac{1}{2}}\left|(b(\xi+i\eta))^{\frac{1}{2}}H^{(1)}_{\mu+q}(b(\xi+i\eta))\right|\left|(\xi+i\eta)^q\left((\xi+i\eta)^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(\xi+i\eta)\right)\right|$$
$$\left|\left(a^{-1}\frac{d}{da}\right)^l\left(h_\mu\psi\right)(a(\xi+i\eta))\right|d\xi$$
$$\leq b^{-\mu-q-\frac{1}{2}}\,C\,e^{-b\eta}\int_{-\infty}^{\infty}\left|(\xi+i\eta)^q\left((\xi+i\eta)^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(\xi+i\eta)\right)\right|\left|\left(a^{-1}\frac{d}{da}\right)^l\left(h_\mu\psi\right)(a(\xi+i\eta))\right|d\xi.$$

Using Lemma 2.1 and putting $\eta = k+1$, we have

$$\left|e^{kb}\left(a^{-1}\frac{d}{da}\right)^l\left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi,\phi)(b,a)\right|$$
$$\leq C.C_\mu\, b^{-\mu-q-\frac{1}{2}}a^{\mu-2r+\frac{1}{2}}e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1)$$
$$\times\int_{-\infty}^{\infty}|\xi+i(k+1)|^{\mu-2r+2l+q+\frac{1}{2}}\left|(\xi+i(k+1))^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(\xi+i(k+1))\right|d\xi.$$

Now, let $z=\xi+i(k+1)$

$$\left|e^{kb}\left(a^{-1}\frac{d}{da}\right)^l\left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi,\phi)(b,a)\right|$$
$$\leq C'_\mu b^{-\mu-q-\frac{1}{2}}a^{\mu-2r+\frac{1}{2}}e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1)$$
$$\times\sup_{|Imz|\leq k+1}\left(1+|z|^2\right)^m\left|z^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(z)\right|\int_{-\infty}^{\infty}\left(1+|z|^2\right)^{-m}|z|^{\mu-2r+2l+q+\frac{1}{2}}dz,$$

is convergent for large value of m. [1, Theorem 2.1], we get

$$\left|e^{kb}\left(a^{-1}\frac{d}{da}\right)^l\left(b^{-1}\frac{d}{db}\right)^q b^{-\mu-\frac{1}{2}}(B_\psi,\phi)(b,a)\right|$$
$$\leq C'_\mu b^{-\mu-q-\frac{1}{2}}a^{\mu-2r+\frac{1}{2}}e^{-b}\sum_{r=0}^{l}\binom{l}{r}\gamma^{\mu}_{k+1,0}(\psi)\,\Gamma(2\mu-2r+2l+1)\left\{\eta^{\mu}_{k+2,0}(\phi)+\eta^{\mu}_{k+2,m}(\phi)\right\}.$$

□

Theorem 2.3. *The Bessel wavelet transform B_ψ is a continuous linear mapping from $\mathcal{Q}_\mu(I)$ to $\mathcal{Q}_\mu(I\times I)$.*

Proof. Let $\phi\in\mathcal{Q}_\mu$. Suppose $\left(B_\psi\phi\right)(z,a)=\Phi(z,a)$ where $z=b+ib'$, $b,b'\in I$ and $a\in I$.

$$a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)=\int_0^\infty(zx)^{-\mu}J_\mu(zx)x^{\mu+\frac{1}{2}}\left(h_\mu\phi\right)(x)(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)dx.$$

Taking the absolute value of the above equation, we get

$$\begin{aligned}
&\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right| \\
&\leq \int_0^\infty \left|(zx)^{-\mu}J_\mu(zx)\right|\left|x^{\mu+\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\left|(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|dx \\
&\leq C\int_0^\infty e^{x|Imz|}\left|x^{\mu+\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\left|(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|dx \\
&\leq C\sup_{x\in I}\left|e^{xk}x^{-\mu-\frac{1}{2}}\left(h_\mu\phi\right)(x)\right|\sup_{x\in I}\left|e^{akx}(ax)^{-\mu-\frac{1}{2}}\left(h_\mu\psi\right)(ax)\right|\int_0^\infty e^{-axk}x^{2\mu+1}dx.
\end{aligned}$$

For $x > 1, l > \mu + \frac{3}{2}$ and in the view of [1, Theorem 2.1,pp.38-39], we have

$$\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right| \leq C\,\omega^\mu_{k+1,l}(\phi)\,\omega^\mu_{k+1,l}(\psi)\frac{\Gamma(2\mu+1)}{(ak)^{2\mu+1}}. \tag{15}$$

For $x \in (0,1)$, using the arguments of [1, Theorem 2.1, pp.38-39], we have

$$\left|a^{-\mu-\frac{1}{2}}z^{-\mu-\frac{1}{2}}\Phi(z,a)\right| \leq C\omega^\mu_{1,n}(\phi)\,\omega^\mu_{1,n}(\psi)\frac{\Gamma(2\mu+1)}{(ak)^{2\mu+1}}, \tag{16}$$

where $n \in \mathbb{N}$ and $n > \mu + 1$. Taking (15) and (16), B_ψ is continuous from $Q_\mu(I)$ to $Q_\mu(I \times I)$. □

Lemma 2.4. *Let ψ be a Bessel wavelet, then it can be written in the terms of Hankel transform as*

$$\tau_b\psi_a(x) = b^{\mu+\frac{1}{2}}\,h_\mu\left[(bu)^{-\mu}J_\mu(bu)(h_\mu\psi)(au)\right](x), \tag{17}$$

where a, b are dilation and translation parameters respectively.

Proof. Using (11) and putting $\frac{t}{a} = u$, we get

$$\begin{aligned}
\tau_b\psi_a(x) &= a^{\mu-\frac{1}{2}}\int_0^\infty \psi(z)\left(\int_0^\infty (au)^{-\mu-\frac{1}{2}}(xu)^{\frac{1}{2}}J_\mu(x\mu)(bu)^{\frac{1}{2}}J_\mu(bu)(zau)^{\frac{1}{2}}J_\mu(azu)adu\right)dz \\
&= \int_0^\infty\left(\int_0^\infty (zau)^{\frac{1}{2}}J_\mu(azu)\psi(z)dz\right)u^{-\mu-\frac{1}{2}}(xu)^{\frac{1}{2}}J_\mu(xu)(bu)^{\frac{1}{2}}J_\mu(bu)du \\
&= \int_0^\infty u^{-\mu-\frac{1}{2}}(xu)^{\frac{1}{2}}J_\mu(xu)(bu)^{\frac{1}{2}}J_\mu(bu)(h_\mu\psi)(au)du \\
&= b^{\mu+\frac{1}{2}}\,h_\mu\left[(bu)^{-\mu}J_\mu(bu)(h_\mu\psi)(au)\right](x).
\end{aligned}$$

□

Theorem 2.5. *If $f \in \chi'_\mu$ and $\psi \in \chi_\mu$ then $b^{-\mu-\frac{1}{2}}\left(B_\psi f\right)(b,a) \in \theta_{\chi_\mu}$, where χ'_μ and θ_{χ_μ} denote the dual and multiplier of χ_μ respectively.*

Proof. Suppose $f \in \chi_\mu'$ and $\varphi \in \chi_\mu$. Then

$$\begin{aligned}\left(B_\psi f\right)(b,a) &= \left(f\#\overline{\psi_a}\right)(b)\\ &= \langle f, \tau_b\psi_a\rangle\\ &= \left\langle \sum_{k=0}^{r} S_\mu^k(e^{rx}x^{-\mu-\frac{1}{2}}f_k), \tau_b\psi_a\right\rangle\\ &= \left\langle \sum_{k=0}^{r} e^{rx}x^{-\mu-\frac{1}{2}}f_k, \tau_b(S_\mu^k\psi_a)\right\rangle.\end{aligned}$$

Using Lemma 2.4, we get

$$\begin{aligned}\left(B_\psi f\right)(b,a) &= \sum_{k=0}^{r}\int_0^\infty e^{rx}x^{-\mu-\frac{1}{2}}f_k(x)b^{\mu+\frac{1}{2}}h_\mu\left[(bt)^{-\mu}J_\mu(bt)h_\mu(S_\mu^k\psi)(at)\right](x)dx\\ &= \sum_{k=0}^{r}\int_0^\infty e^{rx}x^{-\mu-\frac{1}{2}}f_k(x)b^{\mu+\frac{1}{2}}\,h_\mu\left[(bt)^{-\mu}J_\mu(bt)h_\mu\psi(at)(at)^{2k}\right](x)dx.\end{aligned}$$

$$\begin{aligned}&\left|\left(b^{-1}\frac{d}{db}\right)^n b^{-\mu-\frac{1}{2}}\left(B_\psi f\right)(b,a)\right|\\ &= \sum_{k=0}^{r} a^{2k}\int_0^\infty \left|e^{rx}x^{-\mu-\frac{1}{2}}f_k(x)h_\mu\left((bt)^{-\mu-n}J_{\mu+n}(bt)\,t^{2(k+n)}\,(h_\mu\psi)(at)\right)(x)\right|dx\\ &\leq \sup_{x\in I}\left|e^{(r+1)x}x^{-\mu-\frac{1}{2}}h_\mu\left((bt)^{-\mu-n}J_{\mu+n}(bt)\,t^{2(k+n)}\,(h_\mu\psi)(at)\right)(x)\right|\times ra^{2r}\int_0^\infty\left|e^{-x}f_k(x)\right|dx\\ &\leq C'a^{2r}\sup_{x\in I}\left|e^{(r+1)x}x^{-\mu-\frac{1}{2}}h_\mu\left((bt)^{-\mu-n}J_{\mu+n}(bt)t^{2(k+n)}h_\mu\psi(at)\right)(x)\right|.\end{aligned}$$

Since $\psi \in \chi_\mu \Rightarrow (h_\mu\psi)(at) \in Q_\mu$ and $(bt)^{-\mu-n}J_{\mu+n}(bt) \in \theta_{Q_\mu}$ (Multiplier of Q_μ).
Therefore $t^{2(n+k)}(bt)^{-\mu-n}J_{\mu+n}(bt)(h_\mu\psi)(at) \in Q_\mu$. [1, p.42] we get the following expression

$$\left|(b^{-1}\frac{d}{db})^n(1+a^{2r})^{-1}b^{-\mu-1/2}(B_\psi f)(b,a)\right| \leq C'e^{lb}.$$

This shows that $(1+a^{2r})^{-1}b^{-\mu-1/2}(B_\psi f)(b,a) \in \theta_{\chi_\mu}$. □

3. Applications

In this section motivated from [2, p.214], we introduce the Fredholm integral equation associated with Hankel convolution on χ_μ space.
The Fredholm integral equation is defined by

$$\int_0^\infty f(t)g(x,t)dt + \lambda f(x) = u(x), \tag{18}$$

where g(x) and u(x) are given functions and λ is a known parameter. From (3), we can write (18) as

$$(f\#g)(x) + \lambda f(x) = u(x). \tag{19}$$

Theorem 3.1. *Let $f \in L^1(0,\infty)$ and $g \in L^1(0,\infty)$. Then solution of* (19) *is*

$$f(x) = \int_0^\infty (x\xi)^{1/2}J_\mu(x\xi)\frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu g)(\xi)+\lambda)}d\xi. \tag{20}$$

Proof. Taking Hankel transform of (19) and using (5), we get

$$\xi^{-\mu-\frac{1}{2}}(h_\mu f)(\xi)(h_\mu g)(\xi) + \lambda(h_\mu f)(\xi) = (h_\mu u)(\xi)$$

$$(h_\mu f)(\xi) = \frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu g)(\xi) + \lambda)}. \tag{21}$$

From the inversion formula (2), we can find the solution

$$f(x) = \int_0^\infty (x\xi)^{1/2} J_\mu(x\xi) \frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu g)(\xi) + \lambda)} d\xi.$$

□

Theorem 3.2. *Let $f \in \chi_\mu$, $g \in \chi_\mu$. Then*

$$(f\#g)(x) + \lambda f(x) \in \chi_\mu. \tag{22}$$

Proof. From [1, Proposition 3.2], the Hankel convolution is a continuous linear mapping from $\chi_\mu \times \chi_\mu$ into χ_μ. This implies that $(f\#g) \in \chi_\mu$.
Since $f \in \chi_\mu$, therefore

$$(f\#g)(x) + \lambda f(x) \in \chi_\mu.$$

□

Example 3.3. *We take $u(x) = x^{\mu+\frac{1}{2}}e^{-ax^2}$ with $Re\, a > 0, Re\mu > -1$ and $f = g$ in (18). Then from (21), we have*

$$\left[(h_\mu f)(\xi)\right]^2 = \xi^{\mu+\frac{1}{2}}\, h_\mu\left(x^{\mu+\frac{1}{2}}e^{-ax^2}\right)(\xi) = \frac{\xi^{2\mu+1}}{(2a)^{\mu+1}}e^{-\xi^2/4a}.$$

From [3], we have

$$f(x) = 2^{\frac{\mu+1}{2}}x^{\mu+1}e^{-2ax^2}.$$

This $x^{\mu+1}e^{-2ax^2} \in \chi_\mu$ and the solution $f(x) = 2^{\frac{\mu+1}{2}}x^{\mu+1}e^{-2ax^2} \in \chi_\mu$.

Theorem 3.4. *Let $\psi \in \chi_\mu \subset L^1(0,\infty)$ be a Bessel wavelet. Then*

$$f(b) = \int_0^\infty J_\mu(b\xi)(b\xi)^{1/2} \frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu \psi_a)(\xi) + \lambda)} d\xi. \tag{23}$$

Proof. Putting $g = \psi_a(b)$ in (19) and from (21), we have

$$(h_\mu f)(\xi) = \frac{(h_\mu u)(\xi)}{(\xi^{-\mu-\frac{1}{2}}(h_\mu \psi_a)(\xi) + \lambda)}. \tag{24}$$

With the help of inversion formula of Hankel transform, we get (23). □

Example 3.5. *Solve the integral equation*

$$\int_0^\infty f(t)\psi\left(\frac{t}{a}, \frac{b}{a}\right)dt = u(b),$$

where $\psi(x) = 2^\nu\Gamma(\nu+1)x^{-\nu-\frac{1}{2}}J_{\mu+\nu+1}(x)$.

Solution. From (24) and [3, p.26], we get

$$(h_\mu f)(\xi) = (h_\mu u)(\xi)\frac{1}{\xi^{-\mu-\frac{1}{2}} h_\mu (2^\nu \Gamma(\nu+1) x^{-\nu-\frac{1}{2}} J_{\mu+\nu+1}(x))(a\xi)}$$
$$= (h_\mu u)(\xi)\frac{1}{[1-(a\xi)^2]^\nu}, \; Re\,\nu > -1, Re\,\mu > -1.$$

Taking $\nu = (\mu + \frac{1}{2})$, [3, p.32] and (5), we have,

$$(h_\mu f)(\xi) = h_\mu\left(u\,\#\,\pi^{-1/2}2^{-\mu}\Gamma(1/2-\mu)\left(\frac{x}{a}\right)^{\mu-\frac{1}{2}} sin\frac{x}{a}\right)(\xi)$$
$$f(x) = u\,\#\,\pi^{-1/2}2^{-\mu}\Gamma(1/2-\mu)\left(\frac{x}{a}\right)^{\mu-\frac{1}{2}} sin\frac{x}{a}.$$

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