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Essential Norms of Weighted Differentiation Composition Operators between Zygmund Type Spaces and Bloch Type Spaces

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Abstract. We study boundedness of weighted differentiation composition operators $D^k_{\varphi,u}$ between Zygmund type spaces $\mathcal{Z}^\alpha$ and Bloch type spaces $\mathcal{B}^\beta$. We also give essential norm estimates of such operators in different cases of $k \in \mathbb{N}$ and $0 < \alpha, \beta < \infty$. Applying our essential norm estimates, we get necessary and sufficient conditions for the compactness of these operators.

1. Introduction and Preliminaries

Let $\mathbb{D}$ denote the open unit ball of the complex plane $\mathbb{C}$. By a *weight* function ν we mean a continuous, strictly positive and bounded function $\nu : \mathbb{D} \to \mathbb{R}^+$. The weight ν is called *radial* if $\nu(z) = \nu(|z|)$ for all $z \in \mathbb{D}$. Let $H(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. Then, for a weight ν, the *weighted Banach space of analytic functions* H^∞_ν is the space of all analytic functions $f \in H(\mathbb{D})$ for which

$$\|f\|_\nu = \sup_{z\in\mathbb{D}} \nu(z)|f(z)| < \infty.$$

In general, for a weight ν, the associated weight $\widetilde{\nu}$ is defined by

$$\widetilde{\nu}(z) = (\sup\{|f(z)| : f \in H^\infty_\nu, \|f\|_\nu \le 1\})^{-1}, \quad z \in \mathbb{D}.$$

It is known that for *standard weights* $\nu_\alpha(z) = (1 - |z|^2)^\alpha$ $(0 < \alpha < \infty)$, associated weights and weights are the same, i.e. $\widetilde{\nu_\alpha} = \nu_\alpha$.

For each $0 < \alpha < \infty$, the *Bloch type space* $\mathcal{B}^\alpha$ is the space of all analytic functions $f \in H(\mathbb{D})$ for which

$$\sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

The Bloch type space $\mathcal{B}^\alpha$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|, \quad f \in \mathcal{B}^\alpha.$$

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When $\alpha = 1$, we get the classic *Bloch space* $\mathcal{B} = \mathcal{B}^1$. The *little Bloch type space* $\mathcal{B}_0^\alpha$ consists of those functions $f \in \mathcal{B}^\alpha$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f'(z)| = 0.$$

For each $0 < \alpha < \infty$, the *Zygmund type space* $\mathcal{Z}^\alpha$ consists of those functions $f \in H(\mathbb{D})$ satisfying

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)| < \infty.$$

The Zygmund type space $\mathcal{Z}^\alpha$ is a Banach space equipped with the norm

$$\left\|f\right\|_{\mathcal{Z}^\alpha} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f''(z)|, \quad f \in \mathcal{Z}^\alpha.$$

The *little Zygmund type space* $\mathcal{Z}_0^\alpha$ consists of those functions $f \in \mathcal{Z}^\alpha$ for which

$$\lim_{|z|\to 1}(1-|z|^2)^\alpha |f''(z)| = 0.$$

Recall that, for Banach spaces X and Y, a linear operator $T : X \to Y$ is *bounded* if it takes bounded sets to bounded sets. The space of all bounded operators $T : X \to Y$ is denoted by $\mathcal{B}(X, Y)$. The norm of the space $\mathcal{B}(X, Y)$, called *operator norm*, is denoted by $\|T\|_{X\to Y}$ for a bounded operator $T : X \to Y$. An operator $T \in \mathcal{B}(X, Y)$ is *compact* if it takes bounded sets to sets with compact closure. The space of all compact operators $T : X \to Y$ is denoted by $\mathcal{K}(X, Y)$. The *essential norm* of an operator $T \in \mathcal{B}(X, Y)$, denoted by $\|T\|_{e,X\to Y}$, is defined as the distance from T to $\mathcal{K}(X, Y)$. Clearly, an operator $T \in \mathcal{B}(X, Y)$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. Therefore, essential norm estimates of operators in $\mathcal{B}(X, Y)$ lead to necessary and/or sufficient conditions for the compactness of such operators. In this paper we investigate boundedness, and then, essential norm estimates of certain type of operators, defined as follows, between Zygmund type spaces and Bloch type spaces. As a consequence of our essential norm estimates, we obtain necessary and sufficient conditions for the compactness of such operators.

Let $u, \varphi \in H(\mathbb{D})$ where φ is a selfmap of $\mathbb{D}$. The *weighted composition operator* uC_φ on $H(\mathbb{D})$ is defined by

$$(uC_\varphi)(f)(z) = u(z)f(\varphi(z)), \quad z \in \mathbb{D}.$$

Weighted composition operators, which are generalizations of *multiplication operators* and *composition operators*, appear in the study of dynamical systems. Moreover, it is known that isometries on many analytic function spaces are of the canonical forms of weighted composition operators. Boundedness, compactness and essential norm estimates of weighted composition operators have been studied by many authors between different spaces of analytic functions. Weighted composition operator uC_φ from Zygmund type spaces to Bloch type spaces has been studied in [2]. See also [6, 15, 16], for more results on weighted composition operators between certain spaces of analytic functions.

Let φ be an analytic selfmap of $\mathbb{D}$, $u \in H(\mathbb{D})$ and $k \in \mathbb{N}$. The *weighted differentiation composition operator* $D^k_{\varphi,u}$ on $H(\mathbb{D})$ is defined by

$$\left(D^k_{\varphi,u}f\right)(z) = u(z)f^{(k)}(\varphi(z)), \quad z \in \mathbb{D}.$$

Weighted differentiation composition operators [19] are also known as *generalized weighted composition operators* [24]. Boundedness and compactness of these operators between different spaces of analytic functions have been studied by many authors. The operator $D^k_{\varphi,u}$ from Bloch type spaces to weighted-type spaces has been studied in [7]. For the results on the operator $D^k_{\varphi,u}$ from Hardy spaces to Zygmund type spaces, see [10]. Also, the operator $D^k_{\varphi,u}$ between Bloch type spaces has been investigated in [26], and the operator $D^k_{\varphi,u}$ from logarithmic Bloch spaces to Zygmund type spaces has been studied in [14]. For more results on the operator $D^k_{\varphi,u}$ see also [8, 18, 21, 25] and references therein.

Note that by considering special cases of φ, u and k in $D^k_{\varphi,u}$, we can get certain well-known operators. For example, for an analytic selfmap φ of $\mathbb{D}$, by letting $u = \varphi'$ and $k = 1$, $D^1_{\varphi,\varphi'}$ is the well-known operator *composition followed by differentiation* DC_φ, given by

$$\left(DC_\varphi f\right)(z) = \varphi'(z)\left(f'(\varphi(z))\right) = \left(D^1_{\varphi,\varphi'} f\right)(z).$$

Also, if φ is an analytic selfmap of $\mathbb{D}$, then by letting $u = 1$ and $k = 1$, $D^1_{\varphi,1}$ is the well-known operator *composition proceeded by differentiation* $C_\varphi D$, given by

$$\left(C_\varphi D f\right)(z) = f'(\varphi(z)) = \left(D^1_{\varphi,1} f\right)(z).$$

Recently, there has been growing interest in the study of these particular cases. See, for example, [5, 9, 20] and references therein. We also note that weighted forms of operators DC_φ and $C_\varphi D$ [11] are also of the form $D^k_{\varphi,u}$. More precisely,

$$\left(\psi DC_\varphi f\right)(z) = \psi(z)\varphi'(z)\left(f'(\varphi(z))\right) = \left(D^1_{\varphi,\psi\varphi'} f\right)(z),$$

and

$$\left(\psi C_\varphi D f\right)(z) = \psi(z) f'(\varphi(z)) = \left(D^1_{\varphi,\psi} f\right)(z).$$

Therefore, it is worth mentioning that all results in this paper about weighted differentiation composition operators $D^k_{\varphi,u}$ are also valid for the above mentioned operators as particular cases.

In Section 2, we investigate boundedness of weighted differentiation composition operators $D^k_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ in different cases of $k \in \mathbb{N}$ and $0 < \alpha, \beta < \infty$. In Section 3, using results of Section 2, we give essential norm estimates of weighted differentiation composition operators $D^k_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ in different cases of $k \in \mathbb{N}$ and $0 < \alpha, \beta < \infty$. As a consequence of essential norm estimates given in Section 3, we get necessary and sufficient conditions for the compactness of such operators.

We mention that in this paper, for real scalars A and B, the notation $A \preceq B$ means $A \leq cB$ for some positive constant c. Also, the notation $A \asymp B$ means $A \preceq B$ and $B \preceq A$.

2. Boundedness

In this section we characterize boundedness of weighted differentiation composition operators $D^k_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ in different cases of $k \in \mathbb{N}$ and $0 < \alpha, \beta < \infty$. First, we study boundedness of $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$.

We recall the following estimates of $|f(z)|$ and $|f'(z)|$ for functions f in $\mathcal{Z}^\alpha$ (see, [1, Lemma 1.1]).

Lemma 2.1. *For every $f \in \mathcal{Z}^\alpha$ we have*

(i) $|f'(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}^\alpha}$ *and* $|f(z)| \leq \frac{2}{1-\alpha}\|f\|_{\mathcal{Z}^\alpha}$ *for* $0 < \alpha < 1$,

(ii) $|f'(z)| \leq 2\|f\|_{\mathcal{Z}} \log\frac{2}{1-|z|}$ *and* $|f(z)| \leq \|f\|_{\mathcal{Z}}$ *for* $\alpha = 1$,

(iii) $|f'(z)| \leq \frac{2}{\alpha-1}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|)^{\alpha-1}}$, *for* $1 < \alpha < \infty$,

(iv) $|f(z)| \leq \frac{2}{(\alpha-1)(2-\alpha)}\|f\|_{\mathcal{Z}^\alpha}$, *for* $1 < \alpha < 2$,

(v) $|f(z)| \leq 2\|f\|_{\mathcal{Z}^2} \log\frac{2}{1-|z|}$, *for* $\alpha = 2$,

(vi) $|f(z)| \leq \frac{2}{(\alpha-1)(\alpha-2)}\frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|)^{\alpha-2}}$, *for* $2 < \alpha < \infty$.

Before stating next theorems we note that if $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ is a bounded operator, then $u = D^1_{\varphi,u}(1) \in \mathcal{B}^\beta$ and also, since $u\varphi = D^1_{\varphi,u}(\frac{z^2}{2}) \in \mathcal{B}^\beta$, one can see that $u\varphi' \in H^\infty_{v_\beta}$. This fact will be severally used in the proof of theorems in this section.

Theorem 2.2. *Let $0 < \alpha < 1$ and $0 < \beta < \infty$. Then, $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if $u \in \mathcal{B}^\beta$ and*

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|u(z)\varphi'(z)| < \infty. \tag{1}$$

Proof. For $f \in \mathcal{Z}^\alpha$ using Lemma 2.1(i) we have

$$\begin{aligned}|(D^1_{\varphi,u}f)'(z)| \le& |u'(z)f'(\varphi(z))| + |u(z)\varphi'(z)f''(\varphi(z))| \\ \le& \frac{2|u'(z)|}{1-\alpha}\|f\|_{\mathcal{Z}^\alpha} + \frac{|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^\alpha}\|f\|_{\mathcal{Z}^\alpha}.\end{aligned}$$

So, if $u \in \mathcal{B}^\beta$ and (1) holds, then $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ is bounded. Conversely, let $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ be a bounded operator. For each nonzero $a \in \mathbb{D}$ consider the test function $g_a(z) = f_a(z) - h_a(z)$ for all $z \in \mathbb{D}$, where

$$\begin{aligned}f_a(z) =& \frac{1}{\bar{a}^2}\left(\frac{(1-|a|^2)^2}{(1-\bar{a}z)^\alpha} - \frac{1-|a|^2}{(1-\bar{a}z)^{\alpha-1}}\right), \\ h_a(z) =& \frac{1}{\bar{a}}\int_0^z \frac{1-|a|^2}{(1-\bar{a}w)^\alpha}dw.\end{aligned}$$

It is known that for $\varphi(a) \neq 0$, $g_{\varphi(a)} \in \mathcal{Z}^\alpha$, $g'_{\varphi(a)}(\varphi(a)) = 0$, $g''_{\varphi(a)}(\varphi(a)) = \frac{\alpha}{(1-|\varphi(a)|^2)^\alpha}$ and $\sup_{\frac{1}{2}<|\varphi(a)|<1}\|g_{\varphi(a)}\|_{\mathcal{Z}^\alpha} < \infty$. Hence,

$$\begin{aligned}\|D^1_{\varphi,u}g_{\varphi(a)}\|_{\mathcal{B}^\beta} \ge& (1-|a|^2)^\beta|u(a)\varphi'(a)g''_{\varphi(a)}(\varphi(a))| - (1-|a|^2)^\beta|u'(a)g'_{\varphi(a)}(\varphi(a))| \\ =& \frac{\alpha(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^\alpha}|u(a)\varphi'(a)|,\end{aligned}$$

and therefore

$$\sup_{1/2<|\varphi(a)|<1} \frac{\alpha(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^\alpha}|u(a)\varphi'(a)| \le \sup_{1/2<|\varphi(a)|<1} \|D^1_{\varphi,u}g_{\varphi(a)}\|_{\mathcal{B}^\beta} < \infty.$$

On the other hand, since $u\varphi' \in H^\infty_{v_\beta}$, we have

$$\sup_{|\varphi(a)|\le 1/2} \frac{\alpha(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^\alpha}|u(a)\varphi'(a)| < \infty,$$

which completes the proof. □

Theorem 2.3. *If $0 < \beta < \infty$, then $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{B}^\beta$ is bounded if and only if*

(i) $\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u'(z)|\log\frac{2}{1-|\varphi(z)|^2} < \infty$,

(ii) $\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}|u(z)\varphi'(z)| < \infty$.

Proof. If conditions (i) and (ii) hold, then Lemma 2.1(ii) implies that for every $f \in \mathcal{Z}^\alpha$

$$\begin{aligned}|(D^1_{\varphi,u}f)'(z)| =& |u'(z)f'(\varphi(z)) + u(z)\varphi'(z)f''(\varphi(z))| \\ \le& |u'(z)f'(\varphi(z))| + |u(z)\varphi'(z)f''(\varphi(z))| \\ \le& |u'(z)|\|f\|_{\mathcal{Z}}\log\frac{2}{1-|\varphi(z)|^2} + \frac{|u(z)\varphi'(z)|}{1-|\varphi(z)|^2}\|f\|_{\mathcal{Z}}.\end{aligned}$$

So, $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{B}^\beta$ is a bounded operator. Conversely, let $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{B}^\beta$ be a bounded operator. A similar argument as in the proof of Theorem 2.2 shows that (*ii*) holds. In order to prove (*i*), for each $a \in \mathbb{D}$ satisfying $\varphi(a) \neq 0$, consider the test function

$$k_{\varphi(a)}(z) = \frac{h(\overline{\varphi(a)}z)}{\overline{\varphi(a)}}\left(\log\frac{2}{1-|\varphi(a)|^2}\right)^{-1},$$

where

$$h(z) = (z-1)\left((1+\log\frac{2}{1-z})^2+1\right),$$

for all $z \in \mathbb{D}$. It is known that $k_{\varphi(a)} \in \mathcal{Z}$, $k'_{\varphi(a)}(\varphi(a)) = \log\frac{2}{1-|\varphi(a)|^2}$, $k''_{\varphi(a)}(\varphi(a)) = \frac{2}{1-|\varphi(a)|^2}$ and $\sup_{1/2<|\varphi(a)|<1}\|k_{\varphi(a)}\|_{\mathcal{Z}} < \infty$. We also have

$$\begin{aligned}\|D^1_{\varphi,u}k_{\varphi(a)}\|_{\mathcal{B}^\beta} \geq& (1-|a|^2)^\beta|u'(a)k'_{\varphi(a)}(\varphi(a))| - (1-|a|^2)^\beta|u(a)\varphi'(a)k''_{\varphi(a)}(\varphi(a))|\\ =&(1-|a|^2)^\beta|u'(a)|\log\frac{2}{1-|\varphi(a)|^2} - \frac{2(1-|a|^2)^\beta}{1-|\varphi(a)|^2}|u(a)\varphi'(a)|.\end{aligned}$$

Therefore, boundedness of the operator $D^1_{\varphi,u} : \mathcal{Z} \to \mathcal{B}^\beta$ and (*ii*) imply that

$$\begin{aligned}&\sup_{1/2<|\varphi(a)|<1}(1-|a|^2)^\beta|u'(a)|\log\frac{2}{1-|\varphi(a)|^2}\\ \leq& \sup_{1/2<|\varphi(a)|<1}\|D^1_{\varphi,u}k_{\varphi(a)}\|_{\mathcal{B}^\beta} + \sup_{1/2<|\varphi(a)|<1}\frac{2(1-|a|^2)^\beta}{1-|\varphi(a)|^2}|u(a)\varphi'(a)|\\ <&\infty.\end{aligned}$$

Also, since $u \in \mathcal{B}^\beta$, we have

$$\sup_{|\varphi(a)|\leq 1/2}(1-|a|^2)^\beta|u'(a)|\log\frac{2}{1-|\varphi(a)|^2} < \infty,$$

which completes the proof. □

Theorem 2.4. *Let* $1 < \alpha < \infty$ *and* $0 < \beta < \infty$. *Then,* $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ *is bounded if and only if*

(*i*) $\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha-1}}|u'(z)| < \infty$,

(*ii*) $\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}}|u(z)\varphi'(z)| < \infty$.

Proof. Let $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ be a bounded operator. Then, one can prove (*ii*) as in the proof of Theorem 2.2. Also, (*i*) can be proved similar to Theorem 2.3(*i*) and using the test function $f_{\varphi(a)}$ defined in Theorem 2.2. Note that $f'_{\varphi(a)}(\varphi(a)) = \frac{1}{\overline{\varphi(a)}}(1-|\varphi(a)|^2)^{1-\alpha}$, $f''_{\varphi(a)}(\varphi(a)) = \frac{2\alpha}{(1-|\varphi(a)|^2)^\alpha}$ and also $\sup_{\frac{1}{2}<|\varphi(a)|<1}\|f_{\varphi(a)}\|_{\mathcal{Z}^\alpha} < \infty$.

Finally, a similar argument as in the proof of Theorem 2.3 shows that if (*i*) and (*ii*) hold, then $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ is a bounded operator. □

As mentioned in the proof of [23, Proposition 8], for each $k \geq 2$, $0 < \alpha < \infty$ and $f \in \mathcal{B}^\alpha$ we have

$$|f^{(k)}(z)| \leq \frac{\|f\|_{\mathcal{B}^\alpha}}{(1-|z|^2)^{\alpha+k-1}}, \quad z \in \mathbb{D}. \tag{2}$$

Applying (2), for each $k \geq 2$, $0 < \alpha < \infty$ and $f \in \mathcal{Z}^\alpha$ we get

$$|f^{(k+1)}(z)| \leq \frac{\|f\|_{\mathcal{Z}^\alpha}}{(1-|z|^2)^{\alpha+k-1}}, \quad z \in \mathbb{D}. \tag{3}$$

In the next theorem, we give necessary and sufficient conditions for the boundedness of $D^k_{\varphi,u}:\mathcal{Z}^\alpha\to\mathcal{B}^\beta$ for each $k\geq 2$. Note that using the estimate of $|f^{(k)}(z)|$, given in (3), leads to the proof of next theorem in all cases of $0<\alpha,\beta<\infty$.

Theorem 2.5. *Let $0<\alpha,\beta<\infty$ and $k\geq 2$. Then, $D^k_{\varphi,u}:\mathcal{Z}^\alpha\to\mathcal{B}^\beta$ is bounded if and only if*

(i) $\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+k-2}}|u'(z)|<\infty$,

(ii) $\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+k-1}}|u(z)\varphi'(z)|<\infty$.

Proof. Suppose that the operator $D^k_{\varphi,u}:\mathcal{Z}^\alpha\to\mathcal{B}^\beta$ is bounded. For every nonzero $a\in\mathbb{D}$ define the test function $t_a\in\mathcal{Z}^\alpha$ by

$$t_a(z)=\frac{1}{\bar{a}^{k+1}}\left(\frac{(\alpha-1)(1-|a|^2)^2}{(1-\bar{a}z)^\alpha}-\frac{(\alpha+k-1)(1-|a|^2)}{(1-\bar{a}z)^{\alpha-1}}\right).$$

Then, we have $\sup_{\frac{1}{2}<|a|<1}\|t_a\|_{\mathcal{Z}^\alpha}<\infty$, $t_a^{(k)}(a)=0$ and

$$t_a^{(k+1)}(a)=\frac{(\alpha-1)\alpha(\alpha+1)\cdots(\alpha+k-1)}{(1-|a|^2)^{\alpha+k-1}}.$$

Therefore,

$$\begin{aligned}\|D^k_{\varphi,u}t_{\varphi(a)}\|_{\mathcal{B}^\beta}\geq&(1-|a|^2)^\beta|u(a)\varphi'(a)t^{(k+1)}_{\varphi(a)}(\varphi(a))|-(1-|a|^2)^\beta|u'(a)t^{(k)}_{\varphi(a)}(\varphi(a))|\\=&\frac{|\alpha-1|\alpha(\alpha+1)\cdots(\alpha+k-1)(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^{\alpha+k-1}}|u(a)\varphi'(a)|,\end{aligned}$$

and hence

$$\sup_{1/2<|\varphi(a)|<1}\frac{(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^{\alpha+k-1}}|u(a)\varphi'(a)|\leq\sup_{1/2<|\varphi(a)|<1}\|D^k_{\varphi,u}t_{\varphi(a)}\|_{\mathcal{B}^\beta}<\infty.$$

On the other hand, since $u\varphi'\in H^\infty_{\nu_\beta}$, we have

$$\sup_{|\varphi(a)|\leq1/2}\frac{(1-|a|^2)^\beta}{(1-|\varphi(a)|^2)^{\alpha+k-1}}|u(a)\varphi'(a)|<\infty,$$

which completes the proof of (*ii*). One can also prove (*i*) by a similar argument as in the proof of (*ii*) and using the test function $s_a\in\mathcal{Z}^\alpha$, for nonzero $a\in\mathbb{D}$, given by

$$s_a(z)=\frac{1}{\bar{a}^{k}}\left(\frac{(\alpha-1)(1-|a|^2)^2}{(1-\bar{a}z)^\alpha}-\frac{(\alpha+k)(1-|a|^2)}{(1-\bar{a}z)^{\alpha-1}}\right).$$

Note that $\sup_{\frac{1}{2}<|a|<1}\|s_a\|_{\mathcal{Z}^\alpha}<\infty$, $s_a^{(k+1)}(a)=0$ and

$$s_a^{(k)}(a)=\frac{-(\alpha-1)\alpha(\alpha+1)\cdots(\alpha+k-2)}{(1-|a|^2)^{\alpha+k-2}}.$$

Now suppose that (*i*) and (*ii*) hold. Then, for every $f\in\mathcal{Z}^\alpha$, using (3), we have

$$\begin{aligned}|(D^k_{\varphi,u}f)'(z)|\leq&|u'(z)f^{(k)}(\varphi(z))|+|u(z)\varphi'(z)f^{(k+1)}(\varphi(z))|\\\leq&\frac{|u'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k-2}}\|f\|_{\mathcal{Z}^\alpha}+\frac{|u(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\alpha+k-1}}\|f\|_{\mathcal{Z}^\alpha}.\end{aligned}$$

Multiplying both sides by $(1-|z|^2)^\beta$ and taking supremum over $z\in\mathbb{D}$, implies boundedness of $D^k_{\varphi,u}:\mathcal{Z}^\alpha\to\mathcal{B}^\beta$. □

3. Essential Norms

In this section we give estimates for the essential norm of weighted differentiation composition operators $D^k_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ in different cases of $k \in \mathbb{N}$ and $0 < \alpha, \beta < \infty$.

For each $0 < \alpha < \infty$, let $D_\alpha : \mathcal{Z}^\alpha \to \mathcal{B}^\alpha$ and $S_\alpha : \mathcal{B}^\alpha \to H^\infty_{\nu_\alpha}$ denote derivative operators. Then, D_α and S_α are linear isometries on $\widetilde{\mathcal{Z}}^\alpha = \{f \in \mathcal{Z}^\alpha : f(0) = f'(0) = 0\}$ and $\widetilde{\mathcal{B}}^\alpha = \{f \in \mathcal{B}^\alpha : f(0) = 0\}$, respectively. Moreover,

$$S_\beta D^k_{\varphi,u} D^{-1}_\alpha S^{-1}_\alpha = D^k_{\varphi,u'} D^{-1}_\alpha S^{-1}_\alpha + D^k_{\varphi,u\varphi'} S^{-1}_\alpha,$$

which implies that

$$\|D^k_{\varphi,u}\|_{e,\widetilde{\mathcal{Z}}^\alpha\to\mathcal{B}^\beta} \leq \|D^k_{\varphi,u'}\|_{e,\widetilde{\mathcal{Z}}^\alpha\to H^\infty_{\nu_\beta}} + \|D^k_{\varphi,u\varphi'}\|_{e,\widetilde{\mathcal{B}}^\alpha\to H^\infty_{\nu_\beta}}. \tag{4}$$

For any bounded operator $T : \mathcal{B}^\alpha \to H^\infty_{\nu_\beta}$, the operator $f \mapsto T(f(0)1)$ is a compact operator. Similarly, for any bounded operator $T : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ or $T : \mathcal{Z}^\alpha \to H^\infty_{\nu_\beta}$, the operator $f \mapsto T(f(0)1 + f'(0)z)$ is compact. Applying these operators, one can see that $\|D^k_{\varphi,u}\|_{e,\widetilde{\mathcal{B}}^\alpha\to H^\infty_{\nu_\beta}} = \|D^k_{\varphi,u}\|_{e,\mathcal{B}^\alpha\to H^\infty_{\nu_\beta}}$, $\|D^k_{\varphi,u}\|_{e,\widetilde{\mathcal{Z}}^\alpha\to\mathcal{B}^\beta} = \|D^k_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta}$ and $\|D^k_{\varphi,u}\|_{e,\widetilde{\mathcal{Z}}^\alpha\to H^\infty_{\nu_\beta}} = \|D^k_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to H^\infty_{\nu_\beta}}$ (see [17] for a similar approach). Therefore, (4) implies that

$$\|D^k_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta} \leq \|D^k_{\varphi,u'}\|_{e,\mathcal{Z}^\alpha\to H^\infty_{\nu_\beta}} + \|D^k_{\varphi,u\varphi'}\|_{e,\mathcal{B}^\alpha\to H^\infty_{\nu_\beta}}. \tag{5}$$

Before stating main results, we prove the following lemma which is an analogue of [22, Lemma 4.2].

Lemma 3.1. *Let $0 < \alpha < 1$ and (f_n) be a bounded sequence in $\mathcal{Z}^\alpha$ which converges to zero on compact subsets of $\mathbb{D}$. Then, $\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f'_n(z)| = 0$.*

Proof. Suppose that $\varepsilon > 0$ and choose $0 < t < 1$ such that $(1 - t)^{1-\alpha} < \varepsilon$. Then, for each $z \in \mathbb{D}$ with $t < |z| < 1$ we have

$$\begin{aligned}|f'_n(z) - f'_n(\frac{t}{|z|}z)| =& |\int_{t/|z|}^1 z f''_n(zt)dt| \leq c \int_{t/|z|}^1 \frac{|z|}{(1 - |tz|^2)^\alpha} dt \\ \leq& c \int_{t/|z|}^1 \frac{1}{(1 - t)^\alpha} dt \leq \frac{c}{1 - \alpha}(1 - t)^{1-\alpha} < \frac{c}{1 - \alpha}\varepsilon,\end{aligned}$$

where $c = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{Z}^\alpha}$. Therefore,

$$\sup_{t<|z|<1} |f'_n(z)| \leq \frac{c}{1 - \alpha}\varepsilon + \sup_{|z|=t} |f'_n(z)|.$$

Since (f'_n) also converges to zero uniformly on compact subsets of $\mathbb{D}$, we conclude

$$\begin{aligned}\lim_{n\to\infty} \sup_{z\in\mathbb{D}} |f'_n(z)| \leq& \lim_{n\to\infty} \sup_{t<|z|<1} |f'_n(z)| + \lim_{n\to\infty} \sup_{|z|\leq t} |f'_n(z)| \\ \leq& \frac{c}{1 - \alpha}\varepsilon + \lim_{n\to\infty} \sup_{|z|=t} |f'_n(z)| + \lim_{n\to\infty} \sup_{|z|\leq t} |f'_n(z)| \\ =& \frac{c}{1 - \alpha}\varepsilon,\end{aligned}$$

which completes the proof. □

Regarding (5), in order to give upper estimates for the essential norm of $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$, in the next theorem we give essential norm of $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to H^\infty_\nu$.

Theorem 3.2. *Let ν be a radial and non-increasing weight tending to zero at the boundary of $\mathbb{D}$, $0 < \alpha < \infty$ and $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to H^\infty_\nu$ be a bounded operator.*

(i) If $0 < \alpha < 1$*, then* $D^1_{\varphi,u}$ *is a compact operator.*

(ii)

$$\|D^1_{\varphi,u}\|_{e,\mathcal{Z}\to H^\infty_\nu} \asymp \limsup_{|\varphi(z)|\to 1} \nu(z)|u(z)| \log \frac{2}{1-|\varphi(z)|^2}.$$

(iii) If $1 < \alpha < \infty$*, then*

$$\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to H^\infty_\nu} \asymp \limsup_{|\varphi(z)|\to 1} \frac{\nu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}.$$

Proof. Let $0 < \alpha < 1$ and (f_n) be a bounded sequence in $\mathcal{Z}^\alpha$. Then, (f_n) has a subsequence, say (f_n), which converges uniformly on compact subsets of $\mathbb{D}$. Note that, for each $n, k \in \mathbb{N}$, we have

$$\begin{aligned}\|D^1_{\varphi,u}(f_n - f_k)\|_\nu &= \sup_{z\in\mathbb{D}} \nu(z)|u(z)(f_n - f_k)'(\varphi(z))| \\ &\le \|u\|_\nu \sup_{z\in\mathbb{D}} |(f_n' - f_k')(\varphi(z))|.\end{aligned}$$

Therefore, by applying Lemma 3.1, one can see that $(D^1_{\varphi,u} f_n)$ is a Cauchy and hence convergent sequence in H^∞_ν. This implies compactness of the operator $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to H^\infty_\nu$.

Now assume that $1 < \alpha < \infty$. Fix $\delta \in (0,1)$ and let (r_m) be an increasing sequence in $(0,1)$ converging to 1. Then, $D^1_{r_m\varphi,u} : \mathcal{Z}^\alpha \to H^\infty_\nu$ is a compact operator for each $m \in \mathbb{N}$. Indeed, if (f_n) is a bounded sequence in $\mathcal{Z}^\alpha$, then it has a subsequence, say (f_n), which converges uniformly on compact subsets of $\mathbb{D}$. On the other hand, for each $n, k \in \mathbb{N}$, we have

$$\begin{aligned}\|D^1_{r_m\varphi,u}(f_n - f_k)\|_\nu &= \sup_{z\in\mathbb{D}} \nu(z)|u(z)(f_n - f_k)'(r_m\varphi(z))| \\ &\le \|u\|_\nu \sup_{|z|\le r_m} |(f_n - f_k)'(z)|.\end{aligned}$$

This shows that $(D^1_{r_m\varphi,u} f_n)$ is a Cauchy and hence convergent sequence in H^∞_ν implying that the operator $D^1_{r_m\varphi,u}$ is compact. Therefore,

$$\begin{aligned}\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to H^\infty_\nu} &\le \|D^1_{\varphi,u} - D^1_{r_m\varphi,u}\|_{\mathcal{Z}^\alpha\to H^\infty_\nu} \\ &= \sup_{\|f\|_{\mathcal{Z}^\alpha}\le 1} \sup_{z\in\mathbb{D}} \nu(z)|u(z)||f'(\varphi(z)) - f'(r_m\varphi(z))| \\ &\le \sup_{\|f\|_{\mathcal{Z}^\alpha}\le 1} \sup_{|\varphi(z)|<\delta} \nu(z)|u(z)||f'(\varphi(z)) - f'(r_m\varphi(z))| + \sup_{\|f\|_{\mathcal{Z}^\alpha}\le 1} \sup_{|\varphi(z)|\ge\delta} \nu(z)|u(z)||f'(\varphi(z)) - f'(r_m\varphi(z))| \\ &= I + J.\end{aligned}$$

About the term I we have

$$\begin{aligned}|f'(\varphi(z)) - f'(r_m\varphi(z))| &\le \int_{r_m}^1 |\varphi(z)||f''(t\varphi(z))|dt \\ &\le \|f\|_{\mathcal{Z}^\alpha} \int_{r_m}^1 \frac{|\varphi(z)|}{(1-t^2|\varphi(z)|^2)^\alpha} dt \\ &\le \frac{|\varphi(z)|}{(1-|\varphi(z)|^2)^\alpha} \|f\|_{\mathcal{Z}^\alpha}(1-r_m) \\ &\le \frac{\delta}{(1-\delta^2)^\alpha} \|f\|_{\mathcal{Z}^\alpha}(1-r_m),\end{aligned}$$

which tends to zero as $m \to \infty$. Also, about the term J, we have

$$\begin{aligned}
|f'(\varphi(z)) - f'(r_m\varphi(z))| \leq & \int_{r_m}^{1} |\varphi(z)||f''(t\varphi(z))|dt \\
\leq & \|f\|_{\mathcal{Z}^\alpha} \int_{r_m}^{1} \frac{|\varphi(z)|}{(1-t^2|\varphi(z)|^2)^\alpha} dt \\
\leq & \int_{r_m}^{1} \frac{|\varphi(z)|}{(1-t|\varphi(z)|)^\alpha} dt \\
= & \frac{1}{\alpha-1}\left(\frac{1}{(1-|\varphi(z)|)^{\alpha-1}} - \frac{1}{(1-r_m|\varphi(z)|)^{\alpha-1}}\right) \\
\leq & \frac{1}{\alpha-1}\frac{1}{(1-|\varphi(z)|)^{\alpha-1}}.
\end{aligned}$$

Letting $\delta \to 1$ we get

$$J \leq \limsup_{|\varphi(z)|\to 1} \frac{1}{\alpha-1}\frac{\nu(z)|u(z)|}{(1-|\varphi(z)|)^{\alpha-1}},$$

implying that

$$\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to H^\infty_\nu} \leq \limsup_{|\varphi(z)|\to 1} \frac{1}{\alpha-1}\frac{\nu(z)|u(z)|}{(1-|\varphi(z)|^2)^{\alpha-1}}.$$

Now we prove the lower estimate. Let (z_n) be a sequence in $\mathbb{D}$ with $1/2 < |\varphi(z_n)| < 1$ and $|\varphi(z_n)| \to 1$. Consider the sequence of test functions (g_n) defined by

$$g_n(z) = \frac{(1-|\varphi(z_n)|^2)^2}{\alpha\overline{\varphi(z_n)}(1-\overline{\varphi(z_n)}z)^\alpha}.$$

Then (g_n) is a bounded sequence in $\mathcal{Z}^\alpha_0$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. This implies weak convergence of (g_n) to zero in $\mathcal{Z}^\alpha$. Therefore, by letting $c = \sup_{n\in\mathbb{N}} \|g_n\|_{\mathcal{Z}^\alpha}$ and using $g_n'(\varphi(z_n)) = \frac{1}{(1-|\varphi(z_n)|^2)^{\alpha-1}}$, we have

$$\begin{aligned}
c\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to H^\infty_\nu} \geq & \limsup_{n\to\infty} \|D^1_{\varphi,u}g_n\|_{H^\infty_\nu} \\
\geq & \limsup_{n\to\infty} \nu(z_n)|u(z_n)||g_n'(\varphi(z_n))| \\
= & \limsup_{n\to\infty} \frac{\nu(z_n)|u(z_n)|}{(1-|\varphi(z_n)|^2)^{\alpha-1}}.
\end{aligned}$$

This completes the proof of (*iii*).

Finally, we prove (*ii*). The upper estimate in this case can be obtained as in the previous case. For the lower estimate, let (z_n) be a sequence in $\mathbb{D}$ with $1/2 < |\varphi(z_n)| < 1$ and $|\varphi(z_n)| \to 1$. Consider sequence of test functions $k_n = k_{\varphi(z_n)}$ defined in Theorem 2.3. Then, (k_n) is a bounded sequence in $\mathcal{Z}_0$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ and $k_n'(\varphi(z_n)) = \log\frac{2}{1-|\varphi(z_n)|^2}$. Hence, like (*iii*), by letting $c = \sup_{n\in\mathbb{N}} \|k_n\|_{\mathcal{Z}}$, we get

$$\begin{aligned}
c\|D^1_{\varphi,u}\|_{e,\mathcal{Z}\to H^\infty_\nu} \geq & \limsup_{n\to\infty} \|D^1_{\varphi,u}k_n\|_{H^\infty_\nu} \\
\geq & \limsup_{n\to\infty} \nu(z_n)|u(z_n)||k_n'(\varphi(z_n))| \\
= & \limsup_{n\to\infty} \nu(z_n)|u(z_n)|\log\frac{2}{1-|\varphi(z_n)|^2},
\end{aligned}$$

which completes the proof. $\square$

In the rest of this section, in order to simplify the notation in the statement of our results, we use the following simplifications (see, for example, [12]):

$$A(u,\varphi,\alpha,\beta) = \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}|u(z)|,$$

$$B(u,\varphi,\beta) = \limsup_{|\varphi(z)|\to 1}(1-|z|^2)^\beta |u(z)| \log\frac{2}{1-|\varphi(z)|^2}.$$

Theorem 3.3. *Let* $0<\alpha,\beta<\infty$ *and* $D^1_{\varphi,u}:\mathcal{Z}^\alpha\to\mathcal{B}^\beta$ *be a bounded operator.*

(i) If $0<\alpha<1$*, then*

$$\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta} = 0.$$

(ii) If $\alpha=1$*, then*

$$\|D^1_{\varphi,u}\|_{e,\mathcal{Z}\to\mathcal{B}^\beta} \asymp \max\Big\{A(u\varphi',\varphi,1,\beta), B(u',\varphi,\beta)\Big\}.$$

Proof. Since the operator $D^1_{\varphi,u}:\mathcal{Z}^\alpha\to\mathcal{B}^\beta$ is bounded, $u\in\mathcal{B}^\beta$ and therefore by Lemma 2.1(*i*) the operator $D^1_{\varphi,u'}:\mathcal{Z}^\alpha\to H^\infty_{v_\beta}$ is bounded. Theorem 3.2(*i*) implies that $\|D^1_{\varphi,u'}\|_{e,\mathcal{Z}^\alpha\to H^\infty_{v_\beta}}=0$. On the other hand, using a similar argument as in the proof of Theorem 3.2(*i*) one can see that the operator $D^1_{\varphi,u\varphi'}:\mathcal{B}^\alpha\to H^\infty_{v_\beta}$ is compact and hence $\|D^1_{\varphi,u\varphi'}\|_{e,\mathcal{B}^\alpha\to H^\infty_{v_\beta}}=0$. Therefore, by applying (5) we get $\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta}=0$.

In order to prove (*ii*), let (z_n) be a sequence in $\mathbb{D}$ with $1/2<|\varphi(z_n)|<1$ and $|\varphi(z_n)|\to 1$. Consider the sequence $g_n=g_{\varphi(z_n)}$ defined in Theorem 2.2. Then, (g_n) is a bounded sequence in $\mathcal{Z}_0$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $g_n'(\varphi(z_n))=0$ and $g_n''(\varphi(z_n))=\frac{1}{1-|\varphi(z_n)|^2}$. So, by letting $c_1=\sup_{n\in\mathbb{N}}\|g_n\|_{\mathcal{Z}}$ we have

$$\begin{aligned} c_1\|D^1_{\varphi,u}\|_{e,\mathcal{Z}\to\mathcal{B}^\beta} &\geq \limsup_{n\to\infty}\|D^1_{\varphi,u}g_n\|_{\mathcal{B}^\beta}\\ &\geq \limsup_{n\to\infty}(1-|z_n|^2)^\beta|u(z_n)\varphi'(z_n)||g_n''(\varphi(z_n))| - \limsup_{n\to\infty}(1-|z_n|^2)^\beta|u'(z_n)||g_n'(\varphi(z_n))|\\ &= \limsup_{n\to\infty}|u(z_n)\varphi'(z_n)|\frac{(1-|z_n|^2)^\beta}{1-|\varphi(z_n)|^2}. \end{aligned} \tag{6}$$

Now, consider the sequence $k_n=k_{\varphi(z_n)}$ defined in Theorem 2.3. Then, (k_n) is a bounded sequence in $\mathcal{Z}_0$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $k_n'(\varphi(z_n))=\log\frac{2}{1-|\varphi(z_n)|^2}$ and $k_n''(\varphi(z_n))=\frac{2\overline{\varphi(z_n)}}{1-|\varphi(z_n)|^2}$. Let $c_2=\sup_{n\in\mathbb{N}}\|k_n\|_{\mathcal{Z}}$, then

$$\begin{aligned} c_2\|D^1_{\varphi,u}\|_{e,\mathcal{Z}\to\mathcal{B}^\beta} &\geq \limsup_{n\to\infty}\|D^1_{\varphi,u}k_n\|_{\mathcal{B}^\beta}\\ &\geq \limsup_{n\to\infty}(1-|z_n|^2)^\beta|u'(z_n)||k_n'(\varphi(z_n))| - \limsup_{n\to\infty}(1-|z_n|^2)^\beta|u(z_n)\varphi'(z_n)||k_n''(\varphi(z_n))|\\ &= \limsup_{n\to\infty}(1-|z_n|^2)^\beta|u'(z_n)|\log\frac{2}{1-|\varphi(z_n)|^2} - \limsup_{n\to\infty}(1-|z_n|^2)^\beta|u(z_n)\varphi'(z_n)|\frac{2|\overline{\varphi(z_n)}|}{1-|\varphi(z_n)|^2}. \end{aligned}$$

This along with (6) implies that

$$\limsup_{n\to\infty}(1-|z_n|^2)^\beta|u'(z_n)|\log\frac{2}{1-|\varphi(z_n)|^2} \leq (c_2+2c_1)\|D^1_{\varphi,u}\|_{e,\mathcal{Z}\to\mathcal{B}^\beta},$$

and completes the proof of lower estimate.

To prove the upper estimate, fix $\delta \in (0,1)$ and let (r_m) be an increasing sequence in $(0,1)$ converging to 1. Then, $D^1_{r_m\varphi,u} : \mathcal{B} \to H^\infty_{v_\beta}$ is a compact operator, for each $m \in \mathbb{N}$, and therefore

$$\begin{aligned}
\|D^1_{\varphi,u}\|_{e,\mathcal{B}\to H^\infty_{v_\beta}} \leq& \|D^1_{\varphi,u} - D^1_{r_m\varphi,u}\|_{\mathcal{B}\to H^\infty_{v_\beta}} \\
=& \sup_{\|f\|_{\mathcal{B}}\leq 1} \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |u(z)||f'(\varphi(z)) - f'(r_m\varphi(z))| \\
\leq& \sup_{\|f\|_{\mathcal{B}}\leq 1} \sup_{|\varphi(z)|<\delta} (1-|z|^2)^\beta |u(z)||f'(\varphi(z)) - f'(r_m\varphi(z))| \\
&+ \sup_{\|f\|_{\mathcal{B}}\leq 1} \sup_{|\varphi(z)|\geq\delta} (1-|z|^2)^\beta |u(z)||f'(\varphi(z)) - f'(r_m\varphi(z))| \\
=& I + J.
\end{aligned}$$

Using (2) and applying a similar argument as in the proof of Theorem 3.2(*iii*) implies that

$$\|D^1_{\varphi,u}\|_{e,\mathcal{B}\to H^\infty_{v_\beta}} \leq \limsup_{|\varphi(z)|\to 1} |u(z)| \frac{(1-|z|^2)^\beta}{1-|\varphi(z)|^2}.$$

Consequently, Theorem 3.2(*ii*) along with (5) imply the desired upper estimate. □

Theorem 3.4. *Let $1 < \alpha < \infty$, $0 < \beta < \infty$ and $D^1_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$ be a bounded operator. Then,*

$$\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta} \asymp \max\Big\{A(u\varphi', \varphi, \alpha, \beta), A(u', \varphi, \alpha-1, \beta)\Big\}.$$

Proof. First we prove the lower estimate. Let (z_n) be a sequence in $\mathbb{D}$ with $1/2 < |\varphi(z_n)| < 1$ and $|\varphi(z_n)| \to 1$. Then, by considering the sequence (g_n) defined in Theorem 2.2, as in the proof of (6), we get

$$c_1\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta} \geq \limsup_{n\to\infty} \frac{\alpha(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} |u(z_n)\varphi'(z_n)|, \tag{7}$$

where $c_1 = \sup_{n\in\mathbb{N}} \|g_n\|_{\mathcal{Z}^\alpha}$.

Next consider the sequence $f_n = f_{\varphi(z_n)}$ defined in Theorem 2.2. Then (f_n) is a bounded sequence in $\mathcal{Z}^\alpha_0$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $f_n'(\varphi(z_n)) = \frac{1}{\overline{\varphi(z_n)}(1-|\varphi(z_n)|^2)^{\alpha-1}}$ and $f_n''(\varphi(z_n)) = \frac{2\alpha}{(1-|\varphi(z_n)|^2)^\alpha}$. So, by letting $c_2 = \sup_{n\in\mathbb{N}} \|f_n\|_{\mathcal{Z}^\alpha}$, we have

$$\begin{aligned}
c_2\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta} \geq& \limsup_{n\to\infty} \|D^1_{\varphi,u} f_n\|_{\mathcal{B}^\beta} \\
\geq& \limsup_{n\to\infty} (1-|z_n|^2)^\beta |u'(z_n)||f_n'(\varphi(z_n))| - \limsup_{n\to\infty} (1-|z_n|^2)^\beta |u(z_n)\varphi'(z_n)||f_n''(\varphi(z_n))| \\
=& \limsup_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{|\varphi(z_n)|(1-|\varphi(z_n)|^2)^{\alpha-1}} |u'(z_n)| - 2\alpha \limsup_{n\to\infty} \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^\alpha} |u(z_n)\varphi'(z_n)|.
\end{aligned}$$

Consequently, applying (7), we have

$$\begin{aligned}
\limsup_{n\to\infty} |u'(z_n)| \frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha-1}} \leq& \limsup_{n\to\infty} |u'(z_n)| \frac{(1-|z_n|^2)^\beta}{|\varphi(z_n)|(1-|\varphi(z_n)|^2)^{\alpha-1}} \\
\leq& (c_2 + 2c_1)\|D^1_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta}.
\end{aligned}$$

In order to prove the upper estimate, fix $\delta \in (0,1)$ and let (r_m) be an increasing sequence in $(0,1)$ converging to 1. Then, $D^1_{r_m\varphi,u}:\mathcal{B}^\alpha\to H^\infty_{v_\beta}$ is a compact operator for each $m\in\mathbb{N}$, and therefore

$$\begin{aligned}\|D^1_{\varphi,u}\|_{e,\mathcal{B}^\alpha\to H^\infty_{v_\beta}} \le&\|D^1_{\varphi,u}-D^1_{r_m\varphi,u}\|_{\mathcal{B}^\alpha\to H^\infty_{v_\beta}}\\ =&\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|u(z)||f'(\varphi(z))-f'(r_m\varphi(z))|\\ \le&\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{|\varphi(z)|<\delta}(1-|z|^2)^\beta|u(z)||f'(\varphi(z))-f'(r_m\varphi(z))|\\ &+\sup_{\|f\|_{\mathcal{B}^\alpha}\le 1}\sup_{|\varphi(z)|\ge\delta}(1-|z|^2)^\beta|u(z)||f'(\varphi(z))-f'(r_m\varphi(z))|\\ =&I+J.\end{aligned}$$

By a similar argument as in the proof of Theorem 3.2(*iii*) and using (2), one can see that

$$\|D^1_{\varphi,u}\|_{e,\mathcal{B}^\alpha\to H^\infty_{v_\beta}}\le\limsup_{|\varphi(z)|\to 1}\frac{1}{\alpha}|u(z)|\frac{(1-|z|^2)^\beta}{(1-|\varphi(z)|)^\alpha}.$$

Consequently, applying Theorem 3.2(*iii*) and (5), we get the desired upper estimate. □

Theorem 3.5. *Let $0<\alpha,\beta<\infty$, $k\ge 2$ and $D^k_{\varphi,u}:\mathcal{Z}^\alpha\to\mathcal{B}^\beta$ be a bounded operator. Then,*

$$\|D^k_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta}\asymp\max\Big\{A(u\varphi',\varphi,\alpha+k-1,\beta),A(u',\varphi,\alpha+k-2,\beta)\Big\}.$$

Proof. Using Theorem 3.2(*iii*) and (3), the proof of upper estimate is similar to the proof of upper estimate in Theorem 3.4.

To prove the lower estimate let (z_n) be a sequence in $\mathbb{D}$ with $1/2<|\varphi(z_n)|<1$ and $|\varphi(z_n)|\to 1$. Consider the sequence $t_n=t_{\varphi(z_n)}$ given in Theorem 2.5. Indeed, (t_n) is a bounded sequence in $\mathcal{Z}^\alpha_0$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. If $c_1=\sup_{n\in\mathbb{N}}\|t_n\|_{\mathcal{Z}^\alpha}$, then

$$\begin{aligned}c_1\|D^k_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta}\ge&\limsup_{n\to\infty}\|D^k_{\varphi,u}t_n\|_{\mathcal{B}^\beta}\\ \ge&\limsup_{n\to\infty}(1-|z_n|^2)^\beta|u(z_n)\varphi'(z_n)||t_n^{(k+1)}(\varphi(z_n))|-\limsup_{n\to\infty}(1-|z_n|^2)^\beta|u'(z_n)||t_n^{(k)}(\varphi(z_n))|\\ \asymp&\limsup_{n\to\infty}\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha+k-1}}|u(z_n)\varphi'(z_n)|.\end{aligned}\tag{8}$$

Now, consider the sequence $s_n=s_{\varphi(z_n)}$ given in Theorem 2.5. Then, (s_n) is a bounded sequence in $\mathcal{Z}^\alpha_0$ which converges to zero uniformly on compact subsets of $\mathbb{D}$. Therefore, by letting $c_2=\sup_{n\in\mathbb{N}}\|s_n\|_{\mathcal{Z}^\alpha}$ we have

$$\begin{aligned}c_2\|D^k_{\varphi,u}\|_{e,\mathcal{Z}^\alpha\to\mathcal{B}^\beta}\ge&\limsup_{n\to\infty}\|D^k_{\varphi,u}s_n\|_{\mathcal{B}^\beta}\\ \ge&\limsup_{n\to\infty}(1-|z_n|^2)^\beta|u'(z_n)||s_n^{(k)}(\varphi(z_n))|-\limsup_{n\to\infty}(1-|z_n|^2)^\beta|u(z_n)\varphi'(z_n)||s_n^{(k+1)}(\varphi(z_n))|\\ \asymp&\limsup_{n\to\infty}\frac{(1-|z_n|^2)^\beta}{(1-|\varphi(z_n)|^2)^{\alpha+k-2}}|u'(z_n)|.\end{aligned}\tag{9}$$

Applying (8) and (9) we get the desired result for the lower estimate. □

Remark 3.6. *Montes-Rodríguez in [13, Theorem 2.1], and also Hyvärinen et al. in [3, Theorem 2.4], proved that if v and ω are radial and non-increasing weights tending to zero at the boundary of $\mathbb{D}$, then*

(i) the weighted composition operator uC_φ maps H^∞_v into H^∞_ω if and only if

$$\sup_{n\ge 0}\frac{\|u\varphi^n\|_\omega}{\|z^n\|_v}\asymp\sup_{z\in\mathbb{D}}\frac{\omega(z)}{\widetilde{v}(\varphi(z))}|u(z)|<\infty,$$

with norm comparable to the above supremum.

(ii) $\|uC_\varphi\|_{e,H_v^\infty\to H_\omega^\infty} = \limsup_{n\to\infty} \frac{\|u\varphi^n\|_\omega}{\|z^n\|_v} = \limsup_{|\varphi(z)|\to 1} \frac{\omega(z)}{\tilde{v}(\varphi(z))}|u(z)|$.

By applying these facts and using [4, Lemma 2.1], our results in this paper containing terms of the type $\frac{\omega(z)}{\tilde{v}(\varphi(z))}|u(z)|$ *can be restated in terms of* u *and* φ^n*. See, for example, [1, 16] for these types of results.*

Remark 3.7. *Clearly, for Banach spaces* X *and* Y*, a bounded operator* $T : X \to Y$ *is compact if and only if* $\|T\|_{e,X\to Y} = 0$*. Therefore, essential norm estimates of* $D^k_{\varphi,u} : \mathcal{Z}^\alpha \to \mathcal{B}^\beta$*, given in Section 3, lead to necessary and sufficient conditions for the compactness of such operators.*

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