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Fixed Point Results for Multivalued Hardy–Rogers Contractions in b-Metric Spaces

Cristian Chifu[a], **Gabriela Petruşel**[a]

[a] *Babeş-Bolyai University Cluj-Napoca, Faculty of Business*

Abstract. The purpose of this paper is to present some fixed point results in b-metric spaces using a contractive condition of Hardy-Rogers type with respect to the functional H. The data dependence of the fixed point set, the well-posedness of the fixed point problem, as well as, the Ulam-Hyres stability are also studied.

1. Preliminaries

In 1973, Hardy and Rogers ([5]) gave a generalization of Reich fixed point theorem. Since then, many authors have been used different Hardy-Rogers contractive type conditions in order to obtain fixed point results. In what follows we shall recall, pure randomly, some of them.

In 2009, Kadelburg, Radenovic and Rasic ([6]), gave some common fixed point results in cone metric spaces. Radojevic, Paunovic and Radenovic ([7]) have obtained some coincidence point theorems in complete metric spaces. Sgroi and Vetro ([9]) have presented some results for $\mathcal{F}-$contractions in complete and ordered metric spaces. Finally, Roshan, Shobkolaei, Sedghi and Abbas ([8]) gave some common fixed point results in b-metric spaces.

In this paper we shall give some fixed point results for multivalued operators in *b-metric* spaces using a contractive condition of Hardy-Rogers type with respect to the functional H. The data dependence of the fixed point set, the well-posedness of the fixed point problem, as well as, the Ulam-Hyres stability are also studied.

Because we shall work in $b-metric$ spaces, we'll start by presenting some notions about this kind of metric spaces.

Definition 1.1. *Let X be a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is said to be a b-metric if and only if for all $x, y, z \in \mathbb{X}$, the following conditions are satisfied:*

1. $d(x, y) = 0 \Longleftrightarrow x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq s\left[d(x, z) + d(z, y)\right]$.

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Communicated by Calogero Vetro
Email addresses: cristian.chifu@tbs.ubbcluj.ro (Cristian Chifu), gabi.petrusel@tbs.ubbcluj.ro (Gabriela Petruşel)

In this case, the pair (X, d) is called $b-$metric space with constant s.

Remark 1.2. *The class of b-metric spaces is larger than the class of metric spaces since a b-metric space is a metric space when s=1.*

Example 1.3. *Let* $X=\{0,1,2\}$ *and* $d : X \times X \to \mathbb{R}_+$ *such that* $d(0,1) = d(1,0) = d(0,2) = d(2,0) = 1, d(1,2) = d(2,1) = \alpha \geq 2, d(0,0) = d(1,1) = d(2,2) = 0$. *We have*

$$d(x,y) \leq \frac{\alpha}{2}\left[d(x,z) + d(z,y)\right], \text{ for } x,y,z \in X.$$

Then (X, d) is a b-metric space. If $\alpha > 2$ the ordinary triangle inequality does not hold and (X, d) is not a metric space.

Example 1.4. *The set* $l^p(\mathbb{R}) = \left\{(x_n) \subset \mathbb{R} | \lim_{n=1}^{\infty} |x_n|^p < \infty\right\}, 0 < p < 1$, *together with the functional* $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}_+, d(x,y) = \left(\lim_{n=1}^{\infty} \left|x-y\right|^p\right)^{1/p}$, *is a b-metric space with constant* $s = 2^{1/p}$.

Example 1.5. *Let* $X = \mathbb{R}$ *and* $d : X \times X \to \mathbb{R}_+, d(x,y) = \left|x-y\right|^3$. *The (X, d) is a b-metric space with constant $s = 3$.*

Definition 1.6. *Let (X, d) be a $b-$metric space with constant s. Then the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is called:*

1. *convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$, as $n \to \infty$;*
2. *Cauchy if and only if $d(x_n, x_m) \to 0$, as $n, m \to \infty$.*

Definition 1.7. *Let (X, d) be a $b-$metric space with constant s. If Y is a nonempty subset of X, then the closure $\overline{Y}$ of Y is the set of limits of all convergent sequences of points in Y, i.e.,*

$$\overline{Y} := \{x \in X : \exists (x_n)_{n \in \mathbb{N}}, x_n \to x, \text{ as } n \to \infty\}.$$

Definition 1.8. *Let (X, d) be a $b-$metric space with constant s. Then a subset $Y \subset X$ is called:*

1. *closed if and only if for each sequence $(x_n)_{n \in \mathbb{N}} \subset Y$ which converges to x, we have $x \in Y$;*
2. *compact if and only if for every sequence of elements of Y there exists a subsequence that converges to an element of Y;*
3. *bounded if and only if $\delta(Y) := \{d(a,b) : a, b \in Y\} < \infty$.*

Definition 1.9. *The $b-$metric space (X, d) is complete if every Cauchy sequence in X converges.*

Let us consider the following families of subsets of a b-metric space (X, d):

$$\mathcal{P}(X) = \{Y | Y \subset X\}, P(X) := \{Y \in \mathcal{P}(X) | Y \neq \emptyset\}; P_b(X) := \{Y \in \mathcal{P}(X) | \ Y \text{ is bounded }\},$$

$$P_{cl}(X) := \{Y \in \mathcal{P}(X) | \ Y \text{ is closed}\}; P_{cp}(X) := \{Y \in \mathcal{P}(X) | \ Y \text{ is compact}\}$$

Throughout the paper the following fuctionals are used:

- **the gap functional:** $D : P(X) \times P(X) \to \mathbb{R}_+$

  $$D(A,B) = \inf\{d(a,b) \mid a \in A, \ b \in B\}.$$

  In particular, if $x_0 \in X$, then $D(x_0, B) := D(\{x_0\}, B)$.

- **the Pompeiu-Hausdorff generalized functional:** $H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$,

$H(A,B) = \max\{\rho(A,B), \rho(B,A)\}$,

where $\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ defined as

$\rho(A,B) = \sup\{D(a,B) \mid a \in A\}$,

is called **the excess generalized functional**.

Let $T : X \to P(X)$ be a multivalued operator. A point $x \in X$ is called fixed point for T if and only if $x \in T(x)$.

The set $Fix(T) := \{x \in X \mid x \in T(x)\}$ is called the fixed point set of T, while $SFix(T) = \{x \in X \mid \{x\} = T(x)\}$ is called the strict fixed point set of T. Notice that $SFix(T) \subseteq Fix(T)$.

The following properties of some of the functionals defined above will be used throughout the paper (see [1] , [4] for details and proofs):

Lemma 1.10. *Let (X,d) be a b-metric space with constant $s > 1$, $A, B \in P_{cl}(X)$. Then*

1. $D(x,B) \leq d(x,b)$, *for any $b \in B$;*
2. $D(x,B) \leq H(A,B)$, *for any $x \in A$;*
3. $D(x,A) \leq s[d(x,y) + D(y,A)]$, *for all $x, y \in X, A \subset X$;*
4. $D(x,A) = 0$ *if and only if $x \in \overline{A}$;*
5. *For any $q > 1$, $a \in A$, there exists $b \in B$ such that $d(a,b) \leq qH(A,B)$;*
6. $d\left(x_n, x_{n+p}\right) \leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + \ldots + s^{p-1} d\left(x_{n+p-2}, x_{n+p-1}\right) + s^{p-1} d\left(x_{n+p-1}, x_{n+p}\right)$, *for any $n \in \mathbb{N}$ and $p \in \mathbb{N}^*$.*

2. Fixed Point Results

In this section we shall present our main fixed point theorem for multivalued Hardy-Rogers operators.

Theorem 2.1. *Let (X,d) be a complete b-metric space with constant $s > 1$ and $T : X \to P(X)$ a multivalued operator such that:*

(i) there exist $a, b, c \in \mathbb{R}_+, a + b + 2cs < \frac{s-1}{s^2}$ and $b + cs < \frac{1}{s}$ such that

$$H(T(x), T(y)) \leq ad(x,y) + b[D(x,T(x)) + D(y,T(y))] + c[D(x,T(y)) + D(y,T(x))],$$

for all $x, y \in X$;

(ii) T is closed;

In these conditions $Fix(T) \neq \emptyset$.

Proof. (i) It's easy to see that because $a + b + 2cs < \frac{s-1}{s^2}$, $a + b + cs < a + b + 2cs < \frac{s-1}{s^2}$ and hence,

$$s(a + b + cs) < \frac{s-1}{s}.$$

On the other hand, since $b + cs < \frac{1}{s}$, we obtain

$$\frac{1 - b - cs}{s(a + b + cs)} > 1.$$

Let $x_0 \in X$ and $1 < q < \frac{1}{s}\frac{1-b-cs}{a+b+cs}$.

There exists $x_1 \in T(x_0)$ such that

$$H(T(x_0), T(x_1)) \leq ad(x_0, x_1) + b[D(x_0, T(x_0))) + D(x_1, T(x_1))] + c[D(x_0, T(x_1)) + D(x_1, T(x_0))].$$

By Lemma 1.1. we have:

$$\begin{aligned} D(x_0, T(x_0)) &\leq d(x_0, x_1); \\ D(x_1, T(x_1)) &\leq H(T(x_0), T(x_1)); \\ D(x_1, T(x_0)) &= 0; \\ D(x_0, T(x_1)) &\leq s[d(x_0, x_1) + D(x_1, T(x_1))] \leq s[d(x_0, x_1) + H(T(x_0), T(x_1))]. \end{aligned}$$

Hence

$$H(T(x_0), T(x_1)) \leq ad(x_0, x_1) + bd(x_0, x_1) + bH(T(x_0), T(x_1)) + csd(x_0, x_1) + csH(T(x_0), T(x_1))$$

$$(1 - b - cs)H(T(x_0), T(x_1)) \leq (a + b + cs)d(x_0, x_1)$$

Since $b + cs < \frac{1}{s} < 1$ we have

$$H(T(x_0), T(x_1)) \leq \frac{a + b + cs}{1 - b - cs} d(x_0, x_1).$$

Using again Lemma 1.1., there exists $x_2 \in T(x_1)$ such that

$$\begin{aligned} d(x_1, x_2) &\leq qH(T(x_0), T(x_1)) \\ d(x_1, x_2) &\leq q\frac{a + b + cs}{1 - b - cs} d(x_0, x_1). \end{aligned}$$

Let $q\frac{a+b+cs}{1-b-cs} := \alpha < \frac{1}{s} < 1$
Hence

$$d(x_1, x_2) \leq \alpha d(x_0, x_1).$$

Continuing this process we shall obtain that there exists a sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n \in T(x_{n-1})$, such that $d(x_n, x_{n+1}) \leq \alpha^n d(x_0, x_1)$ for each $n \in \mathbb{N}$.

This inequality implies that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, see [3]. Hence there exists $x \in X$ such that $x_n \to x$, as $n \to \infty$.

Now, we shall prove that $x \in T(x)$.

We have:

$$\begin{aligned} D(x, T(x)) &\leq sd(x, x_{n+1}) + sD(x_{n+1}, T(x)) \\ &\leq sd(x, x_{n+1}) + sH(T(x_n), T(x)). \end{aligned}$$

$$\begin{aligned} H(T(x_n), T(x)) &\leq ad(x_n, x) + b[D(x_n, T(x_n)) + D(x, T(x))] + c[D(x, T(x_n)) + D(x_n, T(x))] \\ &\leq ad(x_n, x) + bd(x_n, x_{n+1}) + bD(x, T(x)) + cd(x_{n+1}, x) + csd(x_n, x) + csD(x, T(x)). \end{aligned}$$

Hence

$$\begin{aligned} D(x, T(x)) \leq\; & sd(x, x_{n+1}) + asd(x_n, x) + bsd(x_n, x_{n+1}) + bsD(x, T(x)) + \\ & csd(x_{n+1}, x) + cs^2 d(x_n, x) + cs^2 D(x, T(x)). \end{aligned}$$

If $n \to \infty$ then we obtain $\left(1 - bs - cs^2\right) D(x, T(x)) \leq 0$.

Since $b + cs < \frac{1}{s}$ we have that $bs + cs^2 < 1$ and hence, $D(x, T(x)) = 0$. This implies that $x \in T(x)$ and hence $Fix(T) \neq \emptyset$. $\square$

An existence and uniqueness fixed point result for multivalued Hardy-Rogers operators is the following:

Theorem 2.2. *Let (X, d) be a complete b-metric space with constant $s > 1$ and $T : X \to P(X)$ a multivalued operator such that:*

(i) there exist $a, b, c \in \mathbb{R}_+, a + b + 2cs < \frac{s-1}{s^2}$ and $b + cs < \frac{1}{s}$ such that

$$H(T(x), T(y)) \leq ad(x, y) + b\left[D(x, T(x)) + D(y, T(y))\right] + c\left[D\left(x, T\left(y\right)\right) + D\left(y, T\left(x\right)\right)\right],$$

for all $x, y \in X$;

(ii) T is closed;

If $SFix\,(T) \neq \emptyset$ then $SFix\,(T) = Fix\,(T) = \{x\}$.

Proof. Let $x \in SFix\,(T)$ and suppose that there exist $y \in Fix\,(T)\,, y \neq x$.

$$\begin{aligned} d\left(x, y\right) &= D\left(T\left(x\right), y\right) \leq H\left(T\left(x\right), T\left(y\right)\right) \\ &\leq ad(x, y) + b\left[D(x, T(x)) + D(y, T(y))\right] + c\left[D\left(x, T\left(y\right)\right) + D\left(y, T\left(x\right)\right)\right] \\ &\leq ad(x, y) + 2cd(x, y). \end{aligned}$$

Hence $(1 - a - 2c)\,d(x, y) \leq 0$.

Since $a + 2c < a + b + 2cs < \frac{s-1}{s^2} < 1$, we shall obtain that $d(x, y) = 0$ which implies that $x = y$ and this is a contradiction.

In conclusion $SFix\,(T) = Fix\,(T) = \{x\}$. □

An example illustrating our theorem is given in what follows.

Example 2.3. *Let us consider the following two sets (see [2]):*

$$\begin{aligned} M_1 &= \left\{\frac{m}{n} | m = 0, 1, 3, 9, ...; n = 3k + 1, k \in \mathbb{N}\right\}; \\ M_2 &= \left\{\frac{m}{n} | m = 1, 3, 9, 27, ...; n = 3k + 2, k \in \mathbb{N}\right\}. \end{aligned}$$

Let $X = M_1 \cup M_2$. Define $T : X \to \mathbb{R}_+$,

$$T\left(x\right) = \begin{cases} \{\alpha x, \beta x\}, x \in M_1 \\ \{\beta x\}, \quad x \in M_2 \end{cases},$$

where $0 < \beta \leq \alpha < 1$.

Notice that T is not a Hardy-Rogers operator with respect to the metric $\hat{d}(x, y) := |x - y|$ (see [2]), but it becomes a Hardy-Rogers operator with respect to the b-metric (with constant $s = 3$) defined by $d\left(x, y\right) = \left|x - y\right|^3$.

Proof. We shall prove that there exist $a, b, c \in \mathbb{R}_+$ such that T is a Hardy-Rogers with respect to d. We shall have four cases:

(1) $x, y \in M_1$

In this case $\rho(T\left(x\right), T\left(y\right)) = \left|\alpha x - \alpha y\right|^3 = \alpha^3 d\left(x, y\right)$ and $\rho(T\left(y\right), T\left(x\right)) = \left|\alpha y - \alpha x\right|^3 = \alpha^3 d\left(x, y\right)$ and hence $H(T\left(x\right), T\left(y\right)) = \alpha^3 d\left(x, y\right)$.

(2) $x, y \in M_2$

In this case $\rho(T\left(x\right), T\left(y\right)) = \left|\beta x - \beta y\right|^3 = \beta^3 d\left(x, y\right)$ and $\rho(T\left(y\right), T\left(x\right)) = \left|\beta y - \beta x\right|^3 = \beta^3 d\left(x, y\right)$ and hence $H(T\left(x\right), T\left(y\right)) = \beta^3 d\left(x, y\right) \leq \alpha^3 d\left(x, y\right)$.

(3) $x \in M_1, y \in M_2$

In this case $\rho(T(x), T(y)) = |\alpha x - \beta y|^3$ and $\rho(T(y), T(x)) = |\beta y - \alpha x|^3$ and hence $H(T(x), T(y)) = |\alpha x - \beta y|^3$.

We have to consider the following cases:

3.1. If $x > y$, then $\left|x - \frac{\beta}{\alpha} y\right| < |x - \beta y|$, and hence $H(T(x), T(y)) = |\alpha x - \beta y|^3 = \alpha^3 \left|x - \frac{\beta}{\alpha} y\right|^3 \leq \alpha^3 |x - \beta y|^3 = \alpha^3 D(x, T(y))$.

3.2. If $x < y$, then:

If $x < \beta y$, then $\left|\frac{\alpha}{\beta} x - y\right| < |\alpha x - y|$, and hence $H(T(x), T(y)) = |\alpha x - \beta y|^3 = \beta^3 \left|\frac{\alpha}{\beta} x - y\right|^3 \leq \beta^3 |\alpha x - y|^3 = \beta^3 D(y, T(x)) \leq \alpha^3 D(y, T(x))$.

If $x > \beta y$, then we have another two cases:

If $\alpha x < \beta y$, then $\left|\frac{\alpha}{\beta} x - y\right| < |\alpha x - y|$, and hence $H(T(x), T(y)) = |\alpha x - \beta y|^3 = \beta^3 \left|\frac{\alpha}{\beta} x - y\right|^3 \leq \beta^3 |\alpha x - y|^3 = \beta^3 D(y, T(x)) \leq \alpha^3 D(y, T(x))$.

If $\alpha x > \beta y$, then $\left|x - \frac{\beta}{\alpha} y\right| < |x - \beta y|$, and hence $H(T(x), T(y)) = |\alpha x - \beta y|^3 = \alpha^3 \left|x - \frac{\beta}{\alpha} y\right|^3 \leq \alpha^3 |x - \beta y|^3 = \alpha^3 D(x, T(y))$.

(4) $x \in M_2, y \in M_1$

In this case $\rho(T(x), T(y)) = |\beta x - \alpha y|^3$ and $\rho(T(y), T(x)) = |\alpha y - \beta x|^3$ and hence $H(T(x), T(y)) = |\alpha y - \beta x|^3$.

Just like in the previuos case, we have to consider the following cases:

4.1. $x > y$

If $y < \beta x$, $\left|\frac{\alpha}{\beta} y - x\right| < |\alpha y - x|$, and hence $H(T(x), T(y)) = |\alpha y - \beta x|^3 = \beta^3 \left|\frac{\alpha}{\beta} y - x\right|^3 \leq \beta^3 |\alpha y - x|^3 = \beta^3 D(x, T(y)) \leq \alpha^3 D(x, T(y))$.

If $y > \beta x$, then we have another two cases:

If $\alpha y < \beta x$, then $\left|\frac{\alpha}{\beta} y - x\right| < |\alpha y - x|$, and hence $H(T(x), T(y)) = |\alpha y - \beta x|^3 = \beta^3 \left|\frac{\alpha}{\beta} y - x\right|^3 \leq \beta^3 |\alpha y - x|^3 = \beta^3 D(x, T(y)) \leq \alpha^3 D(x, T(y))$.

If $\alpha y > \beta x$, then $\left|y - \frac{\beta}{\alpha} x\right| < |y - \beta x|$, and hence $H(T(x), T(y)) = |\alpha y - \beta x|^3 = \alpha^3 \left|y - \frac{\beta}{\alpha} x\right|^3 \leq \alpha^3 |y - \beta x|^3 = \alpha^3 D(y, T(x))$.

4.2 $x < y$

In this case we have $\left|y - \frac{\beta}{\alpha} x\right| < |y - \beta x|$, and hence $H(T(x), T(y)) = |\alpha y - \beta x|^3 = \alpha^3 \left|y - \frac{\beta}{\alpha} x\right|^3 \leq \alpha^3 |y - \beta x|^3 = \alpha^3 D(y, T(x))$.

Hence, we can conclude that $H(T(x), T(y)) \leq \alpha^3 d(x, y) + \alpha^3 D(x, T(y)) + \alpha^3 D(y, T(x))$, for all $x, y \in X$.

If, for example $\alpha = \beta = \frac{1}{5}$, then $T : X \to P(X)$. If we consider $a = c = \alpha^3$ and $b = 0$, then, for $s = 3$, all the assumptions on a, b, c in Theorem 2.1 are fulfilled and the operator T defined above satisfies the conditions of the theorem. □

In what follows we shall present a data dependence theorem for multivalued Hardy-Rogers operators in a complete b-metric space.

Theorem 2.4. *Let (X, d) be a complete b-metric space with constant $s > 1$, $T_1, T_2 : X \to P(X)$ be two multivalued closed operators which satisfy the following conditions:*

(a) *there exists* $\eta > 0$ *such that* $H(T_1(x), T_2(x)) \leq \eta$, *for all* $x \in X$;
(b) *there exist* $a_i, b_i, c_i \in \mathbb{R}_+, a_i + b_i + 2c_i s < \frac{s-1}{s^2}$ *and* $b_i + c_i s < \frac{1}{s}$ *such that*

$$H(T_i(x), T_i(y)) \leq a_i d(x, y) + b_i \left[D(x, T_i(x)) + D(y, T_i(y))\right] + c_i \left[D\left(x, T_i\left(y\right)\right) + D\left(y, T_i\left(x\right)\right)\right],$$

for all $x, y \in X, i \in \{1, 2\}$.
In these conditions we have:

$$H(Fix(T_1), Fix(T_2)) \leq \frac{\eta s}{1 - s\max\{A_1, A_2\}},$$

where $A_i = \frac{a_i + b_i + c_i s}{1 - b_i - c_i s}, i \in \{1, 2\}$

Proof. We'll show that for every $x_1^* \in Fix(T_1)$, there exists $x_2^* \in Fix(T_2)$ such that

$$d(x_1^*, x_2^*) \leq \frac{s\eta}{1 - sA_2}.$$

Let $x_1^* \in Fix(T_1)$ arbitrary and let $1 < q < \frac{1-b_2-c_2 s}{a_2+b_2+c_2 s}\frac{1}{s}$. As in the proof of Theorem 2.1. we construct a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ of successive approximations of T_2, with $x_0 := x_1^*$ and $x_1 \in T_2(x_1^*)$ having the property:

$$d(x_n, x_{n+1}) \leq \alpha_2^n d(x_0, x_1)$$

for each $n \in \mathbb{N}$, where $\alpha_2 = q\frac{a_2+b_2+c_2 s}{1-b_2-c_2 s} < \frac{1}{s}$.
If we consider that the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x_2^*, we have that $x_2^* \in Fix(T_2)$. Moreover, for each $n \geq 0$, we have:

$$d(x_n, x_{n+p}) \leq s\alpha_2^n \frac{1 - (s\alpha_2)^p}{1 - s\alpha_2} d(x_0, x_1),\ p \in \mathbb{N}^*.$$

Since $s\alpha_2 < 1$, letting $p \to \infty$ we get that

$$d(x_n, x_2^*) \leq \frac{s\alpha_2^n}{1 - s\alpha_2} d(x_0, x_1), \forall n \in \mathbb{N}.$$

Choosing $n = 0$ in the above relation, we obtain

$$d(x_1^*, x_2^*) \leq \frac{s}{1 - s\alpha_2} d(x_1^*, x_1) \leq \frac{sq}{1 - s\alpha_2} H(T_1(x_1^*), T_2(x_1^*)) \leq \frac{s\eta q}{1 - s\alpha_2}.$$

Interchanging the roles of T_1 and T_2 we obtain that for every $u \in Fix\,(T_2)$, there exists $v \in Fix\,(T_1)$ such that

$$d(u, v) \leq \frac{s\eta q}{1 - s\alpha_1},$$

where $\alpha_1 = q\frac{a_1+b_1+c_1 s}{1-b_1-c_1 s} < \frac{1}{s}$.
Thus, letting $q \searrow 1$, we obtain the conclusion. □

3. Well-Posedness of the Fixed Point Problem

In what follows we shall prove a well-posedness results with respect to the functional D.

Definition 3.1. *Let* (X, d) *be a b-metric space with constant* $s \geq 1$ *and* $T : X \to P(X)$ *be a multivalued operator. By definition, the fixed point problem is well-posed for* T *with respect to* D *if:*

(i) $Fix\,(T) = \{x^*\}$;

(ii) If $(x_n)_{n\in\mathbb{N}}$ is a sequence in X such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$, then $x_n \xrightarrow{d} x^*$, as $n \to \infty$.

Theorem 3.2. *Let (X, d) be a complete b-metric space with constant $s > 1$ and $T : X \to P(X)$ a multivalued operator for which there exist $a, b, c \in \mathbb{R}_+, a + b + 2cs < \frac{s-1}{s^2}$ and $b + cs < \frac{1}{s}$ such that*

$$H(T(x), T(y)) \le ad(x, y) + b\left[D(x, T(x)) + D(y, T(y))\right] + c\left[D\left(x, T\left(y\right)\right) + D\left(y, T\left(x\right)\right)\right],$$

for all $x, y \in X$.

If $SFix(T) \neq \emptyset$, then the fixed point problem is well-posed for T with respect to D.

Proof. Let $x \in SFix\left(T\right)$ and let $(x_n)_{n\in\mathbb{N}}$ such that $D(x_n, T(x_n)) \to 0$, as $n \to \infty$.

We have:

$$\begin{aligned} d\left(x_n, x\right) &\le s\left[D\left(x_n, T\left(x_n\right)\right) + H\left(T\left(x_n\right), T\left(x\right)\right)\right] \le sD\left(x_n, T\left(x_n\right)\right) + asd\left(x_n, x\right) + bsD\left(x_n, T\left(x_n\right)\right) \\ &\quad + bsD\left(x, T\left(x\right)\right) + csD\left(x_n, T\left(x\right)\right) + csD\left(x, T\left(x_n\right)\right) \\ d\left(x_n, x\right) &\le sD\left(x_n, T\left(x_n\right)\right) + asd\left(x_n, x\right) + bsD\left(x_n, T\left(x_n\right)\right) + \\ &\quad + cs^2 d\left(x_n, x\right) + cs^2 D\left(x, T\left(x\right)\right) + cs^2 d\left(x_n, x\right) + cs^2 D\left(x_n, T\left(x_n\right)\right) \end{aligned}$$

$$\left(1 - as - 2cs^2\right) d\left(x_n, x\right) \le s\left(1 + b + cs\right) D\left(x_n, T\left(x_n\right)\right).$$

$a + 2cs < a + b + 2cs < \frac{s-1}{s^2} < \frac{1}{s}$ and hence $1 - as - 2cs^2 > 0$.

Thus, we have

$$d\left(x_n, x\right) \le s\frac{1 + b + cs}{1 - as - 2cs^2} D\left(x_n, T\left(x_n\right)\right).$$

Letting $n \to \infty$, we shall obtain that $x_n \xrightarrow{d} x$. □

4. Ulam-Hyers Stability

Definition 4.1. *Let (X, d) be a b-metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. The fixed point inclusion*

$$x \in T(x),\ x \in X \tag{1}$$

is called generalized Ulam-Hyers stable if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and with $\psi(0) = 0$, such that for each $\varepsilon > 0$ and for each solution $y^* \in X$ of the inequation

$$D(y, T(y)) \le \varepsilon \tag{2}$$

there exists a solution x^* of the fixed point inclusion (4.1) such that

$$d(y^*, x^*) \le \psi(\varepsilon).$$

If there exists $C > 0$ such that $\psi(t) := C \cdot t$, for each $t \in \mathbb{R}_+$, then the fixed point inclusion (4.1) is said to be Ulam-Hyers stable.

Theorem 4.2. *Let (X, d) be a complete b-metric space with constant $s > 1$and $T : X \to P(X)$ a multivalued operator such that:*

(i) there exist $a, b, c \in \mathbb{R}_+, a + b + 2cs < \frac{s-1}{s^2}$ and $b + cs < \frac{1}{s}$ such that

$$H(T(x), T(y)) \le ad(x, y) + b\left[D(x, T(x)) + D(y, T(y))\right] + c\left[D\left(x, T\left(y\right)\right) + D\left(y, T\left(x\right)\right)\right],$$

for all $x, y \in X$;

(ii) T is closed;

If $SFix(T) \neq \emptyset$, then the fixed point inclusion (4.1) *is generalized Ulam-Hyers stable.*

Proof. We are in the conditions of Theorem 2.1. and Theorem 2.2, hence $Fix(T) = SFix(T) = \{x^*\}$. Let $\varepsilon > 0$ and y^* be a solution of (4.2).

We have

$$\begin{aligned} d(x^*, y^*) &= D(T(x^*), y^*) \leq sH(T(x^*), T(y^*)) + sD(y^*, T(y^*)) \\ &\leq sad(x^*, y^*) + sbD(x^*, T(x^*)) + sbD(y^*, T(y^*)) + \\ &\quad + scD(x^*, T(y^*)) + scD(y^*, T(x^*)) + sD(y^*, T(y^*)) \\ &\leq sad(x^*, y^*) + sbD(y^*, T(y^*)) + s^2cd(x^*, y^*) + \\ &\quad + s^2cD(y^*, T(y^*)) + scd(x^*, y^*) + sD(y^*, T(y^*)). \end{aligned}$$

Thus

$$\left(1 - as - cs - cs^2\right) d(x^*, y^*) \leq s(1 + b + cs) D(y^*, T(y^*)).$$

We have that $a + (s+1)c < a + 2cs < a + b + 2cs < \frac{s-1}{s^2} < \frac{1}{s}$, and hence $as + cs + cs^2 < 1$ and now we conclude

$$d(x^*, y^*) \leq \frac{s(1 + b + cs)}{1 - as - cs - cs^2} \varepsilon.$$

Hence, the fixed point problem (4.1) is generalized Ulam-Hyers stable. □

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