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On F_2^ε-Planar Mappings with Function ε of (pseudo-) Riemannian Manifolds

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Abstract. In this paper we study special mappings between n-dimensional (pseudo-) Riemannian manifolds. In 2003 Topalov introduced PQ^ε-projectivity of Riemannian metrics, with constant $\varepsilon \neq 0, 1+n$. These mappings were studied later by Matveev and Rosemann and they found that for $\varepsilon = 0$ they are projective. These mappings could be generalized for case, when ε will be a function on manifold. We show that PQ^ε-projective equivalence with ε is a function corresponds to a special case of F-planar mapping, studied by Mikes and Sinyukov (1983) with $F = Q$. Moreover, the tensor P is derived from the tensor Q and non-zero function ε.

We assume that studied mappings will be also F_2-planar (Mikeš 1994). This is the reason, why we suggest to rename PQ^ε mapping as F_2^ε. For these mappings we find the fundamental partial differential equations in closed linear Cauchy type form and we obtain new results for initial conditions.

1. Introduction

Diffeomorphisms and automorphisms of geometrically generalized manifolds constitute one of the current main direction in differential geometry. Many papers are devoted to geodesic, almost geodesic, quasigeodesic, holomorphically projective, F-planar mappings and many others. Study of special manifold with affine connection, (pseudo-) Riemannian, e-Kählerian and e-Hermitian spaces, give one of the most important area, see [1]–[33]. For example, T. Levi-Civita [15] used geodesic mappings for modeling mechanical processes, A.Z. Petrov [27] used quasigeodesic mapping for modeling in theoretical physics.

More general question were studied by Hrdina, Slovák, Vašík, see [10], [11] and [12]. Others, who deals with question, were Minčić, Stanković, Velimirović, Zlatanović [31].

The PQ^ε-projective equivalence between n-dimensional Riemannian manifolds were introduced by Topalov [32], P and Q are tensors of type $(1,1)$ for which $PQ = \varepsilon\, Id$, $\varepsilon \in \mathbb{R}$, $\varepsilon \neq 1, 1+n$. Moreover, these mappings are special cases of F_2-planar mappings, [8], studied in [19], see [24, p. 225 - 231].

Based on the above and other properties, these mappings were renamed such a F_2^ε-planar [8]. It follows immediately from their definition that PQ^ε-projective equivalence is the correspondence occurring in the earlier studied F-planar mappings (Mikeš, Sinyukov [23]) and $F = Q$.

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In our paper we study F_2^ε-projective mappings between (pseudo-) Riemannian manifolds with non-zero function ε. For these mappings we find a fundamental system of closed linear equations in covariant derivatives in (pseudo-) Riemannian manifolds (M, g, F) with $F^2 \neq \varkappa\, Id$. Moreover, we obtain new results for initial conditions of metrics which are in F_2^ε-planar correspondence.

2. On F-Planar Mappings

Let $A_n = (M, \nabla, F)$ be an n-dimensional manifold M with affine connection ∇, and affinor structure F, i.e. a tensor field of type $(1, 1)$.

Definition 2.1 (Mikeš, Sinyukov [23], see [24, p. 213], [25, p. 385]). A curve ℓ, which is given by the equations $\ell = \ell(t)$, $\lambda(t) = d\ell(t)/dt$ $(\neq 0)$, $t \in I$, where t is a parameter, is called *F-planar*, if its tangent vector $\lambda(t_0)$, for any initial value t_0 of the parameter t, remains, under parallel translation along the curve ℓ, in the distribution generated by the vector functions λ and $F\lambda$ along ℓ.

In accordance with this definition, ℓ is F-planar if and only if the following condition holds: $\nabla_{\lambda(t)}\lambda(t) = \varrho_1(t)\lambda(t) + \varrho_2(t)F\lambda(t)$, where ϱ_1 and ϱ_2 are some functions of the parameter t, see ([23], [24, p. 213]).

We suppose two spaces $A_n = (M, \nabla, F)$ and $\bar{A}_n = (\bar{M}, \bar{\nabla}, \bar{F})$ with torsion-free affine connection ∇ and $\bar{\nabla}$, respectively. Affine structures F and $\bar{F}$ are defined on M, resp. $\bar{M}$.

Definition 2.2 (Mikeš, Sinyukov [23], see [24, p. 214], [25, p. 386]). A diffeomorphism f between manifolds with affine connection A_n and $\bar{A}_n$ is called an *F-planar mapping* if any F-planar curve in A_n is mapped onto an $\bar{F}$-planar curve in $\bar{A}_n$.

Due to the diffeomorphism f we always suppose that ∇, $\bar{\nabla}$, and the affinors F, $\bar{F}$ are defined on M $(\equiv \bar{M})$ where $A_n = (M, \nabla, F)$ and $\bar{A}_n = (M, \bar{\nabla}, \bar{F})$. The following holds.

Theorem 2.3. *An F-planar mapping f from A_n onto $\bar{A}_n$ preserves F-structures (i.e. $\bar{F} = a\,F + b\,Id$, a,b are some functions), and is characterized by the following condition*

$$P(X, Y) = \psi(X) \cdot Y + \psi(Y) \cdot X + \varphi(X) \cdot FY + \varphi(Y) \cdot FX \tag{1}$$

for any vector fields X, Y, where $P = \bar{\nabla} - \nabla$ is the deformation tensor field of f, ψ and φ are some linear forms.

This Theorem was proved by Mikeš and Sinyukov [23] for finite dimension $n > 3$, a more concise proof of this Theorem for $n > 3$ and also a proof for $n = 3$ was given by I. Hinterleitner and Mikeš [3], see [24, p. 214]. Therefore, under Theorem 2.3 we shortly suppose that $\bar{F} = F$ on M. This is the classical assumption that the *structures preserve*.

We remind the following types of F-planar mappings from manifolds A_n with affine connection ∇ onto (pseudo-) Riemannian manifolds $\bar{V}_n$ with metric $\bar{g}$:

Definition 2.4 (Mikeš [18], see [24, p. 225], [25, p. 398]).

1. An F-planar mapping of a manifold $A_n = (M, \nabla, F)$ with affine connection onto a (pseudo-) Riemannian manifold $\bar{V}_n = (M, \bar{g})$ is called an F_1*-planar mapping* if the metric tensor $\bar{g}$ satisfies the condition
$$\bar{g}(X, FX) = 0, \quad \text{for all } X. \tag{2}$$
2. An F_1-planar mapping $A_n \to \bar{V}_n$ is called an F_2*-planar mapping* if the one-form ψ is gradient-like, i.e. $\psi(X) = \nabla_X \Psi$, where Ψ is a function on A_n.
3. An F_1-planar mapping $A_n \to \bar{V}_n$ is called an F_3*-planar mapping* if the one-forms ψ and φ are related by $\psi(X) = \varphi(FX)$.

The F_2-planar mapping $f\colon A_n \to \bar{V}_n$ is characterized by the following equations (Mikeš [18], see [24, p. 230]):

$$\nabla_k a^{ij} = \lambda^i \delta^j_k + \lambda^j \delta^k_i + \xi^i F^k_j + \xi^j F^i_k, \tag{3}$$

where

$$a^{ij} = e^{2\psi} \bar{g}^{ij}, \quad \lambda^i = -a^{i\alpha}\psi_\alpha, \quad \xi^i = -a^{i\alpha}\varphi_\alpha, \tag{4}$$

where ψ_j, φ_i, F^h_i are components of ψ, φ, F and $\bar{g}^{ij}$ are components of the inverse matrix to the metric $\bar{g}$.

It is clear to see that if $A_n = (M, \nabla, F)$ is a (pseudo-) Riemannian manifold $V_n = (M, g, F)$ with metric tensor g and the Levi-Civita connection ∇, after lowering indices in (3), we obtain

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \xi_i F_{jk} + \xi_j F_{ik}, \tag{5}$$

where $a_{ij} = a^{\alpha\beta} g_{i\alpha} g_{j\beta}$, $\lambda_i = g_{i\alpha}\lambda^\alpha$, $\xi_i = g_{i\alpha}\xi^\alpha$, $F_{ik} = g_{i\alpha}F^\alpha_k$.

3. PQ^ε-Projective Riemannian Manifolds

3.1. *Definition of PQ^ε-projective Riemannian manifolds*

Let g and $\bar{g}$ be two (pseudo-) Riemannian metrics on an n-dimensional manifold M. Consider $(1,1)$-tensors P, Q which are satisfying the following conditions:

$$PQ = \varepsilon\, Id,\ g(X, PX) = 0,\ \bar{g}(PX, X) = 0,\ g(X, QX) = 0,\ \bar{g}(QX, X) = 0, \tag{6}$$

for all X and where $\varepsilon \neq 1, n+1$ is a real number. These conditions are written in a different way, see [16].

Definition 3.1 (Topalov, see [32]). The metrics $g, \bar{g}$ are called *PQ^ε-projective* if for the 1-form Φ the Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ satisfy

$$(\bar{\nabla} - \nabla)_X Y = \Phi(X)Y + \Phi(Y)X - \Phi(PX)QY - \Phi(PY)QX \tag{7}$$

for all X, Y.

Remark. Two metrics g and $\bar{g}$ are denoted by the synonym PQ^ε-projective if they are *PQ^ε-projective equivalent*. On the other hand this notation can be seen from the point of view of mappings. Assume two Riemannian manifolds (M, g) and $(\bar{M}, \bar{g})$. A diffeomorphism $f : M \to \bar{M}$ allows to identify the manifolds M and $\bar{M}$. For this reason we can speak about PQ^ε-projective mappings (or more precisely diffeomorphisms) between (M, g) and $(\bar{M}, \bar{g})$, when equations (6) and (7) hold. In these formulas $\bar{g}$ and $\bar{\nabla}$ mean in fact the pullbacks $f^*\bar{g}$ and $f^*\bar{\nabla}$.

3.2. *New results of PQ^ε-projective Riemannian manifolds for function $\varepsilon \neq 0$*

Natural generalization is a case, when ε is a non-zero function, at any point, defined on M. Next, we will study mappings characterized by formula (6) and (7).

Comparing formulas (1) and (7) we make sure that PQ^ε-projective equivalence is a special case of the F-planar mapping between Riemannian manifolds (M, g) and $(M, \bar{g})$. Evidently, this is if $\psi \equiv \Phi$, $F \equiv Q$ and $\varphi(\cdot) = -\Phi(P(\cdot))$.

Moreover, it elementary follows from (7) that ψ is a gradient-like form, see [32], thus a PQ^ε-projective equivalence is a special case of an F_2-planar mapping.

It is see that in above formulas (6) and (7) we can consider ε not only constant but also function on manifold M and we will deal with this problem.

If this PQ^ε-projective mappings $V_n \to \bar{V}_n$ will be F_2-planar, formula (5) has the following linear form:

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} - \lambda_\alpha P^\alpha_i g_{j\beta} Q^\beta_k - \lambda_\alpha P^\alpha_j g_{i\beta} Q^\beta_k. \tag{8}$$

From conditions (4) and (6) we obtain $a(X, PX) = 0$ and $a(X, QX) = 0$ for all X, and equivalently in local form

$$a_{i\alpha}P^\alpha_j + a_{j\alpha}P^\alpha_i = 0 \text{ and } a_{i\alpha}Q^\alpha_j + a_{j\alpha}Q^\alpha_i = 0. \tag{9}$$

Now, from the condition $PQ = \varepsilon Id$, it follows that Q is regular and

$$P = \varepsilon\, Q^{-1}. \tag{10}$$

This implies that P depends on Q and ε. Moreover two conditions in (6) depend on the other ones, i.e. in the definition of PQ^ε-projective mappings we can restrict on the conditions $g(X, QX) = 0$, $\bar{g}(X, QX) = 0$, $PQ = \varepsilon\, Id$. This fact implies the following lemma:

Lemma 3.2. *If Q satisfies the conditions $g(X, QX) = 0$ and $\bar{g}(X, QX) = 0$ for function $\varepsilon \neq 0$, then we obtain $g(X, PX) = 0$ and $\bar{g}(X, PX) = 0$.*

Proof. We can write first conditions (6) for g in the local form as $g_{i\alpha}Q^\alpha_j + g_{j\alpha}Q^\alpha_i = 0$. These equations we contract with $\bar{Q}^i_k\bar{Q}^j_l$, where $\bar{Q} = Q^{-1}$, after some calculations we obtain

$$g_{li}\bar{Q}^i_k + g_{kj}\bar{Q}^j_l = 0,$$

i.e. $g(X, Q^{-1}X) = 0$ for all X. From that follows $g(X, PX) = 0$ for all X. Analogically it holds also for the metric $\bar{g}$. □

4. F^ε_2-Projective Mapping with Function ε

Due to the above properties, from formula (9) and Lemma 3.2, we can simplify the Definition 3.1.

Let g and $\bar{g}$ be two (pseudo-) Riemannian metrics on an n-dimensional manifold M. Consider the regular $(1, 1)$-tensors F which is satisfying the following conditions

$$g(X, FX) = 0 \;\text{ and }\; \bar{g}(X, FX) = 0. \tag{11}$$

for all X.

Definition 4.1. The metrics g and $\bar{g}$ are called *F^ε_2-projective* if for a certain gradient 1-form ψ the Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ satisfy

$$(f^*\bar{\nabla} - \nabla)_X Y = \psi(X)Y + \psi(Y)X - \varepsilon\, \psi(F^{-1}X)FY - \varepsilon\, \psi(F^{-1}Y)FX, \tag{12}$$

for all vector fields X, Y and ε is a function on M, with $\varepsilon(x) \neq 0$, for all $x \in M$.

For $\varepsilon = const \neq 0$ this definition is in [8]. From Lemma 3.2, definition of F_2-planar mapping and comparing formulas (7) and (12) we evidently obtain following proposition:

Proposition 4.2. *A PQ^ε-projective mapping with non-zero function ε and gradient-like form ψ is an F^ε_2-planar mapping with $P = \varepsilon F^{-1}, Q = F$.*

We can rewrite formula (12) into this form:

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \psi_{(i}\delta_{j)} - \varepsilon\psi_\alpha P^\alpha_{(i}F^h_{j)}. \tag{13}$$

Contracting h, j we get

$$\bar{\Gamma}^\alpha_{i\alpha} = \Gamma^\alpha_{i\alpha} + (n + 1 - \varepsilon) \cdot \psi_i. \tag{14}$$

From Voss-Weyl formula, see [24, p. 57], we get $\bar{\Gamma}^\alpha_{i\alpha} - \Gamma^\alpha_{i\alpha} = \partial_i \ln \sqrt{\left|\det \bar{g}/\det g\right|}$. Because ψ_i is gradient-like form, i.e. $\psi_i = \partial_i\Psi$, then ε is a function of argument Ψ, i.e. $\varepsilon = \varepsilon(\Psi)$.

Most important thing in the study of F-planar mapping is preserving covariant derivative of structure F. The structure F *preserves covariant derivative* if $\bar{\nabla}F = \nabla F$. We proof that following lemma holds:

Lemma 4.3. *If F_2^ε-planar mapping f preserves covariant derivative of structure F, then $\varepsilon = -1$ or f is affine, i.e. $f^*\bar{\nabla} = \nabla$.*

Proof. It is known, that $\nabla_j F_i^h = \partial_i F_i^h + F_i^\alpha \Gamma_{\alpha j}^h - F_\alpha^h \Gamma_{ij}^\alpha$ and $\bar{\nabla}_j F_i^h = \partial_i F_i^h + F_i^\alpha \bar{\Gamma}_{\alpha j}^h - F_\alpha^h \bar{\Gamma}_{ij}^\alpha$. Using equation (12), which characterized F_2^ε-planar mapping, we get

$$\bar{\nabla}_j F_i^h = \nabla_j F_i^h + F_i^\alpha \psi_\alpha \delta_j^h - (\varepsilon+1)\psi_i F_j^h + \psi_\alpha P_i^\alpha F_\beta^h F_j^\beta.$$

Evidently, structure F preserves covariant derivative if and only if the following formula holds

$$F_i^\alpha \psi_\alpha \delta_j^h - (\varepsilon+1)\psi_i F_j^h + \psi_\alpha P_i^\alpha F_\beta^h F_j^\beta = 0. \tag{15}$$

We will limit only for calculations at point x_0. We suppose that $\psi_i(x_0) \neq 0$, because from (12) would follow $f^*\bar{\nabla} = \nabla$, i.e. for all points imply f is affine. Thus $\psi_\alpha P_i^\alpha(x_0) \neq 0$ and then from (15) implies:

$$F_\varepsilon^h F_j^\varepsilon = \alpha\delta_j^h + \beta F_j^h. \tag{16}$$

Now, we contract (16) with metric tensor g_{ih} and after that, we symetrize with respect to indices i, j. Because $g_{i\alpha}F_j^\alpha + g_{j\alpha}F_i^\alpha = 0$, we obtain $\beta g_{i\alpha}F_j^\alpha = 0$ and from it, evidently, implies $\beta = 0$.

After substitution (16) to formula (15), we get

$$\left(F_i^\alpha \psi_\alpha + \alpha\psi_\alpha P_i^\alpha\right)\delta_j^h - (\varepsilon+1)\psi_i F_j^h = 0.$$

From this implies that $(\varepsilon+1)\psi_i = 0$ then $\varepsilon = -1$ or $\psi_i \equiv 0$ for all points on M, i.e. in the last case f is affine. □

Theorem 4.4. *Let (M, g, F) be a (pseudo-) Riemannian manifold with regular structure F, for which $F^2 \neq \varkappa\, Id$ and $g(X, FX) = 0$ for all X. If (M, g, F) admits an F_2^ε-projective mapping onto a (pseudo-) Riemannian manifold $(\bar{M}, \bar{g})$ then the complete set of linear differential equations of Cauchy type in covariant derivatives in (M, g, F)*

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} + \Lambda_i g_{j\beta}F_k^\beta + \Lambda_j g_{i\beta}F_k^\beta \tag{17}$$

have a solution respective unknown function a_{ij} which $a_{ij} = a_{ji}$, $\det\|a_{ij}\| \neq 0$ and

$$a_{i\alpha}F_j^\alpha + a_{j\alpha}F_i^\alpha = 0. \tag{18}$$

Whereas $\lambda_i = a_{\alpha\beta}\overset{1}{T}{}_i^{\alpha\beta}$, $\Lambda_i = a_{\alpha\beta}\overset{2}{T}{}_i^{\alpha\beta}$ and $\overset{\sigma}{T}{}_i^{\alpha\beta}$, $\sigma = 1, 2$ are a certain tensors obtained from g_{ij} and F_i^h.

Proof. We will study the fundamental equations of an F_2^ε-planar mapping. From Proposition 4.2 follows, that formula (8) has the form

$$\nabla_k a_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik} - \lambda_\alpha F_i^\alpha g_{j\beta}F_k^\beta - \lambda_\alpha F_j^\alpha g_{i\beta}F_k^\beta. \tag{19}$$

From (4), (9) and Lemma 3.2 we may deduce the validity of condition (18). Now we covariantly differentiate (18) and obtain

$$\nabla_k a_{i\alpha}F_j^\alpha + \nabla_k a_{j\alpha}F_i^\alpha = \overset{3}{T}_{ijk},$$

where $\overset{3}{T}_{ijk} = -a_{i\alpha}\nabla_k F_j^\alpha - a_{j\alpha}\nabla_k F_i^\alpha$.

Using formula (19) and after some calculation, we get

$$(\varepsilon+1)(g_{\alpha k}F_j^\alpha \lambda_i + g_{\alpha k}F_i^\alpha \lambda_j) + \lambda_\alpha F_j^\alpha g_{ik} + \lambda_\alpha F_i^\alpha g_{jk} - \lambda_\alpha P_i^\alpha g_{\beta\gamma}F_j^\beta F_k^\gamma - \lambda_\alpha P_j^\alpha g_{\beta\gamma}F_i^\beta F_k^\gamma = \overset{3}{T}_{ijk}. \tag{20}$$

It is known, that g_{ij}, $g_{\beta\gamma}F_i^\beta F_j^\gamma$ are symmetric and $g_{\alpha k}F_j^\alpha$ is antisymmetric tensors. After skew symmetrization formula (20) with respect to indices j, k and replacing indices i, k we added up the obtained formula with (20) and finally we get:

$$(\varepsilon+1)\cdot(g_{\alpha i}F_j^\alpha\lambda_k+g_{\alpha k}F_j^\alpha\lambda_i)+g_{ik}\lambda_\alpha F_j^\alpha-g_{\beta\gamma}F_i^\beta F_k^\gamma\lambda_\alpha P_j^\alpha=\overset{4}{T}_{ijk}, \tag{21}$$

where $\overset{4}{T}_{ijk}=\frac{1}{2}(\overset{3}{T}_{ijk}+\overset{3}{T}_{kji}-\overset{3}{T}_{kij})$.

Now, we create this homogeneous equation:

$$g_{\alpha i}F_j^\alpha A_k+g_{\alpha k}F_j^\alpha A_i+g_{ik}B_j-g_{\beta\gamma}F_i^\beta F_k^\gamma C_j=0$$

with unknown variables A_i, B_i and C_i. Because rank $\|g_{\alpha i}F_j^\alpha\|>3$ from (21) we get $A_k=0$. From residual case it implies either $B_j=C_j=0$ or $g_{\beta\gamma}F_i^\beta F_k^\gamma=-\varkappa g_{ik}$, the later case is equivalently $F^2=\varkappa Id$. This is a contradiction with the assumption of theorem 4.4.

Therefore $B_j=C_j=0$ and it implies that $\lambda_\alpha F_i^\alpha=a_{\alpha\beta}\overset{5}{T}{}_i^{\alpha\beta}$ and $\lambda_\alpha P_i^\alpha=a_{\alpha\beta}\overset{2}{T}{}_i^{\alpha\beta}$, where $\overset{2}{T}$ and $\overset{5}{T}$ are a certain tensors obtained from g_{ij} and F_i^h. Now, elementary, if $\|\widetilde{F}_i^h\|=\|F_i^h\|^{-1}$, then $\lambda_i=\lambda_\alpha F_j^\alpha\widetilde{F}_i^j=a_{\alpha\beta}\overset{2}{T}{}_j^{\alpha\beta}\widetilde{F}_i^j\equiv a_{\alpha\beta}\overset{1}{T}{}_i^{\alpha\beta}$. Thereby this calculations, from (19) we obtain formula (17). □

5. F_2^ε-Planar Mappings with the $\bar{g}=k\cdot g$ Condition

From the properties of equations (17) and (18) follows the new results for F_2^ε-planar mappings, for which $F^2\neq\varkappa\,Id$. These conditions we suppose for the whole studied (pseudo-) Riemannian manifolds (M,g,F). The theory of differential equations implies that the system of equation (17) for initial condition at the point $x_0\in M$

$$a_{ij}(x_0)=\overset{0}{a}_{ij} \tag{22}$$

has only one unique solution.

Due to this, the general solution of (17) depends on the real parameters which can be, for example, the conditions (22). Because a_{ij} is symmetric, conditions can not be more then $n(n+1)/2$. Moreover, condition (18) implies further reduction of the parameters.

The structure F at the point x_0 can be written in Jordan's form as $F_i^i=\lambda_i$, $F_i^{i+1}=\mu_i=0,1$ and the other components are vanishing. Because det $F\neq 0$, all $\lambda_i\neq 0$. We do not exclude that λ_i are complex numbers (in this case the transformation equations are complex at the point x_0).

Substituting $i=j$ to equation (18), we obtain $a_{ii}\lambda_i+a_{ii+1}\mu_{i+1}=0$ (formally $\mu_{n+1}\equiv 0$), i.e. the diagonal components a_{ii} depend on the other components.

This implies that the maximum number of the independent components of $\overset{0}{a}_{ij}$, which is not greater than $n(n-1)/2-n$, i.e. $n(n-1)/2$ parameters. Therefore the following theorem holds.

Theorem 5.1. *A set of (pseudo-) Riemannian manifolds (M,g,F),* det $F\neq 0$ *and $F^2\neq\varkappa\,Id$, on which some (pseudo-) Riemannian manifold admits an F_2^ε-projective mapping, depends on not more than $n(n-1)/2$ parameters.*

We have the following theorem.

Theorem 5.2. *Let (M,g,F) and $(M,\bar{g},F)$ be (pseudo-) Riemannian manifolds with $F^2\neq\varkappa\,Id$ and which are in F_2^ε-planar correspondence. If the condition $\bar{g}=k\cdot g$ is valid for $x_0\in M$, then g and $\bar{g}$ are homothetic in M, i.e.*

$$\bar{g}(x)=k\cdot g(x), \tag{23}$$

for all $x\in M$, with $k=const$.

Proof. In the assumption of Theorem 5.2, Theorem 4.4 is valid. Then equation (17) holds. For the initial condition (23) there is no more than one unique solution. On the other hand, a trivial solution of equations (17) is $\bar{g}=k\cdot g$, and it satisfies the initial condition (23). The given mapping is homothetic. □

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