Filomat 31:9 (2017), 2821–2825
https://doi.org/10.2298/FIL1709821D

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Exponent of Convergence for Double Sequences and Selection Principles

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Abstract. In this paper, we will define the exponent of convergence for double sequences and in that terminology, we will prove three new theorems in theory of selection principles for double sequences.

1. Introduction

Study of selection principles was initiated in 1920's and 1930's by W. Hurewicz [13, 14], F. Rothberger [25], W. Sierpiński [28] and others. Over the previous decades, selection principles attracted a great number of authors which led to a considerable number of papers. For a brief report to concepts and results related to selection principles and for their interplay with the theory of infinite topological games and other areas of mathematics, we refer to survey papers [17] and [27], articles [4, 7, 8, 16, 31] and references cited there.

In this paper, we will observe some selection principles for double sequences which are convergent under the Pringsheim definition of convergence [23]. Nowadays, theory of double sequences (in particular, theory of convergence of double sequences in Pringsheim's sense) represent an important part of mathematical analysis and it is deeply interfered with many other mathematical disciplines (see, e.g. [12, 15, 21, 22, 26]).

Our attention will be focused on two important classes of positive real double sequences. The first class is consisted of those double sequences which converge to zero by Pringsheim, and the second class is the subclass of the first one consisting of double sequences with zero exponent of convergence. The main aim of this paper is to show that those classes satisfy some important selection principles.

Note that notion of exponents of convergence for double sequences (introduced in Section 2) present a new concept within theory of double sequences and that it is based on well known concept of exponents of convergence for sequences of positive real numbers which converge to zero (see, for example [1]). Exponents of convergence have been introduced by Pringsheim [24], and since then they were widely explored (see, for example [1, 2, 8–10, 18–20, 29, 30]. Also, an important role play generalizations of these concepts, and especially generalization of exponents of convergence which is given by using a sequence of exponents of convergence [1].

The structure of the paper is as follows. After short introduction, in Section 2 we give definitions of basic notions and notation which will be used in further work. In Section 3 we state and prove our main results.

2010 *Mathematics Subject Classification*. Primary 40A05, 54A20
Keywords. Double sequence, Exponent of convergence, Selection principle.
Received: 04 November 2015; Accepted: 03 March 2016
Communicated by Ljubiša Kočinac
Research supported by Ministry of Education and Science, Republic of Serbia.
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2. Notion and Notation

We recall now some notions and notations concerning double sequences and selection principles.

Let $a \in \mathbb{R}$. A real double sequence $x = (x_{m,n})$ converges to a by Pringsheim (denoted by $P - \lim x = a$) [23] if

$$\lim_{\min\{m,n\}\to+\infty} x_{m,n} = a. \tag{1}$$

Denote the class of such double sequences by c_2^a and the class of such positive double sequences by $c_{2,+}^a$ (see [6]). A variation of (1) is given in [5].

Let $x = (x_{m,n})$ be a double sequence of real numbers and let the double sequence $S_2^*(x)$ which represents x be defined by $S_2^*(x) = (S_{m,n}(x))$, where $S_{m,n}(x) = \sum_{k=1}^{m}\left(\sum_{l=1}^{n} x_{k,l}\right)$, for all $m, n \in \mathbb{N}$. Then we say that x has a finite sum in the Pringsheim sense, if there exists $S_2^{(x)} \in \mathbb{R}$ such that $S_2^{(x)} = P - \lim S_2^*(x)$, and we will use notation $S_2^{(x)} = P - \sum x$. Some other types of summarizing of double sequences are presented in [11].

A real number λ is said to be the exponent of convergence of a double sequence $x = (x_{m,n}) \in c_{2,+}^0$ if for every $\varepsilon > 0$, the double sequence $x_\varepsilon^+ = (x_{m,n}^{\lambda+\varepsilon})$ has a finite sum by Pringsheim, while the double sequence $x_\varepsilon^- = (x_{m,n}^{\lambda-\varepsilon})$ does not have. If for every $\varepsilon > 0$, the double sequence $x_\varepsilon = (x_{m,n}^{\varepsilon})$ doesn't have a finite sum by Pringsheim, then we say that $\lambda = +\infty$ is the exponent of convergence of x.

Let $\mathbb{S}_2$ be the set of all double sequences $x = (x_{m,n})$ of positive real numbers and let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of $\mathbb{S}_2$.

(a) Let $S_1^{(d)}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For every double sequence $(A_{m,n})$ of elements of $\mathcal{A}$ there is an element B of $\mathcal{B}$ such that $B = (b_{m,n})$ and $b_{m,n} \in A_{m,n}$, for all $m, n \in \mathbb{N}$;

(b) Let $\alpha_2^{(d)}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For every double sequence $(A_{m,n})$ of elements of $\mathcal{A}$ there is an element B of $\mathcal{B}$ such that the set $B \cap A_{m,n}$ is infinite, for all $m, n \in \mathbb{N}$;

(c) Let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijection. Then $S_\varphi^{(d)}(\mathcal{A}, \mathcal{B})$ denotes the selection principle: For every sequence (A_t) of elements of $\mathcal{A}$ there is an element $B \in \mathcal{B}$ such that $B = (b_{m,n})$ and $b_{m,n} \in A_t$, for $t = \varphi(m, n)$.

Let $\lambda \in [0, +\infty]$ and let $c_{2,+}^0(\lambda)$ denote the set of all double sequences from $\mathbb{S}_2$ which converge to zero under the Pringsheim definition of convergence and which exponent of convergence is λ.

3. Results

In this section, we will state and prove three new theorems in theory of selection principles for double sequences.

Theorem 3.1. *The selection principle $S_1^{(d)}(c_{2,+}^0, c_{2,+}^0(\lambda))$ is satisfied for $\lambda = 0$.*

Proof. Let $(x_{m,n,k,l})$ be a double sequence of double sequences such that for all $(k_0, l_0) \in \mathbb{N} \times \mathbb{N}$ holds $x^{(k_0,l_0)} = (x_{m,n,k_0,l_0}) \in c_{2,+}^0$. Then, we will form a double sequence $y = (y_{k,l})$ in the following way:

Step 1 Take $y_{1,1}$ from double sequence $x^{(1,1)}$ such that $y_{1,1} \le \frac{1}{2}$.

Step 2 For $(k, l) \in \{(1, n), (2, n), \ldots, (n, n), \ldots, (n, 2), (n, 1)\}$, where $n \in \mathbb{N}, n \ge 2$, take $y_{k,l}$ from the double sequence $x^{(k,l)}$ such that $y_{k,l} \le \frac{1}{2^n}$.

For $n \in \mathbb{N}$, let $V_n(y)$ denotes the sum

$$V_n(y) = y_{1,n} + y_{2,n} + \cdots + y_{n,n} + \cdots + y_{n,2} + y_{n,1}.$$

Then we have

$$V_n(y) \leq \frac{2n-1}{2^n}, \quad \text{for any } n \in \mathbb{N},$$

and thus

$$0 < \sum_{n=1}^{+\infty} V_n(y) \leq \sum_{n=1}^{+\infty} \frac{2n-1}{2^n}.$$

Since the series $\sum_{n=1}^{+\infty} \frac{2n-1}{2^n}$ is convergent in $\mathbb{R}$, it follows that the series $\sum_{n=1}^{+\infty} V_n(y)$ is convergent in $\mathbb{R}$, too. Thus, we can conclude that the double sequence y has finite diagonal sum

$$S_1^{(y)} = \sum y = \lim_{n\to+\infty} S_n(y) = \lim_{n\to+\infty} \left(\sum_{k=1}^{n} \sum_{l=1}^{n} y_{k,l} \right)$$

in $\mathbb{R}$ (more about diagonal sums can be seen in [11]). Now, by results obtained in [11], we have that $y \in c^0_{2,+}$.

Further, for arbitrary $\varepsilon > 0$, $n \in \mathbb{N}$ and (k,l) as above, we have $y^{\varepsilon}_{k,l} \leq \frac{1}{2^{\varepsilon n}}$, so for double sequence $y^{\varepsilon} = (y^{\varepsilon}_{k,l})$, holds $V_n(y^{\varepsilon}) \leq \frac{2n-1}{2^{\varepsilon n}}$, and thus

$$S_1^{(y^{\varepsilon})} \leq \sum_{n=1}^{+\infty} \frac{2n-1}{2^{\varepsilon n}}.$$

Since

$$\lim_{n\to+\infty} \frac{(2n+2)2^{\varepsilon n}}{(2n-1)2^{\varepsilon(n+1)}} = \frac{1}{2^{\varepsilon}} < 1,$$

we have that the series $\sum_{n=1}^{+\infty} \frac{2n-1}{2^{\varepsilon n}}$ is convergent, so again by results from [11] we obtain that double sequence y has finite sum by Pringsheim $S_2^{(y)} = P - \sum y$.

Now, let $\varepsilon < 0$. By construction of double sequence y, we have that $\lim_{n\to+\infty} y_{n,n} = 0$, which implies $\lim_{n\to+\infty} y^{\varepsilon}_{n,n} = +\infty$, and thus the double sequence y do not converge to zero by Pringsheim. By results from [11] the double sequence y does not have finite sum by Pringsheim. □

Lj. Kočinac [16] initiated study of selection principles related to α_i-properties. Interplay between α_2-property and classes of double sequences $c^0_{2,+}$ and $c^0_{2,+}(\lambda)$ is presented in the following theorem.

Theorem 3.2. *The selection principle $\alpha_2^{(d)}(c^0_{2,+}, c^0_{2,+}(\lambda))$ is satisfied for $\lambda = 0$.*

Proof. Let primes be sorted in ascending order $2 = p_1 < p_2 < \cdots < p_t < \cdots$, and let $y = (y_{k,l})$ be a double sequence created by procedure presented in proof of Theorem 3.1. Then, for given $n \in \mathbb{N}$ and $(k,l) \in \{(1,n),(2,n),\ldots,(n,n),\ldots,(n,2),(n,1)\}$ holds $y_{k,l} \leq \frac{1}{2^n}$.

Further, let $(x_{m,n,k,l})$ be a double sequence of double sequences, where for every $(k_0,l_0) \in \mathbb{N}\times\mathbb{N}$ holds

$$x^{(k_0,l_0)} = (x_{m,n,k_0,l_0}) \in c^0_{2,+}.$$

We can assume that this double sequence of double sequences is sorted (by some of standard methods) into a sequence of double sequences $(x_{t,k,l})$ by $t \in \mathbb{N}$.

For fixed natural number $t \in \mathbb{N}$, we will observe a sequence (x_{t,p_t^s,p_t^s}) by $s \in \mathbb{N}$. This sequence converge to zero when $s \to +\infty$. Then, there exist $s_{p_t} \in \mathbb{N}$ and a subsequence

$$(x_{t,p_t^{h(s)},p_t^{h(s)}}),$$

such that

$$\sum_{s=s_{p_t}}^{+\infty} x_{t,p_t^{h(s)},p_t^{h(s)}} \leq \frac{1}{2^t}.$$

Now, for every $s \geq s_{p_t}$ holds $x_{t,p_t^{h(s)},p_t^{h(s)}} \leq \frac{1}{2^t}$, which implies $x^{\varepsilon}_{t,p_t^{h(s)},p_t^{h(s)}} \leq \frac{1}{2^{\varepsilon t}}$, for the same s.

In the double sequence y, we will replace elements $y_{k,k}$, where $k = p_t^{h(s)}$ for $s \geq s_{p_t}$, with the elements $x_{t,p_t^{h(s)},p_t^{h(s)}}$. This will be done for every $t \in \mathbb{N}$. In this way, we obtain the double sequence $\overline{y} = (\overline{y}_{k,l})$. Then we have

$$0 < \sum \overline{y} \leq \sum y + \sum_{t=1}^{+\infty} \frac{1}{2^t} < +\infty,$$

and therefore $P - \lim \overline{y} = 0$. Moreover, the following hold:

1° $\overline{y} \cap (x_{t,k,l})$ is an infinite set, for every $t \in \mathbb{N}$;

2° $\overline{y} \in c^0_{2,+}$;

3° $P - \lim \overline{y} \in \mathbb{R}$.

Further, for arbitrary $\varepsilon < 0$, we will observe the double sequence $\overline{y}^{\varepsilon} = (\overline{y}^{\varepsilon}_{k,l})$. Since $\lim_{k\to+\infty} \overline{y}_{k,k} = 0$ implies $\lim_{k\to+\infty} \overline{y}^{\varepsilon}_{k,k} = +\infty$, we can conclude that double sequence $\overline{y}^{\varepsilon}$ does not have finite sum by Pringsheim.

Now, let $\varepsilon > 0$. Then

$$0 < \sum \overline{y}^{\varepsilon} \leq \sum y^{\varepsilon} + \sum_{t=1}^{+\infty} \frac{1}{2^{\varepsilon t}} < +\infty,$$

since $\lim_{t\to+\infty} \frac{2^{\varepsilon t}}{2^{\varepsilon(t+1)}} = 2^{-\varepsilon} < 1$. Thus, $\sum \overline{y}^{\varepsilon}$ is finite, so the double sequence $\overline{y}^{\varepsilon}$ has a finite sum by Pringsheim, according to [11]. This completes the proof of the theorem. □

Theorem 3.3. *The selection principle $S_1^{\varphi}(c^0_{2,+}, c^0_{2,+}(\lambda))$ is satisfied for $\lambda = 0$.*

Proof. Let $(x_{t,k,l})$ be a sequence of double sequences such that for every $(k_0, l_0) \in \mathbb{N}\times\mathbb{N}$ holds $x^{(k_0,l_0)} = (x_{t,k_0,l_0}) \in c^0_{2,+}$. Let $\varphi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a bijective function. Then, for every fixed $t \in \mathbb{N}$, there exists $(m_t, n_t) \in \mathbb{N} \times \mathbb{N}$ such that $\varphi(m_t, n_t) = t$. Let $M_t = \max\{m_t, n_t\}$. Now, we create a double sequence $y = (y_{m_t,n_t})$ such that $y_{m_t,n_t} \leq \frac{1}{2^{M_t}}$ and $y_{m_t,n_t} \in x_{t,k,l}$. Then we have that $y \in c^0_{2,+}$ and this double sequence y for every sequence $(x_{t,k,l})$, $t \in \mathbb{N}$, has exactly one element $y_{m_t,n_t} \in (x_{t,k,l})$.

Similar as in proof of Theorem 3.1, we can obtain that double sequence y has zero exponent of convergence. This means that $y \in c^0_{2,+}(\lambda)$, for $\lambda = 0$. □

Remark 3.4.

(a) The selection principles $\alpha_i^{(d)}(c^0_{2,+}, c^0_{2,+}(\lambda))$, for $i \in \{3, 4\}$ are satisfied for $\lambda = 0$.
Also, the selection principles $\alpha_i(c^0_{2,+}, c^0_{2,+}(\lambda))$ for $i \in \{2, 3, 4\}$ are satisfied for $\lambda = 0$ (more arguments about these selection properties can be found in [3, 6]).

(b) Analogues of Theorems 3.1, 3.2 and 3.3 hold if we replace the first coordinate $c^0_{2,+}$ with the class of all double sequences form $\mathbb{S}_2$ which have at least one Pringsheim's point equal to zero (more about Pringsheim's points can be found in [6]).

(c) By Theorems 3.1, 3.2 and 3.3 we have that corresponding selection principles are mutually equivalent.

References

[1] D. D. Adamović, Michail Petrović's notion of exponent of convergence, Matematički Vesnik 5 (20) (1968), 447–458 (In Serbian).
[2] S. B. Bank and I. Laine, On the oscilation theory of f'' + Af = 0, where A is entire, Transactions of the American Mathematical Society 273 (1982), 351–363.
[3] A. Caserta, D. Djurčić, M. Mitrović, Selection principles and double sequences II, Hacettepe Journal of Mathematics and Statistics, 43 (5) (2015) 725–730.
[4] D. Djurčić, N. Elez, Lj. D. R. Kočinac, On a subclass of the class of rapidly varying sequences, Applied Mathematics and Computations, 251 (2015) 626–632.
[5] D. Djurčić, M. R. Žižović, A. Petojević, Note on selection principles of Kočinac, Filomat 26 (2012), 1291–1295.
[6] D. Djurčić, Lj. D. R. Kočinac, M. R. Žižović, Double sequences and selections, Abstract and Applied Analysis, Vol. 2012, Article ID 497594, 6 pages.
[7] D. Djurčić, Lj. D. R. Kočinac, M. R. Žižović, Summability of sequences and selection properties, Abstract and Applied Analysis, Vol. 2011, Article ID 213816, 8 pages.
[8] D. Djurčić, Lj. D. R. Kočinac, M. R. Žižović, Exponents of Convergence and Games, Advances in Dynamical Systems and Applications, 6 (1) (2011), 41-48.
[9] D. Djurčić, M. R. Žižović, Note on sequence of exponents of SO-regular variability, Mathematica Moravica 12 (2008), 15–18.
[10] D. Djurčić, M. R. Žižović, The sequence of exponents of Karamatas convergence, Filomat 12 (1998), 63–66.
[11] N. Elez, D. Djurčić, A. Šebeković, Summation of Double Sequences and Selection Principles, Filomat 29 (4) (2015), 781–785.
[12] E. W. Hobson, The theory of functions of a real variable, Vol. 2, Cambridge University Press, Cambridge, UK, 2nd. edition, 1926.
[13] W. Hurewicz, Uber die Verallgemeinerung des Borelschen Theorems, Mathematische Zeitschrift 24 (1925), 401–425.
[14] W. Hurewicz, Uber Folgen stetiger Funktionen, Fundamenta Mathematicae 9 (1927), 193–204.
[15] K. Kayduman, C. Cakan, The Cesaro core of double sequences, Abstract and Applied Analysis, Vol. 2011, Article ID 950364, 9 pages.
[16] Lj. D. R. Kočinac, Selection principles related to α_1-properties, Taiwanese Journal of Mathematics 12 (3) (2008) 561–571.
[17] Lj. D. R. Kočinac, Some covering properties in topological and uniform spaces, Proceedings of the Steklov Institute of Mathematics 252 (2006), 122–137.
[18] P. Kostyrko, T. Šalát, On the exponent of convergence, Rendiconti del Circolo Matematico di Palermo II 31 (1982), 187–194.
[19] M. S. Liu and X. M. Zhang, Fixed points of meromorphic solutions of higher order linear differential equations, Annales Academi Scientiarum Fennicæ 31 (2006), 191–211.
[20] A. I. Markushevich, R. A. Silverman, Theory of Functions of a Complex Variable, Vol. 1, Chelsea Publishing Series, 2005.
[21] R.F. Patterson, E. Savas, A cathegory theorem for double sequences, Applied Mathematics Letters 24 (2011), 1680–1684.
[22] R.F. Patterson, E. Savas, On double sequences of continuous functions having continuous P-limits II, Filomat 27:5 (2013), 931–935.
[23] A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, Mathematische Annalen 53 (1900), 289–321.
[24] A. Pringsheim, Elementare Theorie der ganzen transzendenten Funktionen von endlicher Ordnung, Mathematische Annalen 58 (1904), 257342.
[25] F. Rothberger, Eine Versharfung der Eigenschaft C, Fundamenta Mathematicae 30 (1938), 50–55.
[26] E. Savas, (As)-double sequence spaces defined by Orlicz function and double statistical convergence, Computers and Mathematics with Applications 55 (2008), 1293–1301.
[27] M. Scheepers, Selection principles and covering properties in Topology, Note di Matematica 22 (2) (2003/2004), 3–41.
[28] W. Sierpisński, Sur un probleme de M. Menger, Fundamenta Mathematicae 8 (1926), 223–224.
[29] T. Šalát, Exponents of convergence of subsequences, Czechoslovak Mathematical Journal 34 (1984), 362–370.
[30] M. R. Tasković, On parameter convergence and convergence in larger sense, Matematički Vesnik 8 (23) (1971), 55–59.
[31] B. Tsaban, Some new directions in infinite-combinatorial topology, In:Topics in Set Theory and its Applications (J. Bagaria, S. Todorčević, eds.), Birkhäuser, 2006, 225–255.