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Existence and Uniqueness of Positive Solutions for Boundary Value Problems of Fractional Differential Equations

Hojjat Afshari[a], Hamidreza Marasi[b], Hassen Aydi[c]

[a]*Department of Mathematics, Basic Sciences Faculty, University of Bonab, Bonab, Iran.*
[b]*Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.*
[c]*University of Dammam, Department of Mathematics. College of Education of Jubail, P.O: 12020, Industrial Jubail 31961. Saudi Arabia.*
Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan.

Abstract. By using fixed point results of mixed monotone operators on cones and the concept of $\phi-$concavity, we study the existence and uniqueness of positive solutions for some nonlinear fractional differential equations via given boundary value problems. Some concrete examples are also provided illustrating the obtained results.

1. Introduction and Preliminaries

Fractional calculus has recently evolved as an interesting and important field of research. The much interest in the subject owes to its extensive applications in the mathematical modeling of several phenomena in many engineering and scientific disciplines such as physics, chemistry, biophysics, biology, blood flow problems, control theory, aerodynamics, nonlinear oscillation of earthquake, the fluid-dynamic traffic model, polymer rheology, regular variation in thermodynamics, economics, fitting of experimental data, etc, ([14],[15]). A significant feature of a fractional order differential operator, in contrast to its counterpart in classical calculus, is its non local behavior. It means that the future state of a dynamical system or process based on the fractional differential operator depends on its current state as well its past states. It is equivalent to say that differential equations of arbitrary order are capable of describing memory and hereditary properties of certain important materials and processes. There are many methods to deal with the existence of solutions of nonlinear initial value problems of fractional differential equations such as fixed point results, the Leray-Schauder theorem, stability, etc. Mixed monotone operator method is an important concept, which was introduced first by Guo and Lakshmikantham in [9]. Their study has wide applications in the applied sciences such as engineering, biological chemistry technology, nuclear physics and in mathematics (see[11, 12, 21]) and references therein). Various existence (and uniqueness) theorems of fixed points for mixed monotone operators have been discussed extensively, see for example ([1]-[5],[8],[17], [19]). Bhaskar and Lakshmikantham [8], established some coupled fixed point theorems for

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Corresponding Author: H. Aydi
Email addresses: hojat.afshari@yahoo.com (Hojjat Afshari), hamidreza.marasi@gmail.com (Hamidreza Marasi), hmaydi@uod.edu.sa (Hassen Aydi)

mixed monotone operators in partially ordered metric spaces and discussed the existence and uniqueness of a solution for a periodic boundary value problem. In [13], the authors considered

$$\frac{D^{\alpha}}{Dt}u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0 \qquad t \in (0,1), \quad n-1 < \alpha \leq n.$$

According to their limitations, the function $g(t, 0)$ must be a non-zero function. On the other hand, $f(t, u, v)$ depends on $g(t, u)$. In [20], Xu et al. studied the following boundary value problem

$$\frac{D^{\alpha}}{Dt}u(t) = f(t, u(t))$$
$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

They used the following lemma for existence and uniqueness of a solution for the problem.

Lemma 1.1. *[20] Suppose that $A : Q_e \times Q_e \to Q_e$ is a mixed monotone operator and there exists a constant η with $0 \leq \eta < 1$ such that*

$$A(tx, \frac{1}{t}y) \geq t^{\eta}A(x, y), \quad \forall x, y \in Q_e, 0 < t < 1.$$

Then A has a unique fixed point $x^* \in Q_e$.

Recently, Y. Sang [18] proved some new results on existence and uniqueness of a fixed point for mixed monotone operators with perturbations. In this paper, by applying Sang's results, we obtain new results on the existence and uniqueness of positive solutions for some nonlinear fractional differential equations of the form

$$\frac{D^{\alpha}}{Dt}u(t) = f(t, u(t)),$$

via given boundary conditions. Two examples are detailed to show the reliability and efficiency of the considered fixed point theorem. The assumptions we make with the function $f(t, u)$ are straight and clear than previous results. The function f does not need to be contractive, in spite of [7]. In addition, our results also couldn't be studied by the techniques of [13, 20]. For other results on same field, see recent papers in [6, 16, 19].

Definition 1.1. *[14, 15] For a continuous function $f : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined by*

$${}^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds,$$

where $n-1 < \alpha < n, n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 1.2. *[14, 15] The Riemann- Lioville fractional derivative of order α for a continuous function f is defined by*

$$D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{dt})^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n-1}}ds, \qquad (n = [\alpha] + 1),$$

where the right-hand side is pointwise defined on $(0, \infty)$.

Definition 1.3. *[14, 15] Let $[a, b]$ be an interval in $\mathbb{R}$ and $\alpha > 0$. The Riemann-Liouville fractional order integral of a function $f \in L^1([a, b], \mathbb{R})$ is defined by*

$$I_a^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{f(s)}{(t-s)^{1-\alpha}}ds,$$

whenever the integral exists.

Suppose that $(E, \|\ .\ \|)$ is a Banach space partially ordered by a cone $P \subseteq E$, that is, $x \leq y$ if and only if $y - x \in P$. If $x \neq y$, then we denote $x < y$ or $x > y$. We denote the zero element of E by θ. Recall that a non-empty closed convex set $P \subset E$ is a cone if it satisfies
(i) $x \in P,\ \lambda \geq 0 \Longrightarrow \lambda x \in P$,
(ii) $x \in P,\ -x \in P \Longrightarrow x = \theta$.
A cone P is called normal if there exists a constant $N > 0$ such that $\theta \leq x \leq y$ implies $\| x \| \leq N \| y \|$. We also define the order interval $[x_1, x_2] = \{x \in E | x_1 \leq x \leq x_2\}$ for all $x_1, x_2 \in E$. We say that an operator $A : E \to E$ is nondecreasing whenever $x \leq y$ implies $Ax \leq Ay$.

Definition 1.4. *[9, 10] Let $D \subset E$. An operator $A : D \times D \to D$ is said to be a mixed monotone operator if $A(x, y)$ is nondecreasing in x and is nonincreasing in y, i.e., $u_i, v_i (i = 1, 2) \in D$, $u_1 \leq u_2$ and $v_1 \geq v_2$ imply that $A(u_1, v_1) \leq A(u_2, v_2)$.*

An element $x^* \in D$ is called a fixed point of A if it satisfies $A(x^*, x^*) = x^*$. Letting $h > \theta$, we write $P_h = \{x \in E,\ \exists \lambda, \mu > 0 \text{ such that } \lambda h \leq x \leq \mu h\}$.
Let $e > \theta$. An operator $A : P \to P$ is said to be e-concave if it satisfies the following two conditions:
(i) A is e- positive, i.e, $A(P - \{\theta\}) \subset P_e$;
(ii) $\forall x \in P_e$ and $\forall\ 0 < t < 1$, there exists $\eta = \eta(t, x) > 0$ such that $A(tx) \geq (1 + \eta)tAx$, where $\eta = \eta(t, x)$ is called the characteristic function of A.

Theorem 1.1. *[18] Let P be a normal cone of the real Banach space E, $e > \theta$ and $u_0, v_0 \in P$ with $u_0 \leq v_0$. Let $A : P \times P \to P$ be a mixed monotone operator. Suppose that:*
(i) there exists a real positive number r_0 such that $u_0 \geq r_0 v_0$;
(ii) $u_0 \leq A(u_0, v_0)$ and $A(v_0, u_0) \leq v_0$;
(iii) for a fixed v, $A(., v) : P \to P$ is e-concave with its characteristic function $\eta(t, x)$ which is assumed to be monotone in x and continuous in t from the left;
(iv) for a fixed $u \in P$, there exists $N > 0$ such that for $A(u, .) : P \to P$, we have

$$A(u, v_1) - A(u, v_2) \geq -N(v_1 - v_2), \quad \forall v_1 \geq v_2,\ v_1, v_2 \in P.$$

Then A has exactly one fixed point x^* in $[u_0, v_0]$.

2. Main Result

We study the existence and uniqueness of a solution for the fractional differential equation

$$\frac{D^\alpha}{Dt} u(t) = f(t, u(t)),$$

on partially ordered Banach spaces with two types of boundary conditions and two types of fractional derivatives, Riemann-Liouville and Caputo.

2.1. Existence results for the fractional differential equation with the Riemann-Liouville fractional derivative

First, we study the existence and uniqueness of a positive solution for the fractional differential equation

$$\frac{D^\alpha}{Dt} u(t) = f(t, u(t)), \qquad t \in [0, 1],\ 3 < \alpha \leq 4, \tag{1}$$

under the conditions

$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{2}$$

where D^α is the Riemann-Liouville fractional derivative of order α. Consider the Banach space of continuous functions on $[0, 1]$ endowed with the norm

$$\|y\| = \max\{|y(t)|,\ t \in [0, 1]\}.$$

Take

$$P = \{y \in C[0,1] : \min_{t\in[0,1]} y(t) \geq 0\}. \tag{3}$$

Then P is a normal cone. From [20], we have the following lemma.

Lemma 2.1. *Given $f \in C[0,1]$ and $3 < \alpha \leq 4$. The unique solution of the fractional differential equation*

$$\frac{D^\alpha}{Dt}u(t) = f(t,u(t)), \qquad t \in [0,1],\ 3 < \alpha \leq 4, \tag{4}$$

where

$$u(0) = u'(0) = u(1) = u'(1) = 0, \tag{5}$$

is given by $\quad u(t) = \int_0^1 G(t,s)f(s,u(s))ds$, *where*

$$G(t,s) = \begin{cases} \frac{(t-1)^{\alpha-1}+(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \frac{(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

If $f(t,u(t)) = 1$, the unique solution of (4)-(5) is given by

$$u(t) = \int_0^1 G(t,s)ds = \frac{1}{\Gamma(\alpha+1)}t^{\alpha-2}(1-t)^2.$$

Lemma 2.2. *[20] The Green's function $G(t,s)$ defined in lemma 2.1 has the following properties:*
(1) $G(t,s) > 0$ and $G(t,s)$ is continuous for $t,s \in [0,1]$;
(2)

$$\frac{(\alpha-2)h(t)k(s)}{\Gamma(\alpha)} \leq G(t,s) \leq \frac{M_0 k(s)}{\Gamma(\alpha)},$$

where

$$M_0 = \max\{\alpha-1,(\alpha-2)^2\}, \quad h(t) = t^{\alpha-2}(1-t)^2 \quad \text{and} \quad k(s) = s^2(1-s)^{\alpha-2}.$$

Now, we are ready to state and prove our first main result.

Theorem 2.1. *Let $f(t,u(t),v(t)) \in C([0,1]\times[0,\infty]\times[0,\infty])$ be nondecreasing in u and be nonincreasing in v. Let $e > \theta$ and $u_0, v_0 \in P$ with $u_0 \leq v_0$. Suppose that*
(i) there exists a real positive number r_0 such that $u_0 \geq r_0 v_0$;
(ii)

$$u_0(t) \leq \int_0^1 G(t,s)f(s,u_0(s),v_0(s))ds$$

and

$$\int_0^1 G(t,s)f(s,u_0(s),v_0(s))ds \leq v_0(t);$$

(iii) for all $t \in [0,1]$ and $0 < c < 1$, there exists $\eta = \eta(c,x) > 0$ such that

$$f(t,cu(t),v(t)) \geq (1+\eta)c f(t,u(t),v(t)),$$

where $\eta(c,x)$ is nonincreasing in x and is continuous in t from the left;
(iv) for a fixed $u \in P$, there exists $N > 0$ such that for all $v_1, v_2 \in P$ with $v_1 \geq v_2$

$$\int_0^1 G(t,s)f(s,u(s),v_1(s))ds - \int_0^1 G(t,s)f(s,u(s),v_2(t))ds \geq -N(v_1 - v_2).$$

Then the problem (1) with the boundary value condition (2) has a unique solution u^* in $[u_0, v_0]$.

Proof. By using Lemma 2.1, the problem (1)-(2) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds,$$

where

$$G(t,s) = \begin{cases} \frac{(t-1)^{\alpha-1}+(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-2}t^{\alpha-2}[(s-t)+(\alpha-2)(1-t)s]}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases}$$

Define the operator $A : P \times P \to P$ by

$$A(u(t), v(t)) = \int_0^1 G(t,s)f(s,u(s),v(s))ds,$$

then u is a solution of the problem (1) if and only if $u = A(u,u)$. It is easy to see that the operator A is nondecreasing in u and nonincreasing in v on P. By assumptions on u_0 and v_0, we have

$$u_0 \le A(u_0, v_0) \quad \text{and} \quad A(v_0, u_0) \le v_0.$$

Moreover, for a fixed v, $A(., v) : P \to P$ is e-concave with its characteristic function. For a fixed $u \in P$, there exists $N > 0$ such that $A(u, .) : P \to P$ satisfies the following property

$$A(u, v_1) - A(u, v_2) \ge -N(v_1 - v_2), \quad \forall v_1 \ge v_2, v_1, v_2 \in P.$$

Therefore, A satisfies all conditions of Theorem 1.1. Consequently, the operator A has a unique positive solution $u^* \in [u_0, v_0]$ verifying $A(u^*, u^*) = u^*$. This completes the proof. □

We illustrate Theorem 2.1 by the following example which can not be solved by the previous results in the literature.

Example 2.1. *Consider the periodic boundary value problem*

$$D^{\frac{7}{2}}u(t) = f(t, u(t)) = g(t) + u(t) + \frac{1}{u(t)}, t \in [0,1], \tag{6}$$

with

$$u(0) = u'(0) = u(1) = u'(1) = 0,$$

such that g is continuous on $[0,1]$ satisfying $\min_{t\in[0,1]} g(t) = 10^3$ and $\max_{t\in[0,1]} g(t) = \frac{1}{4} \times 10^7$.
So, we seek the solution of the nonlinear integral equation

$$u(t) = \int_0^1 G(t,s)[g(s) + u(s) + \frac{1}{u(s)}]ds. \tag{7}$$

From Lemma 2.2, we have

$$M_1 = \min_{t\in[0,1]} \int_0^1 G(t,s)ds = 0.00001$$

and

$$M_2 = \max_{t\in[0,1]} \int_0^1 G(t,s)ds = 0.004.$$

We shall use Theorem 2.1. Obviously, the integral equation (7) can be written in the form $u = A(u,u)$ such that

$$A(u,v) = A_1(u) + A_2(v),$$

where

$$A_1(u) = \int_0^1 G(t,s)[g(s) + u(s)]ds,$$

and

$$A_2(v) = \int_0^1 G(t,s)[\frac{1}{v(s)}]ds.$$

Now, we show that the operator A satisfies all the conditions of Theorem 2.1. Consider $f(g(t), u(t), v(t)) = g(t) + u(t) + \frac{1}{v(t)}$. *For fixed value functions* u_0 *and* v_0, *it is clear that*

$$\int_0^1 G(t,s)f(s,u_0,v_0)ds = A_1(u_0) + A_2(v_0) \geq 10^{-5}(10^3 + u_0 + \frac{1}{v_0}), \tag{8}$$

$$\int_0^1 G(t,s)f(s,u_0,v_0)ds \leq 4\times 10^{-3}(\frac{1}{4}\times 10^7 + u_0 + \frac{1}{v_0}). \tag{9}$$

Especially, if we choose $u_0 = 10^{-2}$ *and* $v_0 = 100$, *we can easily get*

$$A(u_0, v_0) = \int_0^1 G(t,s)f(s,u_0(s),v_0(s))ds \geq u_0,$$

and

$$A(v_0, u_0) = \int_0^1 G(t,s)f(s,v_0(s),u_0(s))ds \leq v_0.$$

For a fixed v *and* $\forall\ t \in (0,1)$, *there exists* $\eta = \eta(t,u) = \frac{g(t)(1-t)}{t[g(t)+u(t)]} > 0$ *such that*

$$A(tu, v) \geq (1+\eta)tA(u,v),$$

where $\eta(t,u)$ *is nonincreasing in* u *and is continuous in* t *from left.*
Also, for $v_1, v_2 \in [10^{-2}, 10^2]$ *and for a fixed* u, *we have*

$$A(u,v_1) - A(u,v_2) = \int_0^1 G(t,s)(f(s,u(s),v_1(s)) - f(s,u(s),v_2(s)))ds,$$
$$\geq 10^{-5}(\frac{1}{v_1} - \frac{1}{v_2}).$$

Hence, there exists $N = 10^{-9}$ *such that*

$$A(u,v_1) - A(u,v_2) \geq -N\,(v_1 - v_2).$$

*Therefore, the problem (6)-(**??**) has a u unique solution.*

2.2. *Existence results for the fractional differential equation with the Caputo fractional derivative*

In this paragraph, we study the existence and uniqueness of a positive solution for the fractional differential equation

$$\frac{^cD^\alpha}{Dt}u(t) = f(t, u(t)), \qquad t \in [0,1], \tag{10}$$

where

$$u(0) + \int_0^1 u(s)ds = u(1).$$

Mention that $^cD^\alpha$ is the Caputo fractional derivative of order α. Consider the Banach space of continuous functions on $[0,1]$ endowed with the sup norm. Take the normal cone P given by (3). We have the following lemma.

Lemma 2.3. *Let $0 < \alpha \leq 1$ and $h \in C([0,T],\mathbb{R})$ be a given function. Then the boundary value problem*

$$^{c}D^{\alpha}y(t) = h(t), \quad (t \in [0,T],\ T \geq 1),$$

and

$$y(0) + \int_0^T y(s)ds = y(T),$$

has a unique solution given by

$$y(t) = \int_0^T G(t,s)h(s)ds,$$

where $G(t,s)$ is the Green's function given as follows

$$G(t,s) = \begin{cases} \frac{-(T-s)^{\alpha}+\alpha T(t-s)^{\alpha-1}}{T\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & 0 \leq s < t, \\ \frac{-(T-s)^{\alpha}}{T\Gamma(\alpha+1)} + \frac{(T-s)^{\alpha-1}}{T\Gamma(\alpha)}, & t \leq s < T. \end{cases}$$

By using a similar proof, Theorem 2.1 holds for the Green's function defined in lemma 2.3. As a consequence, we have a similar result in this case.

Example 2.2. *Consider the periodic boundary value problem*

$$^{c}D^{\frac{1}{2}}u(t) = f(t,u(t)) = g(t)\sqrt{u(t)} - \frac{1}{4}u(t)^{\frac{1}{3}}, \qquad t \in [0,1], \tag{11}$$

with

$$u(0) + \int_0^1 u(s)ds = u(1),$$

where g is continuous on $[0,1]$ verifying $\min_{t\in[0,1]} g(t) = 4$ and $\max_{t\in[0,1]} g(t) = 6$.
It is easily that $M_1 = \min_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{3}$ and $M_2 = \max_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{80}{51}$. Now, we show that the operator A satisfies all the conditions of Theorem 2.1.
Indeed, let $v_0 = 1$ and $u_0 = 10^{-2}$. We can easily get

$$A(u_0,v_0) = \int_0^1 G(t,s)f(s,u_0(s),v_0(s))ds \geq u_0,$$

and

$$A(v_0,u_0) = \int_0^1 G(t,s)f(s,v_0(s),u_0(s))ds \leq v_0.$$

For a fixed v and for all $t \in (0,1)$, there exists $\eta = \eta(t,u) = \frac{\sqrt{t}-t}{t} > 0$ such that

$$A(tu,v) \geq (1+\eta)t\,A(u,v),$$

where $\eta(t,u)$ is nonincreasing in u and is continuous in t from the left.
For a fixed u, there exists $N = \frac{400}{153}$ such that

$$A(u,v_1) - A(u,v_2) \geq -N(v_1 - v_2).$$

Thus the problem has a unique solution in $[10^{-2},1]$.

References

[1] T. Abdeljawad, H. Aydi, E. Karapinar, *Coupled fixed points for Meir-Keeler contractions in ordered partial metric spaces*, Mathematical Problems in Engineering, Volume 2012, Article ID 327273, 20 pages.

[2] H. Aydi, E. Karapinar, S. Radenović, *Tripled coincidence fixed point results for Boyd-Wong and Matkowski type contractions*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, 107 (2013), 339-353.

[3] H. Aydi, E. Karapinar, W. Shatanawi, *Coupled coincidence points in partially ordered cone metric spaces with a c-distance*, Journal of Applied Mathematics, Volume 2012, Article ID 312078, 15 pages.

[4] H. Aydi, E. Karapinar, Z. Mustafa, *Coupled coincidence point results on generalized distance in ordered cone metric spaces*, Positivity, 17 (4) (2013), 979-993,

[5] H. Aydi, W. Shatanawi, M. Postolache, *Coupled fixed point results for* (ψ, ϕ)*-weakly contractive mappings in ordered G-metric spaces*, Comput. Math. Appl. 63 (2012), 298-309.

[6] D. Baleanu, O.G. Mustafa, R.P. Agarwal, *On the solution set for a class of sequential fractional differential equations.* J. Phys. A, Math. Theor. **43**(38), Article ID 385209 (2010).

[7] M. Benchohra, F. Ouaar, *Existence results for nonlinear Fractional Differential Equations with integral boundary conditions*, Bulletin Math. Anal. Appl. **4**, 7-15 (2010).

[8] T.G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*. Nonlinear Anal. TMA **65**, 1379-1393 (2006).

[9] D. Guo, V. Lakskmikantham, *Coupled fixed points of nonlinear operators with applications*. Nonlinear Anal.TMA. **11** (5), 623-632 (1987).

[10] D. Guo, *Fixed points of mixed monotone operators with application*. Appl. Anal. **34**, 215-224(1988).

[11] D. Guo, V. Lakskmikantham, *Nonlinear Problems in Abstract Cones*. Academic Press, New York, (1988).

[12] D. Guo, *Partial Order Methods in Nonlinear Analysis*. Jinan, Shandong Science and Technology Press, (2000) (in Chinese).

[13] M. Jleli, B. Samet, *Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method*, Nonlinear Analysis: Modelling and Control, **20** (3), 367-376 (2015).

[14] A. Kilbas, H. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies. Elsevier. Amsterdam, (2006).

[15] I. Podlubny, *Fractional Differential Equations*. Academic Press, New york (1999).

[16] H.R. Marasi, H. Piri, H. Aydi, *Existence and multiplicity of solutions for nonlinear fractional differential equations*, J. Nonlinear Sci. Appl. 9 (2016), 4639-4646.

[17] B. Samet, E. Karapinar, H. Aydi, V. Rajić, *Discussion on some coupled fixed point theorems*, Fixed Point Theory Appl. 2013, 2013:50.

[18] Y. Sang, *A class of* ϕ*-concave operators and applications*. Fixed Point Theory Appl. **274**, 1-12(2013).

[19] Z. Xinqiu, L. Lishan, W. Yonghong, *Fixed point theorems for the sum of three classes of mixed monotone operators and applications*. Fixed Point Theory Appl. (2016) 2016:49.

[20] X.J. Xu, D.Q. Jiang, C.J. Yuan, *Multiple positive solutions for the boundary value problems of a nonlinear fractional differential equation*, Nonlinear Anal. **71**, 4676-4688 (2009).

[21] S. Zhang, *Coupled fixed points for mixed monotone condensing operators and an existence theorem of the solution for a class of functional equations arising in dynamic programming*. J. Math. Anal. Appl. **160**, 468-479 (1991).