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Some Inequalities for Submanifolds in a Riemannian Manifold of Nearly Quasi-Constant Curvature

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Abstract. In this paper, we derive a *DDVV*-type inequality for submianifolds in a Riemannian manifold of nearly quasi-constant curvature. Moreover, two inequalities involving the Casorati curvature and the scalar curvature are obtained.

1. Introduction

According to Chen's cornerstone work [3], one of the most important problems in submanifold theory is to establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of submanifolds. The basic relationships discovered until now are inequalities and the study of this topic has attracted a lot of attention during the last two decades [5,8,9,11–15,17–23].

In 1999, P.J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken introduced a new extrinsic invariant called the normal scalar curvature, and posed an inequality for submanifolds in real space forms involving the scalar curvature (intrinsic invariant), the normal scalar curvature (extrinsic invariant) and the squared mean curvature (extrinsic invariant), known as *DDVV* conjecture, which has later been proved by J. Ge, Z. Tang [10] and Z. Lu [16] in different ways. Recently, the similar *DDVV* inequality has been obtained for Lagrangian submanifolds in complex space forms by I. Mihai [17], and the author also proved an analogous inequality for slant submanifolds in complex space forms.

On the other hand, the Casorati curvature of a submanifold is an extrinsic invariant defined as the normalized square of the length of the second fundamental form. And it was preferred by Casorati over the traditional Gauss curvature because it corresponds better with the common intuition of curvature [2]. Therefore it is of great interests to obtain optimal inequalities for Casorati curvatures of submanifolds in different ambient spaces. S. Decu, S. Haesen and L. Verstraelen obtained some optimal inequalities involving the scalar curvature, and they also proved an inequality involving the holomorphic sectional curvature and the Casorati curvature of a Kaehler hypersurface in complex space forms [9].

In [4], B.Y. Chen and K. Yano generalized the notion of real space forms to quasi-constant curvature manifolds, which was further extended to nearly quasi-constant curvature manifolds by U.C. De and A.K. Gazi in [7]. In [18], C. Özgür studied Chen inequalities for submanifolds of a Riemannian manifold of

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quasi-constant curvature. Those inequalities later have been generalized to submanifolds of a Riemannian manifold of nearly quasi-constant curvature by C. Özgür and A. De in [19]. Also, some other basic inequalities involving the squared mean curvature and the Ricci curvature, the scalar curvature and the sectional curvature for submanifolds of this kind of ambient space are obtained in [12,23].

The main purpose of this paper is to continue to establish geometric inequalities for submanifolds in a Riemannian manifold of nearly quasi-constant curvature. In Section 3, we obtain a *DDVV* type inequality in terms of the squared mean curvature, the normalized normal scalar curvature and the scalar curvature. In Section 4, we establish two inequalities which are concerned with the Casorati curvature.

2. Preliminaries

In [4], B.Y. Chen and K. Yano introduced the notion of a Riemannian manifold (N, g) of quasi-constant curvature. Its curvature tensor $\bar{R}$ satisfies

$$\begin{aligned}\bar{R}(X,Y,Z,W) =&\bar{a}[g(X,Z)g(Y,W)-g(X,W)g(Y,Z)]+\bar{b}[g(X,Z)T(Y)T(W)-g(X,W)T(Y)T(Z)\\ &+T(X)T(Z)g(Y,W)-T(X)T(W)g(Y,Z)],\end{aligned} \tag{1}$$

where $\bar{a}, \bar{b}$ are scalar functions, and T is a 1-form defined by

$$g(X,P) = T(X) \tag{2}$$

and P is a unit vector field. It is easy to see, the manifold reduces to a real space form of constant curvature of $\bar{a}$ if $b = 0$.

Later, U. C. De and A. K. Gazi [7] generalized the notion of Riemannian manifold of quasi-constant curvature to Riemannian manifold of nearly quasi-constant curvature whose curvature tensor satisfies

$$\begin{aligned}\overline{R}(X,Y,Z,W) =&a[g(X,Z)g(Y,W)-g(X,W)g(Y,Z)]+b[g(X,Z)B(Y,W)-g(X,W)B(Y,Z)\\ &+B(X,Z)g(Y,W)-B(X,W)g(Y,Z)],\end{aligned} \tag{3}$$

where a, b are scalar functions, and B is a non-zero symmetric tensor filed of type (0,2). If $b = 0$, then the manifold reduces to a real space form. It's known that the outer product of two covariant vectors is a covariant of type (0,2), but the converse is not true, in general [7]. Hence, if $B = T \otimes T$ for a 1-form T, a Riemannian manifold of nearly quasi-constant curvature reduces to a Riemannian manifold of quasi-constant curvature.

Here are two examples of a Riemannian manifold of nearly quasi-constant curvature.

Example 2.1 *Let* $(\mathbb{R}^4, g)$ *be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

Then $(\mathbb{R}^4, g)$ *is a Riemannian manifold of nearly quasi-constant curvature, which is not a Riemannian manifold of quasi-constant curvature. Detailed explanations were given in* [7] (*see also* [19]).

Example 2.2 *Let* $N(c)$ *be a real space form. If* $N(c)$ *has a semi-symmetric metric connection with closed associated* $1-$*form* ω*, then* $N(c)$ *is a space of nearly quasi-constant curvature with respect to the semi-symmetric metric connection. More details can be found in* [19].

Let (M^n, g) be an n-dimensional submanifold in an $(n+m)$-dimensional Riemannian manifold $(N^{n+m}, \bar{g})$ of nearly quasi-constant curvature defined by (3). The Levi-Civita connection on N and M will be denoted by $\bar{\nabla}$ and ∇, respectively.

For vector fields X, Y tangent to M, and a vector field ξ normal to M, the Gauss and Weingarten formulas can be expressed by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),\ \bar{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi,$$

where h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection and A_{ξ} is the shape operator of M which is related with h by

$$g(A_{\xi}X, Y) = \bar{g}(h(X, Y), \xi).$$

Denote by $\bar{R}$ and R the Riemannian curvature tensors associated to $\bar{\nabla}$ and ∇, and we denote by $R^{\perp}$ the normal curvature tensor of M, then the Gauss equation and the Ricci equation are given respectively by

$$R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + \bar{g}(h(X, Z), h(Y, W)) - \bar{g}(h(X, W), h(Y, Z)), \tag{4}$$

$$R^{\perp}(X, Y, \xi, \eta) = \overline{R}(X, Y, \xi, \eta) + \bar{g}(h(X, A_{\xi}Y), \eta) - \bar{g}(h(Y, A_{\xi}X), \eta), \tag{5}$$

for vector fields X, Y, Z, W tangent to M, and vector fields ξ, η normal to M.

In N^{n+m}, we choose a local orthonormal frame

$$e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+m}, \tag{6}$$

such that, restricting to M^n, $e_1, \ldots, e_n$ are tangent to M^n. For convenience, we use the following convention on the range of indies:

$$i, j, \cdots = 1, \cdots, n, \;\; \alpha, \beta, \cdots = n+1, \cdots, n+m.$$

We write $h_{ij}^{\alpha} = \bar{g}(h(e_i, e_j), e_{\alpha})$. Then the mean curvature vector $\overrightarrow{H}$ is given by

$$\overrightarrow{H} = \frac{1}{n} \sum_{\alpha=n+1}^{n+m} (\sum_{i=1}^{n} h_{ii}^{\alpha}) e_{\alpha},$$

and we call $H = \|\overrightarrow{H}\|$ the *mean curvature* of M.

The submanifold is called *totally geodesic* if $h = 0$ and *minimal* if $H = 0$. The submanifold is called *invariantly quasi-umbilical* if there exist m mutually orthogonal unit normal vectors $e_{n+1}, \ldots, e_{n+m}$ such that the shape operators with respect to all directions e_{α} have an eigenvalue of multiplicity $n-1$ and that for each e_{α} the distinguished eigendirection is the same [1].

Let $K(e_i \wedge e_j)$, $1 \leq i < j \leq n$ denote the sectional curvature of the plane section spanned by e_i and e_j. Then the scalar curvature of M^n is defined by

$$\tau = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j), \tag{7}$$

and the normalized scalar curvature ρ, the normalized normal scalar curvature $\rho^{\perp}$ are given respectively by

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j), \tag{8}$$

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1 \leq i < j \leq n} \sum_{n+1 \leq \alpha < \beta \leq n+m} (R^{\perp}(e_{\alpha}, e_{\beta}, e_i, e_j))^2}. \tag{9}$$

By using (3) one can easily get the following.

Lemma 2.1 *Let M^n be a submanifold isometricly immersion into a Riemannian manifold N^{n+m} of nearly quasi-constant curvature whose curvature tensor satisfies (3), then with respect to the frame field defined by (6), we have $\overline{R}(e_{\alpha}, e_{\beta}, e_i, e_j) = 0$.*

3. DDVV-Type Inequality

First, we recall the following theorem ,which is also known as *DDVV* conjecture.

Theorem 3.1 ([8]) *Let M^n be a submanifold isometriclly immersion into a Riemannian manifold N^{n+m} which is a space with constant sectional curvature $\tilde{c}$. Then*

$$\rho + \rho^{\perp} \leq H^2 + \tilde{c}. \tag{10}$$

The equality case holds if and only if, with respect to some suitable orthonormal frame $e_1, \ldots, e_{n+m}$, the shape operators of M^n in N^{n+m} take the following forms

$$A_{e_{n+1}} = \begin{pmatrix} \lambda_1 + \mu & 0 & 0 & \cdots & 0 \\ 0 & \lambda_1 - \mu & 0 & \cdots & 0 \\ 0 & 0 & \lambda_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_1 \end{pmatrix}, \ A_{e_{n+2}} = \begin{pmatrix} \lambda_2 & \mu & 0 & \cdots & 0 \\ \mu & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_2 \end{pmatrix},$$

$$A_{e_{n+3}} = \begin{pmatrix} \lambda_3 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_3 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_3 \end{pmatrix}, \ A_{e_{n+4}} = A_{e_{n+5}} = \cdots = A_{e_{n+m}} = 0. \tag{11}$$

where λ_1, λ_2, λ_3 and μ are real functions on M^n.

In this section, we generalize Theorem 3.1 to submanifolds in Riemannian manifolds of nearly quasi-constant curvature as follows

Theorem 3.2 *Let M^n be a submanifold isometriclly immersion into a Riemannian manifold N^{n+m} of nearly quasi-constant curvature whose curvature tensor satisfies (3). Then we have*

$$\rho + \rho^{\perp} \leq H^2 + a + \frac{2b}{n} \mathrm{tr}(B\,|_M), \tag{12}$$

where $\mathrm{tr}(B\,|_M)$ is the trace of B restricted to M^n. The equality holds if and only if the shape operators take the desired forms as (11) with respect to some suitable frame.

Proof Due to Gauss and Ricci equations, we can get the following from (10) (see [16])

$$\sum_{\alpha} \sum_{i<j} (h_{ii}^{\alpha} - h_{jj}^{\alpha})^2 + 2n \sum_{\alpha} \sum_{i<j} (h_{ij}^{\alpha})^2 \geq 2n \Big[\sum_{\alpha<\beta} \sum_{i<j} (\sum_{k=1}^{n} (h_{ik}^{\alpha} h_{jk}^{\beta} - h_{jk}^{\alpha} h_{ik}^{\beta}))^2 \Big]^{\frac{1}{2}}. \tag{13}$$

From (4) and (7) we have

$$\tau = \sum_{i<j} R_{ijij} = \frac{n(n-1)a}{2} + b(n-1)\mathrm{tr}(B\,|_M) + \sum_{i<j} \sum_{\alpha} [h_{ii}^{\alpha} h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2], \tag{14}$$

which together with (5) and (9) gives

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{i<j} \sum_{\alpha<\beta} \Big[\sum_{k} (h_{ik}^{\alpha} h_{jk}^{\beta} - h_{jk}^{\alpha} h_{ik}^{\beta}) \Big]^2}. \tag{15}$$

On the other hand, we have

$$\begin{aligned} n^2 H^2 &= \sum_{\alpha} (\sum_{i} h_{ii}^{\alpha})^2 \\ &= \frac{1}{n-1} \sum_{\alpha} \sum_{i<j} (h_{ii}^{\alpha} - h_{jj}^{\alpha})^2 + \frac{2n}{n-1} \sum_{\alpha} \sum_{i<j} h_{ii}^{\alpha} h_{jj}^{\alpha}. \end{aligned} \tag{16}$$

Combining (13), (15) and (16), one can easily obtain

$$nH^2 - n\rho^{\perp} \geq \frac{2}{n-1}\sum_{\alpha}\sum_{i<j}\left[h_{ii}^{\alpha}h_{jj}^{\alpha} - (h_{ij}^{\alpha})^2\right]. \tag{17}$$

Plunge (14) into (17), we get,

$$nH^2 - n\rho^{\perp} \geq \frac{2}{n-1}\Big[\tau - \frac{n(n-1)}{2}a - (n-1)b\mathrm{tr}(B\mid_M)\Big],$$

or, equivalently,

$$H^2 - \rho^{\perp} \geq \frac{2}{n(n-1)}\Big[\tau - \frac{n(n-1)}{2}a - (n-1)b\mathrm{tr}(B\mid_M)\Big],$$

which together with (8) gives (12).

The equality case of (12) at a point $p \in M^n$ holds if and only if we have equality in (17). According to [16], the shape operators take the desired forms as (11) with some respect to suitable frame. □

Using $B = T \otimes T$ in (12), we get the following.

Corollary 3.4 *Let M be an n-dimensional submanifold of an $(n+m)$-dimensional manifold N of quasi-constant curvature whose curvature tensor satisfies (1) and (2). Then we have*

$$\rho + \rho^{\perp} \leq H^2 + a + \frac{2b}{n}\|P^{\top}\|^2,$$

where $P^{\top}$ is the tangential components of P on M, and the equality holds if and only if the shape operators take the desired forms as (11) with respect to some suitable frame.

Remark 3.5 If $b = 0$, then we can get Theorem 3.1.

4. Inequalities for Casorati Curvature

The squared norm of h over dimension n is called *Casorati curvature* of the submanifold of M, i.e.,

$$C = \frac{1}{n}\sum_{\alpha=n+1}^{n+m}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2. \tag{18}$$

Suppose $x \in M$, L is a r-dimensional subspace of T_xM spanned by $e_1, \cdots, e_r$, $r \geq 2$. The Casorati curvature of L is defined by

$$C(L) = \frac{1}{r}\sum_{\alpha=n+1}^{n+m}\sum_{i,j=1}^{r}(h_{ij}^{\alpha})^2. \tag{19}$$

Following [9], we can define the normalized $\delta -$ *Casorati* curvatures $\delta_C(n-1)$ and $\hat{\delta}_C(n-1)$ by

$$[\delta_C(n-1)]_x = \frac{1}{2}C_x + \frac{n+1}{2n}\inf\Big\{C(L) \mid L \text{ a hyperplane of } T_xM\Big\}, \tag{20}$$

$$[\hat{\delta}_C(n-1)]_x = 2C_x - \frac{2n-1}{2n}\sup\Big\{C(L) \mid L \text{ a hyperplane of } T_xM\Big\}. \tag{21}$$

In this paper, we obtain two inequalities in term of the normalized $\delta -$ *Casorati* curvature $\delta_C(n-1)$ as follows.

Theorem 4.1 *Let (M^n, g) be a Riemannian submanifold in an $(n+m)$-dimensional Riemannian manifold N of nearly quasi-constant curvature whose curvature tensor satisfies (3). Then we have*

$$\rho \leq \delta_C(n-1) + a + \frac{2b}{n}\mathrm{tr}(B\,|_M). \tag{22}$$

The equality holds if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in N, such that with respect to some suitable frame $e_1, \cdots, e_{n+m}$ the shape operators take the following forms

$$A_{e_1} = \begin{pmatrix} \mu & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \mu & 0 \\ 0 & \cdots & 0 & 2\mu \end{pmatrix}, \; A_{e_2} = A_{e_3} = \cdots = A_{e_{n+m}} = 0,$$

where μ is a real function on M^n.

Remark 4.1 If $b = 0$, (22) is due to the inequality (4.1) in [22].

Proof Consider the following function which is a quadratic polynomial in the components of the second fundamental form

$$\mathcal{P} = \frac{1}{2}n(n-1)C + \frac{1}{2}(n+1)(n-1)C(L) - 2\tau + n(n-1)a + 2(n-1)b\,\mathrm{tr}(B\,|_M). \tag{23}$$

From (18) and (14) we have

$$2\tau = n^2H^2 - nC + n(n-1)a + 2(n-1)b\,\mathrm{tr}(B|_M).$$

Assuming, without loss of generality, that L is spanned by $e_1, \cdots, e_{n-1}$, it follows that

$$\begin{aligned}
\mathcal{P} &= \frac{n(n+1)}{2}C + \frac{(n-1)(n+1)}{2}C(L) - n^2H^2 \\
&= \sum_{\alpha}\left\{n\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + \frac{n-1}{2}(h_{nn}^{\alpha})^2 + 2(n+1)\sum_{1\leq i<j\leq n-1}(h_{ij}^{\alpha})^2 + (n+1)\sum_{i=1}^{n-1}(h_{in}^{\alpha})^2 - 2\sum_{1\leq i<j\leq n}h_{ii}^{\alpha}h_{jj}^{\alpha}\right\} \\
&\geq \sum_{\alpha}\left\{n\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + \frac{n-1}{2}(h_{nn}^{\alpha})^2 - 2\sum_{1\leq i<j\leq n}h_{ii}^{\alpha}h_{jj}^{\alpha}\right\}.
\end{aligned} \tag{24}$$

For $\alpha = n+1, \cdots, n+m$, let us consider the quadratic forms

$$f_\alpha : \mathbb{R}^n \to \mathbb{R},$$

$$f_\alpha(h_{11}^{\alpha}, \cdots, h_{nn}^{\alpha}) = n\sum_{i=1}^{n-1}(h_{ii}^{\alpha})^2 + \frac{n-1}{2}(h_{nn}^{\alpha})^2 - 2\sum_{1\leq i<j\leq n}h_{ii}^{\alpha}h_{jj}^{\alpha}.$$

The matrix of f_α is

$$\mathcal{F}_\alpha = \begin{pmatrix} n & \cdots & -1 & -1 \\ \vdots & \ddots & \vdots & \vdots \\ -1 & \cdots & n & -1 \\ -1 & \cdots & -1 & \frac{n-1}{2} \end{pmatrix}.$$

By a straight calculation, one can get the characteristic polynomial is

$$|\lambda E - \mathcal{F}_\alpha| = \begin{vmatrix} \lambda - n & \cdots & 1 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \cdots & \lambda - n & 1 \\ 1 & \cdots & 1 & \lambda - \frac{n-1}{2} \end{vmatrix} = \lambda(\lambda - \frac{n+3}{2})(\lambda - n - 1)^{n-2}.$$

Hence the eigenvalues of matrix of f_α are

$$\lambda_1 = n+1,\ \lambda_2 = n+1, \cdots, \lambda_{n-2} = n+1,\ \lambda_{n-1} = \frac{n+3}{2},\ \lambda_n = 0.$$

Therefore, f_α is positive semidefinite, i.e.

$$f_\alpha \geq 0. \tag{25}$$

Consequently

$$\mathcal{P} \geq 0. \tag{26}$$

Then, from (23) and (26) we can derive inequality (22).

In the following, we consider the equality case of (22). Equality holds in the inequality (24) if and only if

$$h^\alpha_{ij} = 0,\ i \neq j \in \{1, \ldots, n\}. \tag{27}$$

The critical points $h^c = (h^\alpha_{11}, \cdots, h^\alpha_{nn})$ of f_α are solutions of the following system of linear homogeneous equations

$$\begin{cases} \frac{\partial f_\alpha}{\partial h^\alpha_{11}} = 2nh^\alpha_{11} - 2\sum_{i=2}^{n} h^\alpha_{ii} = 0, \\ \frac{\partial f_\alpha}{\partial h^\alpha_{22}} = 2nh^\alpha_{22} - 2h^\alpha_{11} - 2\sum_{i=3}^{n} h^\alpha_{ii} = 0, \\ \vdots \\ \frac{\partial f_\alpha}{\partial h^\alpha_{n-1,n-1}} = 2nh^\alpha_{n-1,n-1} - 2h^\alpha_{nn} - 2\sum_{i=1}^{n-2} h^\alpha_{ii} = 0, \\ \frac{\partial f_\alpha}{\partial h^\alpha_{nn}} = (n-1)h^\alpha_{nn} - 2\sum_{i=1}^{n-1} h^\alpha_{ii} = 0. \end{cases} \tag{28}$$

On the other hand, we set

$$k^\alpha = h^\alpha_{11} + \cdots + h^\alpha_{nn}, \tag{29}$$

here k^α is the trace of the matrix (h^α_{ij}), which is invariant no matter how h^α_{ij} changes. As for the system of linear homogeneous equations (28), the rank of its coefficient matrix is $n-1$ which is less than n, therefore (28) has non-zero solutions.

By using (28) and (29), the equality in (25) holds if and only if

$$h^\alpha_{11} = h^\alpha_{22} = \cdots = h^\alpha_{n-1,n-1} = \frac{k^\alpha}{n+1},\ h^\alpha_{nn} = \frac{2k^\alpha}{n+1}, \tag{30}$$

for all $\alpha \in \{n+1, \ldots, n+m\}$.

Combining (27) and (30), we see that $\mathcal{P} = 0$ if and only if

$$h^\alpha_{ii} = \frac{k^\alpha}{n+1},\ h^\alpha_{nn} = \frac{2k^\alpha}{n+1},\ i = 1, \ldots, n-1,$$

and

$$h_{ij}^{\alpha} = 0,\ i \neq j \in \{1, \dots, n\},$$

for all $\alpha \in \{n+1, \dots, n+m\}$.

Thus for this case, from (5), (27) and Lemma 2.1, we know that the normal connection of M is flat, that is the normal curvature tensor $R^{\perp}$ vanishes. Then we conclude that the equality holds if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in N, such that with respect to suitable tangent and normal orthonormal frames $e_1, \cdots, e_{n+m}$ the shape operators take the following forms

$$A_{e_{n+1}} = \begin{pmatrix} \mu & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \mu & 0 \\ 0 & \cdots & 0 & 2\mu \end{pmatrix}, A_{e_{n+2}} = A_{e_{n+3}} = \cdots = A_{e_{n+m}} = 0. \tag{31}$$

□

Analogously, working with the function

$$Q = 2n(n-1)C + \frac{1}{2}(1-2n)(n-1)C(L) - 2\tau + n(n-1)a + 2(n-1)b\mathrm{tr}(B|_M),$$

instead of $\mathcal{P}$ in the proof of Theorem 4.1, we obtain a similar inequality involving $\hat{\delta}_C(n-1)$ as follows.

Theorem 4.2 *Let (M^n, g) be a Riemannian submanifold of an $(n+m)$-dimension Riemannian manifold N^{n+m} of nearly quasi-constant curvature whose curvature tensor satisfies (3). Then we have*

$$\rho \leq \hat{\delta}_C(n-1) + a + \frac{2b}{n}\mathrm{tr}(B\ |_M). \tag{32}$$

The equality holds if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in N, such that with respect to some suitable frame $e_1, \cdots, e_{n+m}$ the shape operators take the following forms

$$A_{e_{n+1}} = \begin{pmatrix} 2\mu & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 2\mu & 0 \\ 0 & \cdots & 0 & \mu \end{pmatrix}, A_{e_{n+2}} = A_{e_{n+3}} = \cdots = A_{e_{n+m}} = 0,$$

where μ is a real function on M^n.

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