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A Viscosity Iterative Algorithm for the Optimization Problem System

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Abstract. In this paper, we suggest and analysis a viscosity iterative algorithm for finding a common element of the set of solution of a mixed equilibrium problem and the set the of solutions of a variational inequality and all common fixed points of a nonexpansive semigroup. This algorithm strongly converges to an element which solves an optimization problem system. Finally, some examples and numerical results are also given.

1. Introduction

Throughout this paper, we always assume that H is a Hilbert space and C is a nonempty, closed convex subset of H. Let $T : C \to C$ be a mapping. The fixed points set of T is denoted by $F(T)$, that is

$$F(T) = \{x \in C : x = Tx\}.$$

The mapping T is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. Also, a map $f : C \to C$ is a $\lambda-$contraction on C if there exist a constant $\lambda \in [0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \lambda\|x - y\|$. The strong(weak) convergence of $\{x_n\}$ to x is written by $x_n \to x$ ($x_n \rightharpoonup x$) as $n \to \infty$.
For any $x \in H$, there exists a unique nearest point of x in C, denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \text{ for all } y \in C.$$

P_C is called the metric projection of H onto C.
A family $S = \{T(s) : s \in [0, +\infty)\}$ of mappings of C into itself is called a nonexpansive semigroup on C, if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$;

(ii) $T(s + t) = T(s) \circ T(t)$ for all $s, t \geq 0$;

(iii) $\|T(s)x - T(s)y\| \leq \|x - y\|$ for all $x, y \in C$ and $s \geq 0$;

(iv) for all $x \in C$, $s \longmapsto T(s)x$ is continuous.

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The set of all common fixed points of S is denoted by $F(S)$, that is,

$$F(S) = \{x \in C : T(s)x = x, s \in [0, +\infty)\}.$$

Let $F : C \times C \to R$ be a bi-function and $\psi : C \to R \bigcup\{\infty\}$ be a proper extended real-valued function. The classical mixed equilibrium problem [1] is to find $x \in C$ such that

$$F(x, y) + \psi(y) \geq \psi(x), \text{ for all } y \in C. \tag{1}$$

The solutions set of (1) is denoted by $MEP(F, \psi)$. One can see if x is a solution of problem, then $x \in dom\psi = \{x \in C : \psi(x) < \infty\}$.
If $\psi \equiv 0$, then the mixed equilibrium problem (1) is reduces to the followings equilibrium problem [1]: finding $x \in C$ such that

$$F(x, y) \geq 0, \text{ for all } y \in C. \tag{2}$$

The solutions set of (2) is denoted by $EP(F)$.
Let $\phi : C \to H$ be a mapping. A variational inequality problem (denoted by $VI(C, \phi)$) is to find $x \in C$ such that

$$\langle \phi x, y - x\rangle \geq 0, \text{ for all } y \in C. \tag{3}$$

The map $G : C \to H$ is $\rho-$inverse strongly monotone, if there exists a positive real number $\rho > 0$ such that

$$\langle Gx - Gy, x - y\rangle \geq \rho||Gx - Gy||^2, \text{ for all } x, y \in C.$$

Recall that A is a strongly positive bounded linear operator on H , if there exists a constant $\eta > 0$ such that

$$\langle Ax, x\rangle \geq \eta||x||^2, \text{ for all } x \in H. \tag{4}$$

In 2009, Li et al. [13], motivated and inspired by Marino and Xu [15], introduced the following two iterative algorithms for the approximation of common fixed points of one parameter nonexpansive semigroup $\{T(s) : s \in [0, +\infty)\}$ on a nonempty closed convex subset C in a Hilbert space:

$$x_n = \alpha_n\gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \tag{5}$$

$$y_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)y_n ds, \tag{6}$$

where $A : C \to H$ is a linear bounded strongly positive operator and $f : H \to H$ is $\alpha-$cotraction, $\{\alpha_n\}$ and $\{s_n\}$ are sequences in $[0, 1)$ and $[0, +\infty)$, respectively .
In 2010, Cianciaruso et al. [4] introduced the following iterative method (by improving Plutieng and Punpaeng [18]), that include equilibrium and fixed points problems for nonexpansive semigroups $S = \{T(s)\}_{s\geq 0}$ on a Hilbert space H

$$\begin{cases} x_1 \in H \text{ chosen arbitrary,} \\ G(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \text{ for all } y \in H, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \text{ for all } n \geq 0 \end{cases} \} \tag{7}$$

where $A : C \to H$ is a linear bounded strongly positive operator and $f : H \to H$ is $\alpha-$contraction . They proved, the iterative algorithm $\{x_n\}$ which is defined by (7) strongly converges to a common element of $z \in F(S) \bigcap EP(F)$ and solved the variational inequality

$$\langle(\gamma f - A)z, p - z\rangle \leq 0, \text{ for all } p \in F(S) \bigcap EP(F).$$

Kang et al. [8] considerd an iterative algorithm $\{x_n\}$ in a Hilbert space as follows :

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds.$$

Under the certain condition, the sequence $\{x_n\}$ strongly converges to a unique solution of the variational inequality

$$\langle(\gamma f - A)x^*, x - x^*\rangle \leq 0, \forall x \in F(T).$$

By intuition from [2], we suggest and analysis an iterative algorithm for finding a common element of the set of solution of a mixed equilibrium problem and the set of solutions of a variational inequality and all common fixed points of a nonexpansive semigroup in the framework of a Hilbert space.

2. Preliminaris

Let G be a monotone mapping of C into H. We have :

$$x \in VI(C, G) \Longleftrightarrow x = P_C(x - \lambda Gx),\ \lambda > 0.$$

A set valued mapping $Q : H \to 2^H$ is called monotone, if for all $x, y \in H$, $f \in Qx$ and $g \in Qy$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $Q : H \to 2^H$ is maximal if the graph of Q (denoted by $Graph(Q)$) is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping Q is maximal if and only if for $(x, f) \in H\times H, \langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in Graph(Q)$ implies that $f \in Qx$. Let $N_C v$ be the normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle x - v, w\rangle \leq 0, \forall x \in C\}$ and define

$$Qx = \begin{cases} Gv + N_C v & \text{if } v \in C \\ \emptyset & \text{if } v \notin C \end{cases} \tag{8}$$

then Q is the maximal monotone and $0 \in Qv$ if and only if $v \in VI(C, G)$ [19].
Suppose $\psi : C \to \mathbb{R}\bigcup\{+\infty\}$ is a real-valued function. To solve the mixed equilibrium problem for a bi-function $F : C \times C \to \mathbb{R}$, let us assume the followings:

(A_1) $F(x, x) = 0$ for all $x \in C$;

(A_2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A_3) For each $x, y, z \in C$, $\lim_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x, y)$;

(A_4) For each fixed $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous;

(A_5) For each fixed $y \in C$, $x \mapsto F(x, y)$ is weakly upper semicontinuous;

(B_1) For each $x \in C$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for each $z \in C\backslash D_x$, $F(z, y_x) + \psi(y_x) - \psi(z) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0$,

(B_2) C is a bounded set .

Lemma 2.1. *([16]) Let C be a nonempty closed convex subset of a Hilbert space H and $\{T(s)\}_{s\geq 0}$ be a nonexpansive semigroup on H . Then, for every $h \geq 0$*

$$\lim_{t\to\infty} sup_{x\in C}\left\|\frac{1}{t}\int_0^t T(s)xds - T(h)\frac{1}{t}\int_0^t T(s)xds\right\| = 0.$$

Lemma 2.2. *([22]) Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that*

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)y_n, \quad n \geq 0,$$

where $\{\lambda_n\}$ is a sequence in $[0,1]$ such that

$$0 < \liminf_{t\to\infty} \lambda_n \leq \limsup_{t\to\infty} \lambda_n < 1.$$

Assume

$$\limsup_{t\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0,$$

then $\lim_{t\to\infty}\|y_n - x_n\| = 0$.

Lemma 2.3. *([26]) Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\gamma_n)s_n + \delta_n, \text{ for all } \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequences of real numbers such that

(i) $\lim_{t\to\infty}\gamma_n = 0$ *and* $\sum_{n=1}^{\infty}\gamma_n = \infty$*;*

(ii) $\limsup_{t\to\infty}\frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum|\delta_n| < \infty$*;*

then $\lim_{t\to\infty} s_n = 0$.

Lemma 2.4. *([17]) Let C be a nonempty closed convex subset of H .*
Let $F : C\times \to \mathbb{R}$ be a bi-function satisfies $(A1)-(A5)$ and $\psi : C \to \mathbb{R}\bigcup\{+\infty\}$ be a proper lower semicontinuous and convex function. Assume either $(B1)$ or $(B2)$ holds . For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows :

$$T_r(x) = \{z \in C : F(z,y) + \psi(y) + \frac{1}{r}\langle y-z, z-x\rangle \geq \psi(z), \forall y \in C\},$$

for all $x \in H$. Then the following hold :

(1) $T_r(x) \neq \emptyset$ *,* $\forall x \in H$ *.*

(2) T_r *is singel-valued .*

(3) T_r *is firmly nonexpansive , that is , for any* $x, y \in H$

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x-y\rangle.$$

(4) $F(T_r) = MEP(F,\psi)$*.*

(5) $MEP(F,\psi)$ *is closed and convex.*

3. Viscosity Iterative Algorithm

The viscosity method has been successfully applied to various problems coming from calculus of variations, minimal surface problems, plasticity theory and phase transition. It plays a central role in the study of degenerated elliptic and parabolic second order equations [10], [12], [14]. First abstract formulation of the properties of the viscosity approximation have been given by Tykhonov [23] in 1963 when studying ill-posed problems (see [5] for details). The concept of viscosity solution for Hamilton-Jacobi equations, which plays a crucial role in control theory, game theory and partial differential equations has been introduced by Crandall and Lions [3]. Recently, the viscosity iterative algorithm have received rapid development, see, for example, [6], [9], [11], [21], [20], [24] and [25]. In this section, we introduce a viscosity iterative algorithma for finding a common element of the set of solution for an equilibrium problem (involving a bi-function defined on a closed convex subset) and the set of fixed points of a nonexpansive semigroup.

Theorem 3.1. *Let*

- *H be a real Hilbert space, C be a nonempty closed convex subset of H,*
- *$F_1, F_2, \ldots, F_k$ be bi-functions from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A5)$,*
- *$\psi_1, \psi_2, \ldots, \psi_k$ be proper lower semicontinuous and convex functions form C to $\mathbb{R}\bigcup\{\infty\}$,*
- *$f : C \to C$ be a $\lambda-$contraction,*
- *$F(S) = \{T(s) : s \in [0, +\infty)\}$ is a nonexpansive semigroup on C,*
- *$G : C \to H$ be a $\rho-$inverse strongly monotone map,*
- *A be a strongly positive linear bounded operator on H with coefficient $\eta > 0$ and $0 < \gamma < \frac{\eta}{\lambda}$,*
- *the conditions $(B1)$ or $(B2)$ holds .*
- *$F(S) \bigcap_{i=1}^{k} MEP(F_i, \psi_i) \bigcap VI(C, G) \neq \emptyset$,*
- *$\{x_n\}$ be a sequence generated by $x_1 \in C, u_n^{(i)} \in C$ for all $i \in \{1, 2, \cdots, k\}$ in the following manner:*

$$\begin{cases} x_1 \in C, \\ F_1(u_n^{(1)}, y) + \psi_1(y) - \psi_1(u_n^{(1)}) + \frac{1}{r_n}\langle y - u_n^{(1)}, u_n^{(1)} - x_n\rangle \geq 0, \text{ for all } y \in C, \\ F_2(u_n^{(2)}, y) + \psi_2(y) - \psi_2(u_n^{(2)}) + \frac{1}{r_n}\langle y - u_n^{(2)}, u_n^{(2)} - x_n\rangle \geq 0, \text{ for all } y \in C, \\ \vdots \\ F_k(u_n^{(k)}, y) + \psi_k(y) - \psi_k(u_n^{(k)}) + \frac{1}{r_n}\langle y - u_n^{(k)}, u_n^{(k)} - x_n\rangle \geq 0, \text{ for all } y \in C, \\ \omega_n = \frac{u_n^{(1)} + u_n^{(2)} \cdots u_n^{(k)}}{k}, \\ z_n = P_C(\omega_n - t_n G\omega_n), \\ y_n = \lambda_n\omega_n + (1-\lambda_n)z_n, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)y_n ds, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$ are the sequences in $(0, 1)$ and $\{r_n\} \subset (0, \infty)$, $\{t_n\} \subset (0, 2\rho)$ are a real sequence.

Suppose

(C1) $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty,$

(C2) $\lim_{n\to\infty} \lambda_n = 0,$

(C3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1,$

(C4) $\liminf_{n\to\infty} r_n > 0$ *, and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0,$

(C5) $\{t_n\} \subset [a, b], for a, b \in (0, 2\rho)$ *and* $\lim_{n\to\infty} |t_{n+1} - t_n| = 0,$

(C6) $\lim_{n\to\infty} s_n = \infty$ *, and* $\limsup_{n\to\infty} \|s_{n+1} - s_n\|$ *is finite.*

Then

(i) *the sequence $\{x_n\}$ is bounded,*

(ii) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0,$

(iii) $\lim_{n\to\infty} \|\omega_n - x_n\| = 0$,

(iv) $\lim_{n\to\infty} \|G\omega_n - Gp\| = 0, \ \ p \in MEP(F_i, \psi_i) \bigcap VI(C, G)$

(v) $\lim_{n\to\infty} \|x_n - \frac{1}{s_n}\int_0^{s_n} T(S)y_n ds\| = 0$ *and* $\lim_{n\to\infty} \|y_n - \frac{1}{s_n}\int_0^{s_n} T(S)y_n ds\| = 0$.

Proof. By the same argument in [7],

$$\|(1-\beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n\eta.$$

For any $x, y \in C$, it follows that

$$\begin{aligned} \|(I - t_nG)(x) - (I - t_nG)(y)\|^2 &= \|(x-y) - t_n(Gx - Gy)\|^2 \\ &\leq \|x-y\|^2 - 2t_n\rho - 2t_n\rho\|Gx - Gy\|^2 + t_n^2\|Gx - Gy\|^2\| \\ &= \|x-y\|^2 + t_n(t_n - 2\rho)\|Gx - Gy\|^2. \end{aligned} \tag{9}$$

Therefore, $I - t_nG$ is nonexpansive . Since $t_n \in (0, 2\rho)$, G is a $\rho-$inverse strongly monotone map, we see that

$$\|(I - t_n)Gx - (I - t_n)Gy|^2 \leq \|x - y\|^2.$$

Let $p \in F(S)\bigcap_{i=1}^{k} MEP(F_i, \psi_i) \bigcap VI(C, G)$. Notice that, for all $n \geq 1$, $1 \leq i \leq k$,$u_n^{(i)}$ can be re-written as $u_n^{(i)} = T_{r_n^{(i)}} x_n$, then

$$\|u_n^{(i)} - p\| = \|T_{r_n^{(i)}} x_n - T_{r_n^{(i)}} p\| \leq \|x_n - p\|, \tag{10}$$

hence

$$\|\omega_n - p\| = \|\frac{1}{k}\sum_{i=1}^{k} u_n^{(i)} - p\| \leq \|x_n - p\|. \tag{11}$$

From the fact that P_C and $I - t_nG$ are nonexpansive and $p = P_C(p - t_nGp)$, we obtain

$$\begin{aligned} \|z_n - p\| &= \|P_C(\omega_n - t_nG\omega_n) - P_C(p - t_nGp)\| \\ &\leq \|(I - t_nG)\omega_n - (I - t_nG)p\| \\ &\leq \|\omega_n - p\|. \end{aligned} \tag{12}$$

By (11) and (12), one has

$$\begin{aligned} \|y_n - p\| &= \|\lambda_n\omega_n + (1-\lambda_n)z_n - p\| \\ &\leq \lambda_n\|\omega_n - p\| + (1-\lambda_n)\|z_n - p\| \\ &\leq \|\omega_n - p\| \\ &\leq \|x_n - p\|. \end{aligned} \tag{13}$$

(**i**) : We have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\frac{1}{s_n}\int_0^{s_n} T(s)y_n ds - p\| \\ &\leq \alpha_n\|\gamma f(x_n) - Ap\| + \beta_n\|x_n - p\| + \|((1-\beta_n)I - \alpha_n A\|\|\frac{1}{s_n}\int_0^{s_n} T(s)y_n ds - p\| \\ &\leq \alpha_n\{\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Ap\|\} + \beta_n\|x_n - p\| + (1-\beta_n - \alpha_n\eta)\frac{1}{s_n}\int_0^{s_n} \|T(s)y_n ds - p\| \\ &\leq \alpha_n\lambda\gamma\|x_n - p\| + \alpha_n\|\gamma f(p) - \gamma Ap\| + \beta_n\|x_n - p\| + (1-\beta_n - \alpha_n\eta)\|y_n - p\| \\ &\leq \alpha_n\lambda\gamma\|x_n - p\| + \alpha_n\|\gamma f(p) - \gamma Ap\| + \beta_n\|x_n - p\| + (1-\beta_n - \alpha_n\eta)\|x_n - p\| \\ &= \{1 - (\alpha_n(\eta - \gamma\lambda))\}\|x_n - p\| + \alpha_n\|\gamma f(p) - \gamma Ap\| \\ &\leq max\{\|x_n - p\|, \frac{\|\gamma f(p) - Ap\|}{\eta - \gamma\lambda}\}. \end{aligned}$$

By a simple inductive process,

$$\|x_n - p\| \leq max\{\|x_1 - p\|, \frac{\|\gamma f(p) - Ap\|}{\eta - \gamma\lambda}\}. \tag{14}$$

This implies that the sequence $\{x_n\}$ is bounded. Also , $\{f(x_n)\}, \{y_n\}$ and $\{\frac{1}{s_n}\int_0^{s_n} T(s)y_n ds\}$ are bounded.

(ii) : For $i \in \{1, 2, \cdots, k\}$, $u_n^{(i)} = T_{r_{n,i}} x_n \in dom\psi_i$ and $u_{n+1}^{(i)} = T_{r_{n+1,i}} x_{n+1} \in dom\psi_i$. Also , for all $y \in C$

$$F_i(u_n^{(i)}, y) + \psi_i(y) - \psi_i(u_n^{(i)}) + \frac{1}{r_n}\langle y - u_n^{(i)}, u_n^{(i)} - x_n\rangle \geq 0, \tag{15}$$

$$F_i(u_{n+1}^{(i)}, y) + \psi_i(y) - \psi_i(u_{n+1}^{(i)}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}^{(i)}, u_{n+1}^{(i)} - x_{n+1}\rangle \geq 0. \tag{16}$$

In view of $(A2)$ and take $y = u_{n+1}^{(i)}$ in (15) and $y = u_n^{(i)}$ in (16), we obtain

$$\langle u_{n+1}^{(i)} - u_n^{(i)}, \frac{u_n^{(i)} - x_n}{r_n} - \frac{u_{n+1}^{(i)} - x_{n+1}}{r_{n+1}}\rangle \geq 0. \tag{17}$$

It follows that

$$\langle u_{n+1}^{(i)} - u_n^{(i)}, u_n^{(i)} - u_{n+1}^{(i)} + x_{n+1} - x_n + (1 - \frac{r_n}{r_{n+1}})(u_{n+1}^{(i)} - x_{n+1})\rangle \geq 0. \tag{18}$$

Since $\liminf_{n\to\infty} r_n > 0$, without loss of generality, there exists $\alpha > 0$ such that $r_n > \alpha$ for all $n \geq 1$. Hence

$$\|u_{n+1}^{(i)} - u_n^{(i)}\|^2 \leq \|u_{n+1}^{(i)} - u_n^{(i)}\|\{\|x_{n+1} - x_n\| + |1 - \frac{r_n}{r_{n+1}}|\|u_{n+1}^{(i)} - x_{n+1}\|\}, \tag{19}$$

so

$$\|u_{n+1}^{(i)} - u_n^{(i)}\| \leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|M_i, \tag{20}$$

where $M_i = max\{\|u_n^{(i)} - x_n\|, n \in \mathbb{N}\}$.
Therefore, we obtain

$$\|\omega_{n+1} - \omega_n\| \leq \frac{1}{k}\sum_{i=1}^{k}\|u_{n+1}^{(i)} - u_n^{(i)}\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n|, \tag{21}$$

where $M = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{\alpha}M_i$.
From which it follows that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|P_C(\omega_{n+1} - t_{n+1}G\omega_{n+1}) - P_C(\omega_n - t_nG\omega_n)\| \\
&\leq \|(\omega_{n+1} - t_{n+1}G\omega_{n+1}) - (\omega_n - t_{n+1}G\omega_n) + (\omega_n - t_{n+1}G\omega_n) - (\omega_n - t_nG\omega_n)\| \\
&\leq \|\omega_{n+1} - \omega_n\| + |t_{n+1} - t_n|\|G\omega_n\|.
\end{aligned}$$

In view of (21), we obtain that

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + M|r_{n+1} - r_n| + |t_{n+1} - t_n|\|G\omega_n\|. \tag{22}$$

It follows

$$\begin{aligned}\|y_{n+1}-y_n\| &= \|\lambda_{n+1}\omega_{n+1}+(1-\lambda_{n+1})z_{n+1}-\lambda_n\omega_n-(1-\lambda_n)z_n\|\\ &= \|\lambda_{n+1}(\omega_{n+1}-\omega_n)+(\lambda_{n+1}-\lambda_n)\omega_n+(1-\lambda_{n+1})z_{n+1}-(1-\lambda_{n+1})z_n+(1-\lambda_{n+1})z_n-(1-\lambda_n)z_n\|\\ &= \|\lambda_{n+1}(\omega_{n+1}-\omega_n)+(\lambda_{n+1}-\lambda_n)\omega_n+(1-\lambda_{n+1})(z_{n+1}-z_n)+(\lambda_n-\lambda_{n+1})z_n\|\\ &\le \lambda_{n+1}\|\omega_{n+1}-\omega_n\|+|\lambda_{n+1}-\lambda_n|(\|\omega_n\|+\|z_n\|)+(1-\lambda_{n+1})\|z_{n+1}-z_n\|\\ &\le \lambda_{n+1}\|\omega_{n+1}-\omega_n\|+|\lambda_{n+1}-\lambda_n|(\|\omega_n\|+\|z_n\|)+(1-\lambda_{n+1})\{\|x_{n+1}-x_n\|+M|r_{n+1}-r_n|+|t_{n+1}-t_n|\|G\omega_n\|\},\end{aligned}$$

from which it follows that

$$\|y_{n+1}-y_n\|\le\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|(\|\omega_n\|+\|z_n\|)+M|r_{n+1}-r_n|+|t_{n+1}-t_n|\|G\omega_n\|. \tag{23}$$

On the other hand

$$\begin{aligned}&\|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}T(s)y_{n+1}ds-\tfrac{1}{s_n}\int_0^{s_n}T(s)y_nds\|\\ =\;&\|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}[T(s)y_{n+1}-T(s)y_n]ds+(\tfrac{1}{s_{n+1}}-\tfrac{1}{s_n})\int_0^{s_n}[T(s)y_n-T(s)p]ds+\tfrac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}[T(s)y_n-T(s)p]ds\|\\ \le\;&\|y_{n+1}-y_n\|+\tfrac{2|s_{n+1}-s_n|}{s_{n+1}}\|y_n-p\|.\end{aligned}$$

Thus, we obtian from (23)

$$\begin{aligned}&\|\tfrac{1}{s_{n+1}}\int_0^{s_{n+1}}T(s)y_{n+1}ds-\tfrac{1}{s_n}\int_0^{s_n}T(s)y_nds\|\\ &\le\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|(\|\omega_n\|+\|z_n\|)+M|r_{n+1}-r_n|+|t_{n+1}-t_n|\|G\omega_n\|+\tfrac{2|s_{n+1}-s_n|}{s_{n+1}}\|y_n-p\|.\end{aligned} \tag{24}$$

Now, suppose $\Sigma_n=\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)\Lambda_n}{1-\beta_n}$, where $\Lambda_n=\frac{1}{s_n}\int_0^{s_n}T(s)y_nds$. It follows from (24)

$$\begin{aligned}\|\Sigma_{n+1}-\Sigma_n\| &= \|\frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}A)\Lambda_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_nA)\Lambda_n}{1-\beta_n}\|\\ &= \|\frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}}+\frac{(1-\beta_{n+1})\Lambda_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_{n+1}A\Lambda_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_n\gamma f(x_n)}{1-\beta_n}-\frac{(1-\beta_n)\Lambda_n}{1-\beta_n}+\frac{\alpha_nA\Lambda_n}{1-\beta_n}\|\\ &= \|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})-A\Lambda_{n+1})+\frac{\alpha_n}{1-\beta_n}(A\Lambda_n-\gamma f(x_n))+(\Lambda_{n+1}-\Lambda_n)\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-A\Lambda_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|A\Lambda_n-\gamma f(x_n)\|+\|\Lambda_{n+1}-\Lambda_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1})-A\Lambda_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|A\Lambda_n-\gamma f(x_n)\|\\ &\quad+\|x_{n+1}-x_n\|+M|r_{n+1}-r_n|+\frac{2|s_{n+1}-s_n|}{s_{n+1}}\|y_n-p\|+|t_{n+1}-t_n|\|G\omega_n\|+|\lambda_{n+1}-\lambda_n|(\|\omega_n\|+\|z_n\|).\end{aligned}$$

Thanks to the conditions $(C1)$- $(C2)$ and $(C4)-(C6)$, we conclude that

$$\limsup_{n\to\infty}(\|\Sigma_{n+1}-\Sigma_n\|-\|x_{n+1}-x_n\|)\le 0.$$

By lemma 2.2, we arrive at

$$\lim_{n\to\infty}\|\Sigma_n-x_n\|=0. \tag{25}$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|\Sigma_n-x_n\|=0. \tag{26}$$

(**iii**) : For any $i = 1, 2, \cdots, k$, we have

$$\begin{aligned} \|u_n^{(i)} - p\|^2 &= \|T_{r_n^{(i)}} x_n - T_{r_n^{(i)}} p\|^2 \\ &\leq \langle T_{r_n^{(i)}} x_n - T_{r_n^{(i)}} p, x_n - p\rangle \\ &= \frac{1}{2}(\|u_n^{(i)} - p\|^2 + \|x_n - p\|^2 - \|u_n^{(i)} - p - x_n + p\|^2) \\ &= \frac{1}{2}(\|u_n^{(i)} - p\|^2 + \|x_n - p\|^2 - \|u_n^{(i)} - x_n\|^2), \\ &\leq \|x_n - p\|^2 - \|u_n^{(i)} - x_n\|^2. \end{aligned}$$

Hence

$$\begin{aligned} \|\omega_n - p\| = &\ \| \textstyle\sum_{i=1}^k \frac{1}{k}(\|u_n^{(i)} - p)\|^2 \\ \leq &\ \tfrac{1}{k} \textstyle\sum_{i=1}^k \|u_n^{(i)} - p\|^2 \\ \leq &\ \|x_n - p\|^2 - \tfrac{1}{k} \textstyle\sum_{i=1}^k \|u_n^{(i)} - x_n\|^2. \end{aligned} \tag{27}$$

In viwe of (27) and (13), we obtain that

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)\Lambda_n - p\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - Ap) + \beta_n(x_n - p) + ((1 - \beta_n)I - \alpha_n A)(\Lambda_n - p)\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\eta)\|\Lambda_n - p\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\eta)\|y_n - p\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\eta)\|\omega_n - p\|^2 \\ &\leq \alpha_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 + (1 - \beta_n - \alpha_n\eta)(\|x_n - p\|^2 - \frac{1}{k}\sum_{i=1}^{k} \|u_n^{(i)} - x_n\|^2) \\ &\leq \alpha_n\|\gamma f(x_n) - Ap\|^2 + \|x_n - p\|^2 - (1 - \beta_n - \alpha_n\eta)\frac{1}{k}\sum_{i=1}^{k} \|u_n^{(i)} - x_n\|^2. \end{aligned}$$

It follows that

$$\begin{aligned} (1 - \beta_n - \alpha_n\eta)\frac{1}{k}\sum_{i=1}^{k} \|u_n^{(i)} - x_n\|^2 &\leq \|x_n - p\|^2 + \alpha_n\|\gamma f(x_n) - Ap\|^2 - \|x_{n+1} - p\|^2 \\ &= \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) + \alpha_n\|\gamma f(x_n) - Ap\|^2. \end{aligned}$$

Thanks to the condition ($C1$) and (26), we conclude that

$$\lim_{n\to\infty} \|u_n^{(i)} - x_n\| = 0, \tag{28}$$

also

$$\lim_{n\to\infty} \|\omega_n - x_n\| = 0. \tag{29}$$

(**iv**) : Let $z_n^* = P_C(p - t_n Gp)$ it follows from (9) that

$$\begin{aligned} \|z_n - z_n^*\|^2 &= \|P_C(\omega_n - t_n G\omega_n) - P_C(p - t_n Gp)\|^2 \\ &\leq \|(\omega_n - t_n G\omega_n) - (p - t_n Gp)\|^2 \\ &= \|(\omega_n - p) - t_n(G\omega_n - Gp)\|^2 \\ &\leq \|x_n - p\|^2 + t_n(t_n - 2\rho)\|G\omega_n - Gp\|^2. \end{aligned}$$

Observe that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)\|\Lambda_n-p\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)\|y_n-p\|^2\\
&= \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)\|\{\lambda_n(\omega_n-p)+(1-\lambda_n)(z_n-z_n^*)\|^2\}\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)\lambda_n\|\omega_n-p\|^2\\
&\quad +(1-\beta_n-\alpha_n\eta)(1-\lambda_n)\|z_n-z_n^*\|^2\\
&= \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)\lambda_n\|\omega_n-p\|^2\\
&\quad +(1-\beta_n-\alpha_n\eta)(1-\lambda_n)\|(\omega_n-p)-t_n(G\omega_n-Gp))\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)\lambda_n\|\omega_n-p\|^2\\
&\quad +(1-\beta_n-\alpha_n\eta)(1-\lambda_n)\{\|x_n-p\|^2+t_n(t_n-2\rho)\|G\omega_n-Gp\|^2\}\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+(1-\alpha_n\eta)\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)(1-\lambda_n)t_n(t_n-2\rho)\|G\omega_n-Gp\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)(1-\lambda_n)a(b-2\rho)\|G\omega_n-Gp\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
0 &\leq (1-\beta_n-\alpha_n\eta)(1-\lambda_n)a(2\rho-b)\|G\omega_n-Gp\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|).
\end{aligned}$$

The condition $(C1)$ and (26) imply

$$\lim_{n\to\infty}\|G\omega_n-Gp\|=0. \tag{30}$$

(**v**) : Notice that

$$\begin{aligned}
\|z_n-p\|^2 &= \|z_n-z_n^*\|^2\\
&\leq \langle(\omega_n-t_nG\omega_n)-(p-t_nGp),z_n-p\rangle\\
&= \frac{1}{2}\{\|(\omega_n-t_nG\omega_n)-(p-t_nGp)\|^2+\|z_n-p\|^2\}-\frac{1}{2}\{\|(\omega_n-t_nG\omega_n)-(p-t_nGp)-(z_n-p)\|^2\}\\
&= \frac{1}{2}\{\|\omega_n-p\|^2+\|z_n-p\|^2-\|(\omega_n-z_n)-t_n(G\omega_n-Gp)\|^2\}\\
&= \frac{1}{2}\{\|\omega_n-p\|^2+\|z_n-p\|^2-(\|\omega_n-z_n\|^2+t_n^2\|G\omega_n-Gp\|^2-2t_n\langle\omega_n-z_n,G\omega_n-Gp\rangle)\}\\
&\leq \frac{1}{2}\{\|\omega_n-p\|^2+\|z_n-p\|^2-\|\omega_n-z_n\|^2-t_n^2\|G\omega_n-Gp\|^2+2t_n\langle\omega_n-z_n,G\omega_n-Gp\rangle)\}.
\end{aligned}$$

From which follows that

$$\|z_n-p\|^2\leq\|\omega_n-p\|^2-\|\omega_n-z_n\|^2-t_n^2\|G\omega_n-Gp\|^2+2t_n\langle\omega_n-z_n,G\omega_n-Gp\rangle.$$

On the other hand, by some manipulation,

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)\|y_n-p\|^2\\
&= \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+(1-\beta_n-\alpha_n\eta)\|\lambda_n\omega_n+(1-\lambda_n)z_n-p\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+\lambda_n(1-\beta_n-\alpha_n\eta)\|\omega_n-p\|^2\\
&\quad +(1-\lambda_n)(1-\beta_n-\alpha_n\eta)\|z_n-p\|^2\\
&\leq \alpha_n\|\gamma f(x_n)-Ap\|^2+\beta_n\|x_n-p\|^2+\lambda_n(1-\beta_n-\alpha_n\eta)\|\omega_n-p\|^2\\
&\quad +(1-\beta_n-\alpha_n\eta)\{\|\omega_n-p\|^2-\|z_n-\omega_n\|^2-t_n^2\|G\omega_n-Gp\|^2+2t_n\langle\omega_n-z_n,G\omega_n\rangle\}
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|\gamma f(x_n) - Ap\|^2 + \beta_n\|x_n - p\|^2 + \lambda_n(1-\beta_n-\alpha_n\eta)\|\omega_n - p\|^2 + (1-\beta_n-\alpha_n\eta)\|x_n \\
&\quad -p\|^2 - (1-\beta_n-\alpha_n\eta)\|\omega_n - z_n\|^2 - t_n^2(1-\beta_n-\alpha_n\eta)\|G\omega_n - Gp\|^2 + 2t_n(1-\beta_n-\alpha_n\eta)\langle\omega_n - z_n, G\omega_n - Gp\rangle\} \\
&\leq \alpha_n\|\gamma f(x_n) - Ap\|^2 + \|x_n - p\|^2 + \lambda_n(1-\beta_n-\alpha_n\eta)\|\omega_n - p\|^2 \\
&\quad -(1-\beta_n-\alpha_n\eta)\|\omega_n - z_n\|^2 - t_n^2(1-\beta_n-\alpha_n\eta)\|G\omega_n - Gp\|^2 + 2t_n(1-\beta_n-\alpha_n\eta)\|\omega_n - z_n\|\|G\omega_n - Gp\|,
\end{aligned}$$

then

$$\begin{aligned}
(1-\beta_n-\alpha_n\eta)\|\omega_n - z_n\|^2 &\leq \alpha_n\|\gamma f(x_n) - Ap\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \lambda_n(1-\beta_n-\alpha_n\eta)\|\omega_n - p\|^2 \\
&\quad -t_n^2(1-\beta_n-\alpha_n\eta)\|G\omega_n - Gp\|^2 + 2t_n(1-\beta_n-\alpha_n\eta)\|\omega_n - z_n\|\|G\omega_n - Gp\|.
\end{aligned}$$

By vitrue of (26) and (30) with conditions $(C1)-(C2)$, we arrive at

$$\lim_{n\to\infty}\|\omega_n - z_n\| = 0. \tag{31}$$

Since $\|y_n - z_n\| = \lambda_n\|\omega_n - z_n\|$ then $\lim_{n\to\infty}\|y_n - z_n\| = 0$. We observe that

$$\|y_n - \Lambda_n\| \leq \|\Lambda_n - x_n\| + \|x_n - \omega_n\| + \|\omega_n - z_n\| + \|z_n - y_n\|.$$

From which it follows that $\lim_{n\to\infty}\|y_n - \Lambda_n\| = 0$ or

$$\lim_{n\to\infty}\|y_n - \frac{1}{s_n}\int_0^{s_n} T(s)y_n ds\| = 0. \tag{32}$$

On the other hand, it follows from definition of $\{x_n\}$ that

$$\begin{aligned}
\|\Lambda_n - x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - \Lambda_n\| \\
&\leq \|x_{n+1} - x_n\| + \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\Lambda_n - \Lambda_n\| \\
&\leq \|x_{n+1} - x_n\| + \alpha_n\|\gamma f(x_n) - A\Lambda_n\| + \beta_n\|x_n - \Lambda_n\|,
\end{aligned}$$

That is,

$$\|\Lambda_n - x_n\| \leq \frac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\beta_n}\gamma f(x_n) - A\Lambda_n\|.$$

In view of $(C1)$ and (26), we see that

$$\lim_{n\to\infty}\|\Lambda_n - x_n\| = 0. \tag{33}$$

□

Theorem 3.2. *Suppose that all assumptions of Theorem 3.1 are hold. Then the sequence $\{x_n\}$ strongly converges to a point $\bar{x}$, where $\bar{x} \in F(S)\bigcap_{i=1}^k MEP(F_i, \psi_i)\bigcap VI(C, G)$ solves the variational inequality*

$$\langle (A - \gamma f)\bar{x}, x - \bar{x}\rangle \geq 0.$$

Proof. Let $\Gamma = \bigcap_{i=1}^k F(S)\bigcap MEP(F_i, \psi_i)\bigcap VI(C, G)$. Thus $P_\Gamma(I - A + \gamma f)$ is a contraction of H into itself . Since H is complete , then there exists a unique element $\bar{x} \in H$ such that

$$\bar{x} = P_\Gamma(I - A + \gamma f)(\bar{x}).$$

Next, we show

$$\limsup_{n\to\infty}\langle (A - \gamma f)\bar{x}, \bar{x} - \frac{1}{s_n}\int_0^{s_n} T(s)y_n ds\rangle \leq 0.$$

Let $\tilde{x} = P_\Gamma x_1$. Set

$$\Pi = \{\bar{y} \in H : \|\bar{y} - \tilde{x}\| \leq \|x_1 - \tilde{x}\| + \frac{\|\gamma f(\tilde{x}) - A\tilde{x}\|}{\eta - \gamma\lambda}\}.$$

It is clear, Π is nonempty closed bounded convex subset of H which is $T(s)$-invariant for each $s \in [0, \infty)$ and $\{x_n\} \subset \Pi$. We may assume $S = \{T(s) : s \in [0, \infty)\}$ is a nonexpansive semigroup on Π. It follows from Lemma 2.1 that

$$\limsup_{n\to\infty} \|\frac{1}{s_n} \int_0^{s_n} T(s)y_n ds - T(h)\frac{1}{s_n} \int_0^{s_n} T(s)y_n ds\| = 0. \tag{34}$$

Let $\Lambda_n = \frac{1}{s_n} \int_0^{s_n} T(s)y_n ds$, since $\{\Lambda_n\} \subset \Pi$ is bounded, there is a subsequence $\{\Lambda_{n_j}\}$ of $\{\Lambda_n\}$ such that

$$\limsup_{n\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_n\rangle = \lim_{j\to\infty} \langle (A - \gamma f)\bar{x}, \bar{x} - \Lambda_{n_j}\rangle. \tag{35}$$

As $\{\Lambda_{n_j}\}$ is also bounded , there exists a subsequence $\{\Lambda_{n_{j_l}}\}$ of $\{\Lambda_{n_j}\}$ such that $\Lambda_{n_{j_l}} \rightharpoonup \xi$. Without loss of generality, let $\Lambda_{n_j} \rightharpoonup \xi$. Now, we prove the following items :

(i) $\xi \in F(S) = \bigcap_{s\geq 1} F(T(s))$:

Assume $\xi \neq T(h)\xi$ for some $h \in [0, \infty)$. In view of (32) and Opial's condition, we obtain

$$\begin{aligned}
\liminf_{j\to\infty} \|\Lambda_{n_j} - \xi\| &< \liminf_{j\to\infty} \|\Lambda_{n_j} - T(h)\xi\| \\
&\leq \liminf_{j\to\infty} (\|\Lambda_{n_j} - T(h)\Lambda_{n_j}\| + \|T(h)\Lambda_{n_j} - T(h)\xi\|) \\
&\leq \liminf_{j\to\infty} \|\Lambda_{n_j} - \xi\|.
\end{aligned}$$

That is a contradiction . Hence $\xi = T(h)\xi$, i.e., $\xi \in F(S) = \bigcap_{s\geq 1} F(T(s))$

(ii) $\xi \in \bigcap_{i=1}^k MEP(F_i, \psi_i)$.

For all $1 \leq i \leq k$, we see that

$$F_i(u_n^{(i)}, y) + \psi_i(y) - \psi_i(u_n^{(i)}) + \frac{1}{r_n}\langle y - u_n^{(i)}, u_n^{(i)} - x_n\rangle \geq 0, \text{ for all } y \in C.$$

It follows from $(A2)$ that

$$\psi_i(y) - \psi_i(u_n^{(i)}) + \langle y - u_n^{(i)}, \frac{u_{n_j}^{(i)} - x_{n_j}}{r_{n_j}}\rangle \geq F_i(u_{n_j}^{(i)}, y), \text{ for all } y \in C.$$

Thanks to the condition $(C4)$ and (28), we conclude that

$$\liminf_{n\to\infty} \|\frac{x_n - u_n^i}{r_n}\| = \liminf_{n\to\infty} \frac{1}{r_n}\|x_n - u_n^i\| = 0. \tag{36}$$

In view of (30) and (28), we obtain $u_{n_j}^{(i)} \to \xi$.

Since $\psi_i, 1 \leq i \leq k$ are weakly lower semicontinus, it follows from $(A4)$ and (36) that

$$F_i(y, \xi) + \psi_i(\xi) - \psi_i(y) \leq 0, \text{ for all } y \in C.$$

Let $x_r = rx + (1 - r)\xi$ such that $0 < r \leq 1$ and $x \in C$. It is clear that $x_r \in C$, thus

$$F_i(x_r, \xi) + \psi_i(\xi) - \psi_i(x_r) \leq 0.$$

In view of $(A1)$-$(A4)$ and convexity of ψ_i, we obtain

$$\begin{aligned} 0 &= F_i(x_r, x_r) + \psi_i(x_r) - \psi_i(x_r) \\ &\leq rF_i(x_r, x) + (1-r)F_i(x_r, \xi) + r\psi_i(x) + (1-r)\psi_i(x_r) \\ &= r(F_i(x_r, x) + \psi_i(x) - \psi_i(x_r)) + (1-r)(F_i(x_r, \xi) + \psi_i(\xi) - \psi_i(x_r)) \\ &\leq r(F_i(x_r, x) + \psi_i(x) - \psi_i(x_r)). \end{aligned}$$

From which it follows

$$F_i(x_r, x) + \psi_i(x) - \psi_i(x_r) \geq 0.$$

By virtue of weakly lower semi-continuity of ψ_i and $(A3)$, we get

$$F_i(\xi, y) + \psi_i(y) - \psi_i(\xi) \geq 0, \text{ for all } y \in C.$$

This shows that $\xi \in \bigcap_{i=1}^k MEP(F_i, \psi_i)$.

(iii) $\xi \in VI(C, G)$:
Let N_C be the normal cone of C at $v \in C$. We define a set-valued $\Phi : H \to 2^H$ as follows:

$$\Phi v = \begin{cases} Gv + N_C v, & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C \end{cases}$$

We knowe Φ is maximal monotone and $0 \in \Phi_v$ if and only if $v \in VI(C, G)$. Let $(v, w) \in Graph(\Phi)$. Since $w - Gv \in N_C v$ and $z_n \in C$, we have

$$\langle v - z_n, w - Gv\rangle.$$

Hence

$$\langle v - z_n, z_n - (\omega_n - t_n G\omega_n)\rangle \geq 0,$$

that is

$$\langle v - z_n, \frac{z_n - \omega_n}{t_n} + G\omega_n\rangle \geq 0.$$

Moreover,

$$\begin{aligned} \langle v - z_{n_j}, w\rangle &\geq \langle v - z_{n_j}, Gv\rangle \\ &\geq \langle v - z_{n_j}, Gv\rangle - \langle v - z_{n_j}, \frac{z_{n_j} - \omega_{n_j}}{t_{n_j}} + G\omega_{n_j}\rangle \\ &= \langle v - z_{n_j}, Gv - \frac{z_{n_j} - \omega_{n_j}}{t_{n_j}} - G\omega_{n_j}\rangle \\ &= \langle v - z_{n_j}, Gv - Gz_{n_j}\rangle + \langle v - z_{n_j}, Gz_{n_j} - G\omega_{n_j}\rangle - \langle v - z_{n_j}, \frac{z_{n_j} - \omega_{n_j}}{t_{n_j}}\rangle \\ &\geq \langle v - z_{n_j}, Gz_{n_j} - G\omega_{n_j}\rangle - \langle v - z_{n_j}, \frac{z_{n_j} - \omega_{n_j}}{t_{n_j}}\rangle \\ &\geq \|v - z_{n_j}\|\|Gz_{n_j} - G\omega_{n_j}\| - \|v - z_{n_j}\|\|\frac{z_{n_j} - \omega_{n_j}}{t_{n_j}}\|. \end{aligned}$$

Since G is $\rho-$inverse strongly monotone and $\lim_{n\to\infty}\|z_{n_j} - \omega_{n_j}\| = 0$, then $\langle v - \xi, w\rangle \geq 0$. Also Φ is maximal monotone, then $\xi \in \Phi^{-1}0$. Thus $\xi \in VI(C, G)$.
From which it follows that $\xi \in F(S) \bigcap_{i=1}^k MEP(F_i, \Psi_i) \bigcap VI(C, G)$.

Since $\xi = P_{\Gamma}(I - A + \gamma f)(\xi)$, we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle (A-\gamma f)\bar{x}, \bar{x} - \frac{1}{s_n}\int_0^{s_n} T(s)y_n ds\rangle \\ &= \limsup_{n\to\infty}\langle (A-\gamma f)\bar{x}, \bar{x} - \Lambda_{n_j}\rangle \\ &= \langle (A-\gamma f)\bar{x}, \bar{x} - \xi\rangle \leq 0.\end{aligned} \tag{37}$$

Finally, we prove that the sequence $\{x_n\}$ strongly converges to $\bar{x}$. We have

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)\Lambda_n - \bar{x}\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - A\bar{x}) + \beta_n(x_n - \bar{x}) + ((1-\beta_n)I - \alpha_n A)(\Lambda_n - \bar{x})\|^2 \\
&= \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 + \|\beta_n(x_n - \bar{x}) + ((1-\beta_n)I - \alpha_n A)(\Lambda_n - \bar{x})\|^2 \\
&\quad + 2\langle \beta_n(x_n - \bar{x}) + ((1-\beta_n)I - \alpha_n A)(\Lambda_n - \bar{x}), \alpha_n(\gamma f(x_n) - A\bar{x})\rangle \\
&\leq \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 + \{\beta_n\|x_n - \bar{x}\| + (1-\beta_n-\alpha_n\eta)\|y_n - \bar{x}\|\}^2 \\
&\quad + 2\alpha_n\beta_n\langle x_n - \bar{x}, \gamma f(x_n) - A\bar{x}\rangle + 2\alpha_n(1-\beta_n-\alpha_n\eta)\langle \Lambda_n - \bar{x}, \gamma f(x_n) - A\bar{x}\rangle \\
&\leq \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 + \{\beta_n\|x_n - \bar{x}\| + (1-\beta_n-\alpha_n\eta)\|x_n - \bar{x}\|\}^2 \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\eta)\langle \Lambda_n - \bar{x}, \gamma f(x_n) - \gamma f(\bar{x})\rangle + 2\alpha_n(1-\beta_n-\alpha_n\eta)\langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle \\
&\quad + 2\alpha_n\beta_n\langle x_n - \bar{x}, \gamma f(x_n) - \gamma f(\bar{x})\rangle + 2\alpha_n\beta_n\langle x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle \\
&\leq \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 + (1-\alpha_n\eta)^2\|x_n - \bar{x}\|^2 \\
&\quad + 2\alpha_n\beta_n\gamma\|x_n - \bar{x}\|\|f(x_n) - f(\bar{x})\| + 2\alpha_n\beta_n\langle x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\eta)\gamma\|\Lambda_n - \bar{x}\|\|f(x_n) - f(\bar{x})\| + 2\alpha_n(1-\beta_n-\alpha_n\eta)\langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle \\
&\leq \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 + (1-\alpha_n\eta)^2\|x_n - \bar{x}\|^2 + 2\alpha_n\beta_n\gamma\lambda\|x_n - \bar{x}\|^2 \\
&\quad + 2\alpha_n\beta_n\langle x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle + 2\alpha_n(1-\beta_n-\alpha_n\eta)\gamma\lambda\|x_n - \bar{x}\|^2 \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\eta)\langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle \\
&= \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 + (1 - 2\alpha_n\eta + \alpha_n^2\eta^2 + 2\alpha_n\gamma\lambda - 2\alpha_n^2\eta\gamma\lambda)\|x_n - \bar{x}\|^2 \\
&\quad + 2\alpha_n\beta_n\langle\, x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle + 2\alpha_n(1-\beta_n-\alpha_n\eta)\langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle \\
&\leq \{1 - \alpha_n(2\eta - \alpha_n\eta^2 - 2\gamma\lambda + 2\alpha_n\eta\gamma\lambda)\}\|x_n - \bar{x}\|^2 \\
&\quad + \alpha_n^2\|\gamma f(x_n) - A\bar{x}\|^2 + 2\alpha_n\beta_n\langle x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle \\
&\quad + 2\alpha_n(1-\beta_n-\alpha_n\eta)\langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle.
\end{aligned}$$

Suppose that

$$\varepsilon_n = \alpha_n\|\gamma f(x_n) - A\bar{x}\|^2 + 2\beta_n\langle x_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle + 2(1-\beta_n-\alpha_n\eta)\langle \Lambda_n - \bar{x}, \gamma f(\bar{x}) - A\bar{x}\rangle,$$

thus

$$\|x_{n+1} - \bar{x}\|^2 \leq \{1 - \alpha_n(2\eta - \alpha_n\eta^2 - 2\gamma\lambda + 2\alpha_n\eta\gamma\lambda)\}\|x_n - \bar{x}\|^2 + \alpha_n\varepsilon_n \tag{38}$$

It follows from (37) and Lemma 2.3 that

$$\limsup_{n\to\infty}\varepsilon_n \leq 0$$

which implies that the sequence $\{x_n\}$ strongly converges to $\bar{x}$. □

4. An Application in Optimal Problem Systems

In this section , we consider the following optimization problem :

$$min_{x\in F(T)}\frac{1}{2}\langle Ax, x\rangle - g(x), \tag{39}$$

where $F(T)$ is the set of fixed points of $T : C \to C$ and g is a potential function for γ (i.e, $g(x) = \gamma f(x)$ for $x \in H$).

Theorem 4.1. *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $f : C \to C$ be a $\lambda-$contraction. Let A be a strongly positive linear bounded operator on H with coefficient $\eta > 0$ and $0 < \gamma < \frac{\eta}{\lambda}$ and $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Suppose $\{x_n\}$ be a sequence generated by $x_1 \in C$ and:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)Tx_n, \tag{40}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences in Theorem 3.1. Also, the conditions (C1) and (C3), in Theorem 3.1, are hold. If $F(T)$ is a compact subset of C , then the sequence $\{x_n\}$ strongly converges to a point $\bar{x}$, where $\bar{x} \in F(T)$ which solves the optimization problem (39).

Proof. For $\{T(s)\}_{s\geq 0} = T$, $A = I$, $\lambda_n = 0$, $\psi_i = 0$, $F_i = 0, \forall i \in \{1, 2, \dots, k\}$, and $G \equiv 0$, $P_C = I$, $y_n = x_n$ in Theorem 3.1, the sequence $\{x_n\}$ strongly converge to a point $\bar{x}$, where $\bar{x} \in F(T)$ which is the unique solution of the following variational inequality

$$\langle (A - \gamma f)\bar{x}, x - \bar{x}\rangle \geq 0, \forall x \in F(T). \tag{41}$$

Note that $F(T)$ is a compact and convex subset of C and

$$\frac{1}{2}\langle Ax, x\rangle - g(x) : C \to R \tag{42}$$

is a continuous mapping. By Weierstrass theorem , there exists $\varepsilon \in F(T)$ which is a minimal point of optimization problem (39). On the other hand , (41) is the optimality necessary condition for the optimization problem (39) [26]. This implies

$$\langle (A - \gamma f)\varepsilon, x - \varepsilon\rangle \geq 0, \forall x \in F(T). \tag{43}$$

It is clear $\bar{x} = \varepsilon$, since $\bar{x}$ is the unique solution of (41). □

5. Numerical Examples

First , we present an example for Theorem 3.2 .

Example 5.1. Suppose $H = \mathbb{R}, C = [-1, 1]$ and

$$F_1(x, y) = -3x^2 + xy + 2y^2 \text{ and } F_2(x, y) = -5x^2 + xy + 4y^2,$$
$$F_3(x, y) = -7x^2 + xy + 6y^2 \text{ and } F_4(x, y) = -9x^2 + xy + 8y^2.$$

Also , we consider $\psi_1(x) = \psi_2(x) = \psi_3(x) = \psi_4(x) = 3x^2$, $G(x) = \frac{x}{10}$, $A = I, f(x) = \frac{x}{5}$ with coefficient $\eta = \frac{1}{2}$, $\gamma = 1$ and $T(s) = e^{-s}$ as a nonexpansive semigroup on C. It is easy to check that $\psi_1, \psi_2, \psi_3, \psi_4, A, f$ and $T(s)$ satisfy all conditions in Theorem 3.1 . For each $r > 0$ and $x \in C$, there exists $z \in C$ such that, for any $y \in C$,

$$\begin{aligned} F(z, y) + \psi(y) - \psi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad &\Leftrightarrow \quad -3z^2 + zy + 5y^2 - 3z^2 + \frac{1}{r}(y - z)(z - x) \geq 0 \\ &\Leftrightarrow \quad 5ry^2 + ((r + 1)z - x)y - 6rz^2 - z^2 + zx \geq 0 \end{aligned}$$

Set $B(y) = 5ry^2 + ((r + 1)z - x)y - 6rz^2 - z^2 + zx$. Then $B(y)$ is a quadratic function of y with coefficients $a = 5r, b = (r + 1)z - x$ and $c = -6rz^2 - z^2 + zx$. So

$$\begin{aligned} \Delta &= [(r + 1)z - x]^2 - 20r(zx - z^2 - 6rz^2) \\ &= (r + 1)^2z^2 - 2(r + 1)xz + x^2 + 120r^2z^2 + 20rz^2 - 20rzx \\ &= x^2 - 2(11rz + z)x + (121r2z^2 + 22rz^2 + z^2) \\ &= [(x - (11rz + z))]^2. \end{aligned}$$

$B(y) \geq 0$ for all $y \in C$, if and only if $\Delta = [(x - (11rz + z))]^2 \leq 0$. Therefore , $z = \frac{x}{11r+1}$, which yields $T_{r_n^{(1)}} = u_n^{(1)} = \frac{x_n}{11r_n+1}$.
By the same argument , for F_2, one can conclude

$$T_{r_n^{(2)}} = u_n^{(2)} = \frac{x_n}{15r_n + 1},$$
$$T_{r_n^{(3)}=u_n^{(2)}} = \frac{x_n}{19r_n + 1},$$
$$T_{r_n^{(4)}=u_n^{(4)}} = \frac{x_n}{23r_n + 1}.$$

Hence

$$\omega_n = \frac{u_n^{(1)} + u_n^{(2)} + u_n^{(3)} + u_n^{(4)}}{4}.$$

Let $t_n = \frac{n}{n+2}, s_n = n, r_n = \frac{n}{n+1}$ and $\lambda_n = \frac{1}{10n}$, $\alpha_n = \frac{1}{20n}$ and $\beta_n = \frac{2n-1}{10n-9}$, we have the following algorithm for the sequence $\{x_n\}$

$$\begin{cases} z_n = \frac{19n+20}{10n+20}\omega_n, \\ y_n = \frac{1}{10n}\omega_n + \frac{10n-1}{10n}z_n, \\ x_{n+1} = \frac{200n^2-190n+9}{1000n^2-900n}x_n + \frac{160n^2-170n+9}{200n^3-180n^2}(1-e^{-n})\omega_n. \end{cases}$$

By using MATLAB software , we obtain the following table and figure of the result , with initial point $x_1 = 1$.

n	x_n	n	x_n	n	x_n
1	1	11	2.287053908×10^{-6}	21	$3.762255275\times10^{-13}$
2	1.006470149	12	4.88149991×10^{-7}	22	$7.780342962\times10^{-14}$
3	0.3059037033	13	1.036076343×10^{-7}	23	$1.606482782\times10^{-14}$
4	0.07945190726	14	2.188663486×10^{-8}	24	$3.312377978\times10^{-15}$
5	0.01922449408	15	4.604863982×10^{-9}	25	$6.820908513\times10^{-16}$
6	0.004465484024	16	$9.654838368\times10^{-10}$	26	$1.402906653\times10^{-16}$
7	0.001010074517	17	$2.018167853\times10^{-10}$	27	$2.882304587\times10^{-17}$
8	0.0002242641019	18	$4.207385568\times10^{-11}$	28	$5.915770281\times10^{-18}$
9	0.00004911325747	19	$8.750678216\times10^{-12}$	29	$1.213039518\times10^{-18}$
10	0.00001064273173	20	$1.816167376\times10^{-12}$	30	2.48518864×10^{-19}

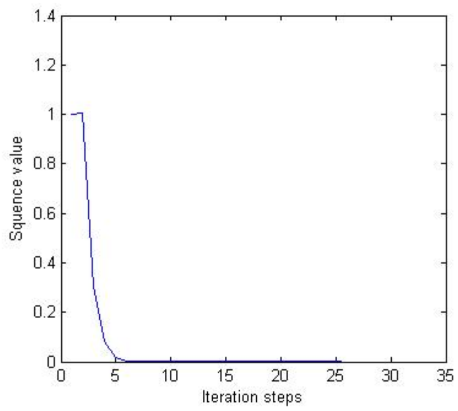

Figure 1: The graph of $\{x_n\}$ with initial value $x_1 = 1$.

Example 5.2. *Let $H = \mathbb{R}$ and $C = [1, 2]$. For each $x \in C$, we know $f(x) = \frac{x^2}{4}$ is a contractive mapping, $T(x) = \frac{1}{x}$ is a nonexpansive mapping on C and $F(T) = \{1\}$, $A(x) = 2x$ is a strongly positive linear bounded operator on H . Let $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n}{2n-1}$ and $\gamma = 1$. Substituting all of the given conditions to the scheme (40), we have*

$$x_{n+1} = \frac{n}{2n-1}x_n + \frac{1}{4n}x_n^2 + \frac{n^2-5n+2}{2n^2-n}\frac{1}{x_n}$$

Following the proof of Theorem 4.1, we easily obtain the sequence $\{x_n\}$ strongly converge to $1 \in F(T)$, which is the solution of the optimization problem

$$min_{x\in F(T)}\frac{1}{2}\langle Ax, x\rangle - g(x).$$

We obtain the following table and figure of the result, with the initial point $x_1 = 2$.

n	x_n	n	x_n	n	x_n
1	2	11	0.7992975259	21	0.9068849985
2	1.5	12	0.8201697889	22	0.9116063592
3	0.9097222222	13	0.8370906671	23	0.9158706523
4	0.4930490245	14	0.8510851663	24	0.9197413424
5	0.3762130308	15	0.8628533176	25	0.9232706421
6	0.5332945703	16	0.8728886137	26	0.9265019037
7	0.6270101604	17	0.8815486756	27	0.9294714244
8	0.6922463705	18	0.8890989313	28	0.9322098276
9	0.7385773109	19	0.8957405173	29	0.9347431316
10	0.7729182527	20	0.901628642	30	0.9370935844

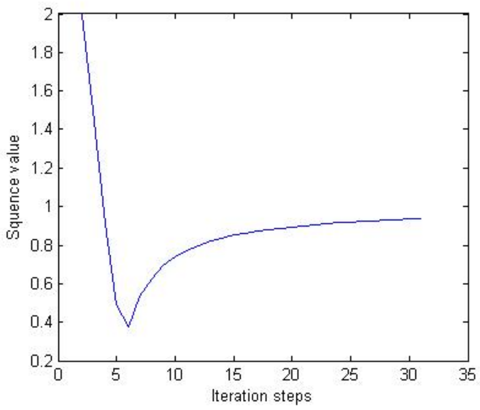

Figure 2: The graph of $\{x_n\}$ with initial value $x_1 = 2$.

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