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Planarity and Outerplanarity Indexes of the Unit, Unitary and Total Graphs

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Abstract. In this paper, we consider the problem of planarity and outerplanarity of iterated line graphs of the unit, unitary and total graphs when R is a finite commutative ring. We give a full characterization of all these graphs with respect to their planarity and outerplanarity indexes.

1. Introduction

In recent years, the investigation of iterated line graphs has recorded a large progress. The kth iterated line graph of G is denoted by $L^k(G)$ and these graphs are defined inductively as follows: $L^0(G) = G$, $L^1(G) = L(G)$ is the line graph of G and $L^k(G) = L(L^{k-1}(G))$.

The planarity index of graph G was defined as the smallest k such that $L^k(G)$ is non-planar. We denote the planarity index of G by $\xi(G)$. If $L^k(G)$ is planar for all $k \geqslant 0$, we define $\xi(G) = \infty$. In [6], the authors gave a full characterization of graphs with respect to their planarity index.

Theorem 1.1. *[Theorem 10, [6]] Let G be a connected graph. Then:*

(i) $\xi(G) = 0$ *if and only if G is non-planar.*

(ii) $\xi(G) = \infty$ *if and only if G is either a path, a cycle, or* $K_{1,3}$*.*

(iii) $\xi(G) = 1$ *if and only if G is planar and either* $\Delta(G) \geqslant 5$ *or G has a vertex of degree* 4 *which is not a cut-vertex.*

(iv) $\xi(G) = 2$ *if and only if L(G) is planar and G contains one of the graphs* H_i *in Figure 1 as a subgraph.*

(v) $\xi(G) = 4$ *if and only if G is one of the graphs* X_k *or* Y_k *(Figure 1) for some* $k \geqslant 2$*.*

(vi) $\xi(G) = 3$ *otherwise.*

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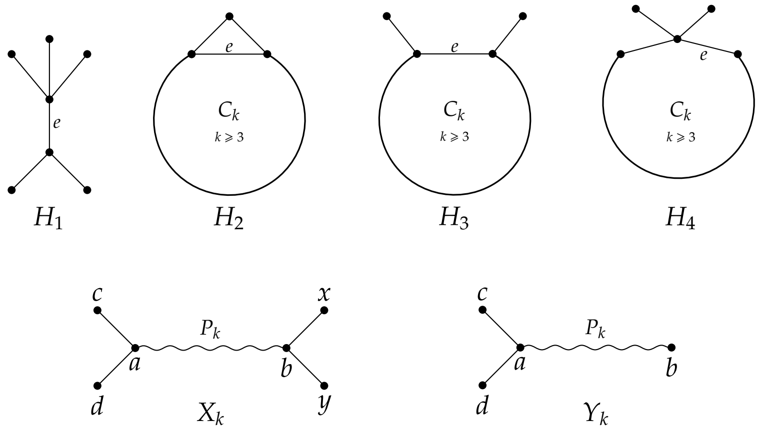

Figure 1

The outerplanarity index of a graph G, which is denoted by $\zeta(G)$, is the smallest integer k such that the kth iterated line graph of G is non-outerplanar. If $L^k(G)$ is outerplanar for all $k \geqslant 0$, we define $\zeta(G) = \infty$. In [7], the authors gave a full characterization of all graphs with respect to their outerplanar index.

Theorem 1.2. *Let G be a connected graph. Then:*

(i) $\zeta(G) = 0$ *if and only if G is non-outerplanar.*

(ii) $\zeta(G) = \infty$ *if and only if G is a path, a cycle, or $K_{1,3}$.*

(iii) $\zeta(G) = 1$ *if and only if G is planar and G has a subgraph homeomorphic to $K_{1,4}$ or $K_1 + P_3$ in Figure 2.*

(iv) $\zeta(G) = 2$ *if and only if $L(G)$ is outerplanar and G has a subgraph isomorphic to one of the graphs G_2 and G_3 in Figure 2.*

(v) $\zeta(G) = 3$ *if and only if $G \in I(d_1, d_2, \ldots, d_t)$ where $d_i \geqslant 2$ for $i = 2, \ldots, t-1$, and $d_1 \geqslant 1$ (Figure 2).*

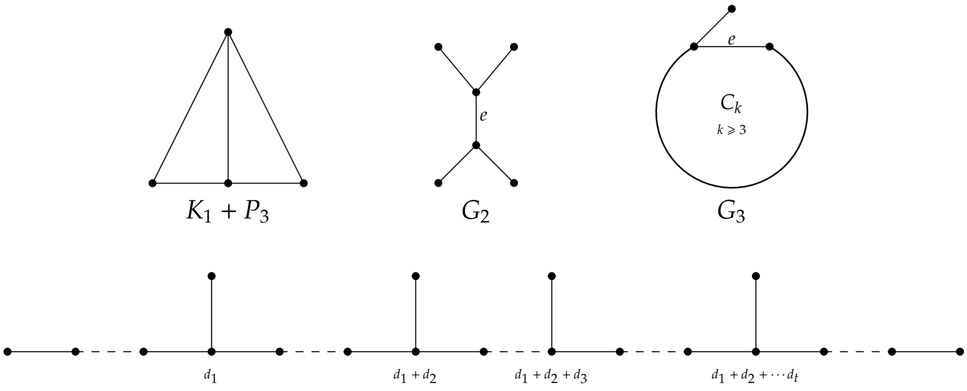

Figure 2

In Sections 2, 3 and 4, we study the planarity and outerplanarity indexes of the iterated line graphs of the unit, unitary and total graphs, respectively. We give a full characterization of all these graphs with respect to their indexes. Also, For all these graphs, we show that it's iterated line graph is outerplanar if and only if it is outerplanar, that is the outerplanarity index of these graphs is zero or infinity.

Now, we start to remind a belief necessary background of graphs. We use the standard terminology of graphs in [5]. Let G be a graph. We say that G is the *empty graph* when its vertex set is empty. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. A *complete graph* is a graph such that each pair of distinct vertices are adjacent. We use the notation K_n to denote the complete graph with n vertices. A *bipartite graph* is one whose vertices are partitioned into two disjoint parts such that the vertices of each edge belong to different partitions. A *complete bipartite* graph is a bipartite graph in which each vertex in the first set is joined to each vertex in the second set by exactly one edge. The complete bipartite graph with n and m vertices is denoted by $K_{n,m}$. A graph G is *connected* if there exists a path between every two vertices a and b of G, and otherwise we say G is *disconnected*. A vertex v is called a *cut vertex* if the number of connected components in $G \setminus \{v\}$ (a subgraph of G with removing the vertex v) is larger than that of G.

Throughout the paper, R is a finite commutative ring with non-zero identity. Also, we denote the set of all zero-divisor and unit elements of R by $Z(R)$ and $U(R)$, respectively.

2. Planarity and Outerplanarity Indexes of the Unit Graphs

The unit graph of R, denoted by $G(R)$, is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if and only if $x + y \in U(R)$. By [4, Theorem 2.4], if $2 \notin U(R)$, then the unit graph $G(R)$ is a $|U(R)|$-regular graph. Otherwise, for every $x \in U(R)$, we have $\deg(x) = |U(R)| - 1$ and, for every $x \in R \setminus U(R)$, we have that $\deg(x) = |U(R)|$.

First, we want to characterize the planarity index of the all unit graphs. By [4, Theorem 5.14], we have that the unit graph $G(R)$ is planar if and only if R is isomorphic to one of the following rings:

(i) $R \cong \mathbb{Z}_5$,

(ii) $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$,

(iii) $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times S$, where $\ell \geq 0$ and $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$, $S \cong \mathbb{Z}_4$, $S \cong \mathbb{F}_4$, or $S \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$.

Theorem 2.1. *Let R be a finite ring. Then:*

(i) $\xi(G(R)) = \infty$ *if and only if* $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times S$, *where* $\ell \geq 0$ *and* $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$, $S \cong \mathbb{Z}_4$, *or* $S \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$.

(ii) $\xi(G(R)) = 1$ *if and only if* $R \cong \mathbb{Z}_5$ *or* $\mathbb{Z}_3 \times \mathbb{Z}_3$.

(iii) $\xi(G(R)) = 2$ *if and only if* $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{F}_4$ *where* $\ell \geq 0$.

(iv) $\xi(G(R)) = 0$ *otherwise.*

Proof. First, since for every non-planar graphs we have that $\xi(G(R)) = 0$, it is sufficient to study the cases which the graph $G(R)$ is planar. Thus we have the following cases:

Case 1. $R \cong \mathbb{Z}_5$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$.
The graphs $G(\mathbb{Z}_5)$ and $G(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is pictured in Figure 3. The vertex 0 in the graph $G(\mathbb{Z}_5)$ and the vertex 00 in the graph $G(\mathbb{Z}_3 \times \mathbb{Z}_3)$ have degree 4 and they are not cut vertices. So $\xi(G(\mathbb{Z}_5)) = \xi(G(\mathbb{Z}_3 \times \mathbb{Z}_3)) = 1$.

$G(\mathbb{Z}_5)$ $\quad$ $G(\mathbb{Z}_3 \times \mathbb{Z}_3)$

Figure 3

Case 2. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_2$, where $\ell \geq 0$.

If $\ell = 0$, then $R \cong \mathbb{Z}_2$. The graph $G(\mathbb{Z}_2)$ is a path and so $\xi(G(\mathbb{Z}_2)) = \infty$. Now, we assume that $\ell > 0$. We have that $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1)\}$. Since $2 \notin U(R)$, $G(R)$ is a 1-regular graph. Thus $L(G(\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_2))$ is a totally disconnected graph and so $L^k(G(\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_2))$ for $k \geqslant 2$, is an empty graph. Thus $\xi(G(\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_2)) = \infty$.

Case 3. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_3$, where $\ell \geq 0$.

If $\ell = 0$, then $R \cong \mathbb{Z}_3$. Since $G(\mathbb{Z}_3)$ is a path, we have that $\xi(G(\mathbb{Z}_3)) = \infty$. Now, suppose that $\ell > 0$. Since $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1), (\underbrace{1, 1, \ldots, 1}_{\ell}, 2)\}$, $G(R)$ is a 2-regular graph. So $G(R)$ is the union of cycles. In fact every connected component of this graph is a cycle. Therefore $G(\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_3) \cong L^k(G(\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_3))$ for all $k \geqslant 1$. So $\xi(G(\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_3)) = \infty$.

Case 4. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_4$

If $\ell = 0$, then $R \cong \mathbb{Z}_4$. Since $U(R) = \{1, 3\}$ and $2 \notin U(\mathbb{Z}_4)$, the graph $G(\mathbb{Z}_4)$ is a 2-regular graph. So, this graph is a cycle on 4 vertices which implies that $\xi(G(\mathbb{Z}_4)) = \infty$. Otherwise $\ell > 0$. Since $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1), (\underbrace{1, 1, \ldots, 1}_{\ell}, 3)\}$ and $2 \notin U(R)$, the graph $G(R)$ is a 2-regular graph. Now similar to Case 3, we have that $\xi(G(\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_4)) = \infty$.

Case 5. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{F}_4$.

Consider the field $\mathbb{F}_4 = \{0, f_1, f_2, f_3\}$. Since $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, f_1), (\underbrace{1, 1, \ldots, 1}_{\ell}, f_2), (\underbrace{1, 1, \ldots, 1}_{\ell}, f_3)\}$, $G(R)$ is a 3-regular graph. We also have $\mathrm{char}(\mathbb{F}_4) = 2$. Thus the sum of each pair of distinct non-zero elements in $\mathbb{F}_4$ is non-zero. Therefore the vertex $(a_1, a_2, \ldots, a_\ell, f_i)$ is adjacent to $(1 - a_1, 1 - a_2, \ldots, 1 - a_\ell, f_j)$ for all $a_k \in \mathbb{Z}_2$ and $f_i, f_j \in \mathbb{F}_4$, where $1 \leq k \leq \ell$ and $1 \leq i \neq j \leq 3$. So every component of $G(R)$ has a form similar to that we show in the following figure.

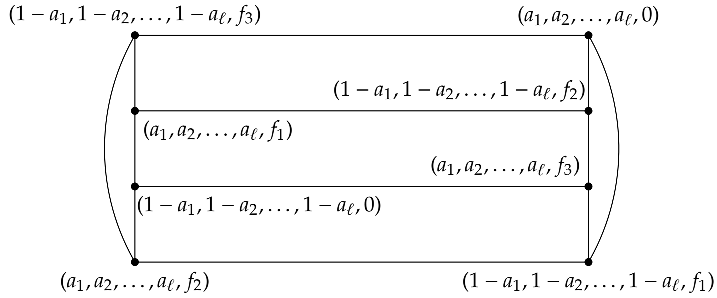

Figure 4: One of the connected component of $G(\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \mathbb{F}_4)$

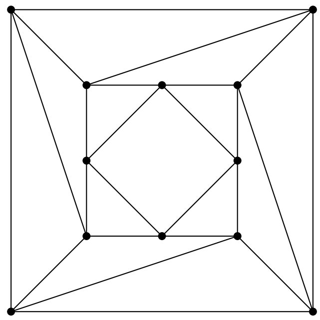

Figure 5: The line graph of the connected component of the graph $G(\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \mathbb{F}_4)$

Now, by Figure 5, the line graph of every component of $G(R)$ is planar and so $L(G(R))$ is planar. Also, in view of Figure 4, we can see that $G(R)$ contains H_3 as a subgraph. So, in this situation, $\xi(G(R)) = 2$.

Case 6. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$.

In this case, we have that

$$U(R) = \left\{ (\underbrace{1, 1, \ldots, 1}_{\ell}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (\underbrace{1, 1, \ldots, 1}_{\ell}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) \right\}.$$

Since $2 \notin U(R)$, $G(R)$ is a 2-regular graph. Similar to Case 3, $\xi(G(R)) = \infty$.

The converse statement follows easily. □

Now, we consider the outerplanarity index of the unit graph. In the following theorem, we give a full characterization of the unit graphs with respect to outerplanarity index.

Theorem 2.2. *Let R be a finite ring. Then:*

(i) $\zeta(G(R)) = \infty$ *if and only if* $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times S$, *where* $\ell \geq 0$ *and* $S \cong \mathbb{Z}_2$, $S \cong \mathbb{Z}_3$, $S \cong \mathbb{Z}_4$ *or* $S \cong \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}$.

(ii) $\zeta(G(R)) = 0$ *otherwise.*

Proof. Since $\zeta(G(R)) = 0$ when $G(R)$ is not outerplanar, we may suppose that $G(R)$ is outerplanar. In [1, Theorem 2.2], the authors showed that the unit graph $G(R)$ is outerplanar if and only if R is one of the following rings:

$$R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times S, \text{ where } \ell \geq 0 \text{ and } S \cong \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4 \text{ or } \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}.$$

Thus we have the following cases:

Case 1. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_2$, where $\ell \geq 0$.

Since $G(R)$ is a 1-regular graph, so $L^k(G(\underbrace{\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{\ell} \times \mathbb{Z}_2))$ is outerplanar for all $k \geq 0$. Therefore $\zeta(G(R)) = \infty$.

Case 2. R is isomorphic to one of the following rings:

$$\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times S, \text{ where } \ell \geq 0 \text{ and } S \cong \mathbb{Z}_3, \mathbb{Z}_4 \text{ or } \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \mid a, b \in \mathbb{Z}_2 \right\}.$$

In these cases, $G(R)$ is a union of cycles and so the graph $L^k(G(R))$, for all $k \geqslant 0$, is an outerplanar graph. Thus $\zeta(G(R)) = \infty$.

The converse statement follows easily. □

By previous theorem, we can conclude the following corollary.

Corollary 2.3. *Let R be a finite ring. Then the following statements are equivalent:*

(i) $G(R)$ *is outerplanar.*

(ii) $L^k(G(R))$ *is outerplanar for some* $k \geqslant 1$.

(iii) $L^k(G(R))$ *is outerplanar for all* $k \geqslant 1$.

3. Planarity and Outerplanarity Index of the Unitary Graphs

The unitary graph $G_R = Cay(R, U(R))$ is defined to be the graph whose vertex-set is R, with an edge between x and y if $x - y \in U(R)$. It is easy to see that G_R is a $|U(R)|$-regular graph. In this section, we provide a characterization of all finite rings with respect to planarity and outerplanarity indexes of iterated line graphs of their unitary graphs.

Theorem 3.1. *Let R be a finite ring. Then:*

(i) $\xi(G_R) = \infty$ *if and only if* $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_2$, $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_3$, $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_4$.

(ii) $\xi(G_R) = 2$ *if and only if* $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell} \times \mathbb{F}_4$.

(iii) $\xi(G_R) = 0$ *otherwise.*

Proof. We know that $\xi(G_R) = 0$ if and only if G_R is not planar. So it is sufficient to study the planar unitary graph. By [2, Theorem 8.2], G_R is planar if and only if R is one of the following rings:

$$\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_2, \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_3, \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_4 \text{ and } \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{F}_4.$$

Therefore we have the following cases:

Case 1. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_2$.

Since $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1)\}$, G_R is a 1-regular graph. Therefore $\xi(G_R) = \infty$.

Case 2. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_3$.

We have that $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, 1), (\underbrace{1, 1, \ldots, 1}_{\ell}, 2)\}$. Hence G_R is a 2-regular graph and thus every connected component of G_R is a cycle. Thus $G_R \cong L^k(G_R)$, for all $k \geqslant 1$, which implies that $\xi(G_R)) = \infty$.

Case 3. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_4$.

Similar to Case 2, one can see that $\xi(G_R) = \infty$

Case 4. $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{F}_4$.

Let $\mathbb{F}_4 = \{0, f_1, f_2, f_3\}$. Since $U(R) = \{(\underbrace{1, 1, \ldots, 1}_{\ell}, f_i) | \, i = 1, 2, 3\}$, the graph G_R is a 3-regular graph. So $L(G_R))$ is planar. Also, for all $1 \leq i \neq j \leq 3$, the vertices $(a_1, a_2, \ldots, a_\ell, f_i)$ and $(1 - a_1, 1 - a_2, \ldots, 1 - a_\ell, f_j)$ are adjacent in G_R, where $a_k \in \mathbb{Z}_2$. So every connected component of the graph G_R has a similar form of Figure 6.

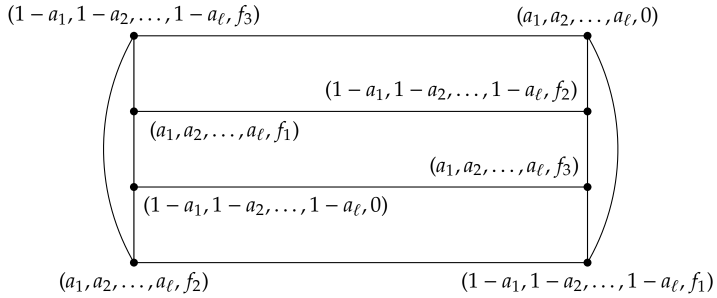

Figure 6: One of the connected component of the unitary graph of the ring $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2 \times \mathbb{F}_4$

One can easily see that G_R has H_3 as a subgraph. Now, by Theorem 1.1, we have that $\xi(G_R) = 2$. □

In the following of this section, we study the outerplanarity index of the unitary graphs and give a full characterization for thses graphs with respect to their outerplanarity index.

Theorem 3.2. *Let R be a finite ring. Then:*

(i) $\zeta(G_R) = \infty$ *if and only if* $R \cong \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0}$, $\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_3$ *and* $\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_4$.

(ii) $\zeta(G_R) = 0$ *otherwise.*

Proof. Since the line graph of a non-outerplanar graph is non-outerplanar, we need to calculate the outerplanarity index of G_R, whenever G_R is an outerplanar graph. By Theorem 3.2 of [1], we know that the unitary graph G_R is outerplanar if and only if R is isomorphic to one of the following rings:

$$\underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0}, \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_3 \text{ or } \underbrace{\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2}_{\ell \geq 0} \times \mathbb{Z}_4.$$

But for the above cases G_R is a 1-regular or a 2-regular graph. Thus $\zeta(G_R) = \infty$.

The converse statement follows easily. □

By Theorem 3.2, we can consider the following corollary.

Corollary 3.3. *Let R be a finite ring. Then the following statements are equivalent:*

(i) G_R *is outerplanar.*

(ii) $L^k(G_R)$ *is outerplanar for some* $k \geqslant 1$.

(iii) $L^k(G_R)$ *is outerplanar for all* $k \geqslant 1$.

4. Planarity and Outerplanarity of the Total Graphs

The total graph $T(\Gamma(R))$ is a graph with vertex-set R and two distinct vertices a and b are adjacent if and only if $a + b \in Z(R)$. In this section, we investigate the planarity and outerplanarity index of the total graphs.

Theorem 4.1. *Let R be a finite ring. Then:*

(i) $\xi(T(\Gamma(R))) = \infty$ *if and only if R is a field or isomorphic to* $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[X]}{(X^2)}$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) $\xi(T(\Gamma(R))) = 2$ *if and only if R is isomorphic to one of the rings:*

$$\frac{\mathbb{F}_4[X]}{(X^2)}, \frac{\mathbb{Z}_4[X]}{(X^2+X+1)}, \frac{\mathbb{Z}_2[X]}{(X^3)}, \frac{\mathbb{Z}_2[X,Y]}{(X,Y)^2}, \frac{\mathbb{Z}_4[X]}{(2X,X^2)}, \frac{\mathbb{Z}_4[X]}{(2X,X^2-2)}, \mathbb{Z}_8 \text{ or } \mathbb{Z}_6.$$

(iii) $\xi(T(\Gamma(R))) = 0$ *otherwise.*

Proof. First suppose that $T(\Gamma(R))$ is planar. Then we have the following cases:

Case 1. R is local. Then, by [8, Theorem 1.5(a)] , R is a field or it is isomorphic to one of the following rings:

$$\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{(X^2)}, \frac{\mathbb{Z}_2[X]}{(X^3)}, \frac{\mathbb{Z}_2[X,Y]}{(X,Y)^2}, \frac{\mathbb{Z}_4[X]}{(2X,X^2)}, \frac{\mathbb{Z}_4[X]}{(2X,X^2-2)}, \mathbb{Z}_8, \frac{\mathbb{F}_4[X]}{(X^2)}, \frac{\mathbb{Z}_4[X]}{(X^2+X+1)}.$$

If R is a field, then $T(\Gamma(R))$ is union of a totally disconnected graph and a 1-regular graph and so $\xi(T(\Gamma(R))) = \infty$. For rings $\frac{\mathbb{F}_4[X]}{(X^2)}$ and $\frac{\mathbb{Z}_4[X]}{(X^2+X+1)}$ we have $T(\Gamma(R))$ is the union of four K_4. Since $L(T(\Gamma(R)))$ is planar and $T(\Gamma(R))$ has H_2 as a subgraph, we have that:

$$\xi(T(\Gamma(\tfrac{\mathbb{F}_4[X]}{(X^2)}))) = \xi(T(\Gamma(\tfrac{\mathbb{Z}_4[X]}{(X^2+X+1)}))) = 2.$$

Let R is isomorphic to one of the following rings:

$$\frac{\mathbb{Z}_2[X]}{(X^3)}, \frac{\mathbb{Z}_2[X,Y]}{(X,Y)^2}, \frac{\mathbb{Z}_4[X]}{(2X,X^2)}, \frac{\mathbb{Z}_4[X]}{(2X,X^2-2)}, \mathbb{Z}_8.$$

Then for these rings we have that $T(\Gamma(R))$ is the union of two K_4, and so

$$\begin{aligned}\xi(T(\Gamma(\frac{\mathbb{Z}_2[X]}{(X^3)}))) &= \xi(T(\Gamma(\frac{\mathbb{Z}_2[X,Y]}{(X,Y)^2})))\\ &= \xi(T(\Gamma(\frac{\mathbb{Z}_4[X]}{(2X,X^2)})))\\ &= \xi(T(\Gamma(\frac{\mathbb{Z}_4[X]}{(2X,X^2-2)})))\\ &= \xi(T(\Gamma(\mathbb{Z}_8)))\\ &= 2.\end{aligned}$$

For the remaining two rings $\mathbb{Z}_4$ and $\frac{\mathbb{Z}_2[X]}{(X^2)}$, we have $T(\Gamma(R)) \cong 2K_2$ and so:

$$\xi(T(\Gamma(\mathbb{Z}_4))) = \xi(T(\Gamma(\frac{\mathbb{Z}_2[X]}{(X^2)}))) = \infty.$$

Case 2. R is non-local. Then since $T(\Gamma(R))$ is planar and R is finite, by [8, Theorem 1.5(b)], we have that R is isomorphic to the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_6$. If $R \cong \mathbb{Z}_6$, then $T(\Gamma(R))$ is a 3-regular graph which is shown that in Figure 7.

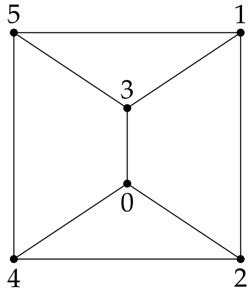

Figure 7: $T(\Gamma(\mathbb{Z}_6))$

Thus the line graph of the graph $T(\Gamma(\mathbb{Z}_6)))$ is planar. Also $T(\Gamma(\mathbb{Z}_6)))$ has H_3 as a subgraph. So $\xi(T(\Gamma(\mathbb{Z}_6))) = 2$. In the case that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have $T(\Gamma(R)) \cong C_4$ which implies that $\xi(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = \infty$. □

Theorem 4.2. *Let R be a finite ring. Then:*

(i) $\zeta(T(\Gamma(R))) = \infty$ *if and only if R is a field or isomorphic to one of the rings* $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[X]}{(X^2)}$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2$,

(ii) $\zeta(T(\Gamma(R))) = 0$ *otherwise.*

Proof. Assume that $T(\Gamma(R))$ is outerplanar. By [1, Theorem 4.2], $T(\Gamma(R))$ is outerplanar if and only if R is a field or isomorphic to one of the rings $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[X]}{(X^2)}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. For two rings $\mathbb{Z}_4$ and $\frac{\mathbb{Z}_2[X]}{(X^2)}$, we have $T(\Gamma(R)) \cong 2K_2$, and therefore $\zeta(T(\Gamma(\mathbb{Z}_4))) = \zeta(T(\Gamma(\frac{\mathbb{Z}_2[X]}{(X^2)}))) = \infty$. Also, $T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2)) \cong C_4$ which implies that $\zeta(T(\Gamma(\mathbb{Z}_2 \times \mathbb{Z}_2))) = \infty$. If R is a field, then $T(\Gamma(R))$ is the union of a totally disconnected graph and a 1-regular graph, and so $L(T(\Gamma(R)))$ is a totally disconnected graph and $L^k(T(\Gamma(R)))$ is an empty graph for all $k \geqslant 2$. Thus $\zeta(T(\Gamma(R))) = \infty$.

The converse statement follows easily. □

By previous theorem, we deduce that the outerplanarity index of the total graph is zero or infinity. Also, we can see the following corollary is true for all total graphs when R is a finite ring.

Corollary 4.3. *Let R be a finite ring. Then the following statements are equivalent:*

(i) $T(\Gamma(R))$ *is outerplanar.*

(ii) $L^k(T(\Gamma(R)))$ *is outerplanar for some* $k \geqslant 1$.

(iii) $L^k(T(\Gamma(R)))$ *is outerplanar for all* $k \geqslant 1$.

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