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On T-slant, N-slant and B-slant Helices in Pseudo-Galilean Space $\mathbb{G}_3^1$

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Abstract. In this paper, we introduce T-slant, N-slant and B-slant helices in the pseudo-Galilean space $\mathbb{G}_3^1$ and define an angle between the spacelike and the timelike isotropic vector lying in the pseudo-Euclidean plane $x = 0$. In particular, we obtain the explicit parameter equations of the T-slant helices and prove that there are no N-slant and B-slant helices in $\mathbb{G}_3^1$. We also prove that there are no Darboux helices in the same space.

1. Introduction

In Euclidean space $\mathbb{E}^3$, a regular curve whose the tangent vector T and the principal normal vector N make a constant angle with some fixed direction, is called the *general helix* (or curve of the constant slope) and the *slant helix*, respectively. It is well-known that a regular curve α in $\mathbb{E}^3$ with the curvature $\kappa \neq 0$ and the torsion τ in $\mathbb{E}^3$ is the general helix if and only if it has constant conical curvature τ/κ. In particular, slant helices have constant geodesic curvature of the spherical image of their principal normal indicatrix ([8]). Some characterizations of the slant helices can be found in [9–11]. *Darboux helices* in $\mathbb{E}^3$ are defined in [20] as the curves whose Darboux vector makes a constant angle with some fixed direction. In Minkowski space $\mathbb{E}_1^3$, the Darboux helices are studied in [12, 16].

The general helices in Galilean space $\mathbb{G}_3$ are defined in [18] as admissible curves which have a constant conical curvature τ/κ. In particular, the general helices in $\mathbb{G}_3$ with the natural equations $\tau(x) = b/ax$ and $\kappa(x) = 1/ax$, where $a, b = constant \neq 0$ lie on a cone and have a property that they are isogonal trajectories of the cone generators ([17]). In pseudo-Galilean space $\mathbb{G}_3^1$, the general helices are defined in [4] in terms of an angle between two isotropic vectors which lie in the pseudo-Euclidean plane $x = 0$. In particular, it is proved in [4] that an admissible curve in $\mathbb{G}_3^1$ is the general helix if and only if it has constant conical curvature τ/κ. Some characterizations of the general helices can be found in [1, 3, 5, 6, 15].

In this paper, we introduce T-slant, N-slant and B-slant helices in $\mathbb{G}_3^1$ as admissible curves whose the tangent, the principal normal and the binormal vector respectively makes a constant angle with some fixed straight line (an axis of the helix). Since the notion of an angle between two vectors plays an important role in the definitions of the mentioned three kinds of slant helices, in this paper we define an angle between the

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spacelike isotropic vector and the timelike isotropic vector (the Definition 2.4 in Section 2) in the pseudo-Galilean space $\mathbb{G}_3^1$. In particular, we obtain the explicit parameter equations of the T-slant helices and prove that there are no N-slant and B-slant helices in $\mathbb{G}_3^1$. Finally, we prove that there are no Darboux helices in the same space.

2. Preliminaries

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries of projective signature $(0,0,+,-)$. The absolute of the pseudo-Galilean geometry is an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f line in w and I is the fixed hyperbolic involution of the points of f ([5]).

According to the motion group of the pseudo-Galilean space, there are *non-isotropic* vectors $\mathbf{x} = (x, y, z)$ and four types of *isotropic* vectors: *spacelike* $(x = 0, y^2 - z^2 > 0)$, *timelike* $(x = 0, y^2 - z^2 < 0)$ and two types of *lightlike* vectors $(x = 0, y = \pm z)$. A non-lightlike (spacelike or timelike) isotropic vector is called the *unit* vector, if $y^2 - z^2 = \pm 1$.

The *scalar product* of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{G}_3^1$ is given by

$$\langle \mathbf{u}, \mathbf{v} \rangle = \begin{cases} u_1 v_1 & , \text{if } u_1 \neq 0 \ \vee v_1 \neq 0; \\ u_2 v_2 - u_3 v_3 & , \text{if } u_1 = 0 \ \wedge \ v_1 = 0. \end{cases}$$

This scalar product leaves invariant the pseudo-Galilean norm of the vector $\mathbf{u} = (u_1, u_2, u_3)$ defined by

$$\|\mathbf{u}\| = \begin{cases} |u_1| & , \text{if } u_1 \neq 0; \\ \sqrt{|u_2^2 - u_3^2|} & , \text{if } u_1 = 0. \end{cases}$$

The *cross product* of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is given by

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} 0 & -e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where $e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$. The *angle measure* between two unit non-isotropic vectors is defined as the length of their difference in the following way ([5]).

Definition 2.1. *Let* $\mathbf{a} = (1, a_2, a_3)$ *and* $\mathbf{b} = (1, b_2, b_3)$ *be the unit non-isotropic vectors in general position in pseudo-Galilean space* $\mathbb{G}_3^1$. *An angle* φ *between* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by*

$$\varphi = \sqrt{|(a_2 - b_2)^2 - (a_3 - b_3)^2|}.$$

Since the spacelike and the timelike isotropic vectors lie in the pseudo-Euclidean plane of the projective signature $(0, +, -)$, i.e. in Minkowski plane with signature $(+, -)$, an angle between them is equal to the hyperbolic angle between them. For the definitions of the hyperbolic angles between two spacelike or timelike vectors, or between the spacelike and the timelike vector in Minkowski plane, see [2] (page 44), [13] and [14].

Definition 2.2. *An angle* ω *between two timelike isotropic vectors* $\mathbf{c} = (0, c_2, c_3)$ *and* $\mathbf{d} = (0, d_2, d_3)$ *in* $\mathbb{G}_3^1$ *is given by*

$$\cosh \omega = \epsilon_1 \frac{c_2 d_2 - c_3 d_3}{\sqrt{|c_2^2 - c_3^2|}\sqrt{|d_2^2 - d_3^2|}}.$$

where $\epsilon_1 = 1$ *if* $sgn(c_3) \neq sgn(d_3)$, *or* $\epsilon_1 = -1$ *if* $sgn(c_3) = sgn(d_3)$.

Definition 2.3. *An angle φ between two spacelike isotropic vectors $\mathbf{c} = (0, c_2, c_3)$ and $\mathbf{d} = (0, d_2, d_3)$ in $\mathbb{G}_3^1$ is given by*

$$\cosh \varphi = \epsilon_1 \frac{c_2 d_2 - c_3 d_3}{\sqrt{|c_2^2 - c_3^2|}\sqrt{|d_2^2 - d_3^2|}}.$$

where $\epsilon_1 = 1$ if $sgn(c_2) = sgn(d_2)$, or $\epsilon_1 = -1$ if $sgn(c_2) \neq sgn(d_2)$.

Definition 2.4. *An angle ψ between spacelike isotropic vector $\mathbf{c} = (0, c_2, c_3)$ and timelike isotropic vector $\mathbf{d} = (0, d_2, d_3)$ in $\mathbb{G}_3^1$ is given by*

$$\sinh \psi = \epsilon_1 \frac{c_2 d_2 - c_3 d_3}{\sqrt{|c_2^2 - c_3^2|}\sqrt{|d_2^2 - d_3^2|}}.$$

where $\epsilon_1 = 1$ if $sgn(c_2) = sgn(d_3)$, or $\epsilon_1 = -1$ if $sgn(c_2) \neq sgn(d_3)$.

The curve $\alpha(t) = (x(t), y(t), z(t))$ with $x, y, z \in C^3$ in pseudo-Galilean space $\mathbb{G}_3^1$ is said to be *admissible*, if $\dot{x}(t) \neq 0$ ([5]). Each admissible curve can be written as $\alpha(x) = (x, y(x), z(x))$, where in addition we assume $y''(x)^2 - z''(x)^2 \neq 0$. The arc-length parameter of α is defined by $ds = |\dot{x}(t)dt| = |dx|$. For simplicity, we assume $ds = dx$ and $s = x$ as the arc-length parameter of α.

The curvature κ and the torsion τ of $\alpha(x)$ are given by

$$\kappa(x) = \sqrt{|y''^2(x) - z''^2(x)|} \quad \text{and} \quad \tau(x) = \frac{y''(x)z'''(x) - y'''(x)z''(x)}{\kappa^2(x)}. \tag{1}$$

The Frenet frame $\{T, N, B\}$ of an admissible curve $\alpha(x) = (x, y(x), z(x))$, has the form

$$\begin{aligned} T(x) &= (1, y'(x), z'(x)), \\ N(x) &= \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \\ B(x) &= \frac{1}{\kappa(x)}(0, \epsilon z''(x), \epsilon y''(x)), \end{aligned} \tag{2}$$

where T, N and B are called the *tangent*, the *principal normal* and the *binormal vector field* of α, respectively. Here $\epsilon = 1$ or $\epsilon = -1$ is chosen by the criterion $\det(T, N, B) = 1$. This means

$$|y''^2(x) - z''^2(x)| = \epsilon\left(y''^2(x) - z''^2(x)\right). \tag{3}$$

An admissible curve $\alpha(x)$ is *timelike* or *spacelike*, if the principal normal vector N is spacelike or timelike, respectively. The Frenet equations of the curve $\alpha(x)$ are given by ([5])

$$\begin{bmatrix} T'(x) \\ N'(x) \\ B'(x) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(x) & 0 \\ 0 & 0 & \tau(x) \\ 0 & \tau(x) & 0 \end{bmatrix} \begin{bmatrix} T(x) \\ N(x) \\ B(x) \end{bmatrix}. \tag{4}$$

Also, the Frenet's frame vectors of α satisfy the equations

$$T \times N = \epsilon B, \quad N \times B = 0, \quad B \times T = -\epsilon N. \tag{5}$$

By using the relations $\alpha(x) = (x, y(x), z(x))$ and (2), we get

$$y''' = \frac{\kappa'}{\kappa} y'' + \epsilon\tau z'', \quad z''' = \frac{\kappa'}{\kappa} z'' + \epsilon\tau y''. \tag{6}$$

When the Frenet frame $\{T, N, B\}$ moves along an admissible curve α in $\mathbb{G}_3^1$, there exists an axis of the frame's rotation. The direction of such axis is given by *Darboux* vector (centrode), which has the equation

$$D(x) = \epsilon\tau(x)T(x) - \epsilon\kappa(x)B(x), \tag{7}$$

where $\epsilon = 1$ or $\epsilon = -1$, if N is spacelike or timelike, respectively. The Darboux vector satisfies *Darboux equations* given by

$$\begin{aligned} T'(x) &= D(x) \times T(x), \\ N'(x) &= D(x) \times N(x), \\ B'(x) &= D(x) \times B(x). \end{aligned}$$

Throughout the next sections, let $\mathbb{R}_0$ denote $\mathbb{R}\backslash\{0\}$.

3. T-slant, N-slant and B-slant Helices in the Pseudo-Galilean Space $\mathbb{G}_3^1$

In this section we introduce T-slant, N-slant and B-slant helices in pseudo-Galilean space $\mathbb{G}_3^1$ and obtain explicit parameter equations of the T-slant helices. We also prove that there are no N-slant and B-slant helices in $\mathbb{G}_3^1$.

Definition 3.1. *An admissible curve α in pseudo-Galilean space $\mathbb{G}_3^1$ is called T-slant helix, if its tangent vector T makes a constant angle with some non-isotropic fixed direction.*

Remark 3.2. *The notion of T-slant helices in $\mathbb{G}_3^1$ corresponds to the notion of the general helices given in reference [7] (Remark 5).*

Definition 3.3. *An admissible curve α in pseudo-Galilean space $\mathbb{G}_3^1$ is called N-slant and B-slant helix, if its principal normal and binormal vectors N and B respectively make a constant angle with some isotropic fixed direction.*

The fixed direction in the Definitions 3.1 and 3.3 is called an *axis* of the helix. We will exclude the case when the Frenet vectors T, N and B are constant, since they trivially make a constant angle with any fixed direction. Let us first characterize T-slant helices.

Theorem 3.4. *Let α be an admissible curve in $\mathbb{G}_3^1$ with the curvature $\kappa \neq 0$ and the torsion τ. Then α is T-slant helix if and only if it has a non-zero constant conical curvature τ/κ.*

Proof. Assume that an admissible curve $\alpha(x) = (x, y(x), z(x))$ is T-slant helix with non-isotropic axis spanned by the unit constant vector $U = (1, u_2, u_3)$. Then its tangent vector $T = (1, y', z')$ makes the constant angle φ with U. According to the Definition 2.1, we have

$$\varphi^2 = \left|(y' - u_2)^2 - (z' - u_3)^2\right| = c^2, \quad c \in \mathbb{R}_0^+.$$

We may consider two cases:

(i) $(y' - u_2)^2 - (z' - u_3)^2 = c^2$. Then

$$y' - u_2 = c\cosh\psi, \quad z' - u_3 = c\sinh\psi, \tag{8}$$

for some differentiable function $\psi = \psi(x)$. Differentiating the last two equations two times with respect to x, we get

$$y'' = c\psi'\sinh\psi, \quad z'' = c\psi'\cosh\psi, \tag{9}$$

$$y''' = c\,(\psi')^2\cosh\psi + c\psi''\sinh\psi, \quad z''' = c\psi''\cosh\psi + c\,(\psi')^2\sinh\psi. \tag{10}$$

Relation (9) gives $y''^2 - z''^2 < 0$, so the curve α is a spacelike. Substituting (9) and (10) in (6) and using the linear independence of the hyperbolic functions $\sinh x$ and $\cosh x$, we obtain

$$\psi'' = \frac{\kappa'}{\kappa}\psi', \quad \psi'^2 = \tau\psi'. \tag{11}$$

If $\psi' = 0$, then $y''^2 - z''^2 = 0$ which is a contradiction. Hence $\psi' \neq 0$. From the relation (11), we find

$$\psi' = \tau, \quad \frac{\kappa'}{\kappa} = \frac{\tau'}{\tau}. \tag{12}$$

Since $\psi' = \tau \neq 0$, the second equation of (12) implies $\tau/\kappa = constant \neq 0$.

(ii) $(y' - u_2)^2 - (z' - u_3)^2 = -c^2$. Then

$$y' - u_2 = c\sinh\psi, \quad z' - u_3 = c\cosh\psi, \tag{13}$$

for some differentiable function $\psi = \psi(x)$. Differentiating the last two equations two times with respect to x, analogously as in the case (i) we get that α is a timelike curve with the conical curvature $\tau/\kappa = constant \neq 0$.

Conversely, assume that an admissible curve α has the constant conical curvature τ/κ. Let us put $\tau/\kappa = -1/a$, $a \in \mathbb{R}_0$. Consider the unit non-isotropic vector U given by

$$U = T(x) + aB(x).$$

Differentiating the previous equation with respect to x and using the Frenet equations (4), we find $U' = 0$. Hence U is a fixed vector. By using the Definition 2.1, it can be easily checked that an angle φ between the vectors T and U reads $\varphi = |a| = constant$. According to the Definition 3.1, the curve α is T-slant helix. □

Corollary 3.5. *The non-isotropic axis of the general helix α is given by*

$$U = T(x) + aB(x),$$

where $a = -\frac{\kappa}{\tau} \in \mathbb{R}_0$.

In the next theorem, we obtain explicit parameter equations of T-slant helices.

Theorem 3.6. *Let α be an admissible curve in $\mathbb{G}_3^1$ with the curvature κ and the torsion $\tau \neq 0$. Then α is T-slant helix with an axis determined by the unit non-isotropic fixed vector $U = (1, u_2, u_3)$, if and only if it has parameter equation given by*

$$\begin{cases} \alpha(x) = \left(x, u_2x + c\int\cosh\psi(x)dx, u_3x + c\int\sinh\psi(x)dx\right), & \text{if } \alpha \text{ is spacelike} \\ \alpha(x) = \left(x, u_2x + c\int\sinh\psi(x)dx, u_3x + c\int\cosh\psi(x)dx\right), & \text{if } \alpha \text{ is timelike} \end{cases} \tag{14}$$

where $\psi(x) = \varepsilon\int\tau(x)\,dx + c_0$, $c_0, u_2, u_3 \in \mathbb{R}$, $\epsilon = \pm 1$ *and* $c \in \mathbb{R}_0^+$.

Proof. Assume that $\alpha(x) = (x, y(x), z(x))$ is T-slant helix with an axis determined by the unit non-isotropic vector $U = (1, u_2, u_3)$. From the relation (8), we have

$$y'(x) = u_2 + c\cosh\psi(x), \quad z'(x) = u_3 + c\sinh\psi(x),$$

where $c \in \mathbb{R}_0^+$ is the constant angle between T and U. Integrating the last two equations, we find

$$\begin{cases} y(x) = u_2x + c\int\cosh\psi(x)dx + c_1, & c_1 \in \mathbb{R}, \\ z(x) = u_3x + c\int\sinh\psi(x)dx + c_2, & c_2 \in \mathbb{R}. \end{cases} \tag{15}$$

Up to a translation, we may take $c_1 = c_2 = 0$. By using the relation (15), we get that α is a spacelike T-slant helix with parameter equation

$$\alpha(x) = \left(x, u_2 x + c \int \cosh \psi(x) dx, u_3 x + c \int \sinh \psi(x) dx\right).$$

In particular, by using the first equation in relation (12), we get

$$\psi(x) = \int \tau(x) dx + c_0, \quad c_0 \in \mathbb{R}. \tag{16}$$

Analogously, by using the relation (13) we get that α is a timelike T-slant helix parameterized by

$$\alpha(x) = \left(x, u_2 x + c \int \sinh \psi(x) dx, u_3 x + c \int \cosh \psi(x) dx\right),$$

where $\psi(x) = -\int \tau(x)\, dx + c_0$.

Conversely, if an admissible curve α has parameter equation given by (14), by using the Definition 2.1 it can be easily checked that an angle φ between T and the fixed vector $U = (1, u_2, u_3)$ is a constant. Then the Definition 3.1 implies that α is T-slant helix. □

Example 3.7. *Consider a spacelike admissible curve α in $\mathbb{G}_3^1$ with parameter equation*

$$\alpha(x) = \left(x, u_2 x + c \int \cosh \psi(x) dx, u_3 x + c \int \sinh \psi(x) dx\right),$$

where $\psi(x) = -\int \tau(x)\, dx + c_0$, $c_0, u_2, u_3 \in \mathbb{R}$ and $c \in \mathbb{R}_0^+$. Assume that α has the torsion $\tau(x) = \frac{1}{x}$ and let us put $c_0 = 0$. Then α is given by (Figure 1)

$$\alpha(x) = \left(x, u_2 x + \frac{c}{4}\left(2\ln(x) + x^2\right), u_3 x + \frac{c}{4}\left(2\ln(x) - x^2\right)\right).$$

It can be easily verified that an angle φ between the tangent vector $T(x) = (1, u_2 + c\cosh\psi(x), u_3 + c\sinh\psi(x))$ and a fixed vector $U = (1, u_2, u_3)$ reads $\varphi = |c| =$ constant. Hence α is a spacelike T-slant helix with an axis determined by a fixed vector U.

Figure 1: T-slant helix

Next, let us consider N-slant helices. Let α be N-slant helix whose the principal normal vector $N(x)$ makes a constant angle ω with an isotropic axis determined by the unit isotropic fixed vector $U = (0, u_2, u_3)$. If $N(x)$ and U are both spacelike or timelike vectors, according to the Definitions 2.2 and 2.3 we have

$$\cosh \omega = \epsilon_1 \frac{1}{\kappa}(u_2 y'' - u_3 z'') = c_0,$$

where $c_0 \in \mathbb{R}_0$, $\epsilon_1 = \pm 1$ and $u_2^2 - u_3^2 = \pm 1$. The previous relation gives

$$y'' = \frac{1}{u_2}(\epsilon_1 c_0 \kappa + u_3 z''). \tag{17}$$

Differentiating the previous equation with respect to x, we get

$$y''' = \frac{1}{u_2}(\epsilon_1 c_0 \kappa' + u_3 z'''). \tag{18}$$

Substituting (17) and (18) in the first equation of (6), we get

$$z''' = \frac{\kappa'}{\kappa} z'' + \epsilon\tau \frac{u_2}{u_3} z''.$$

By using the last equation and the second equation of (6), we find

$$\epsilon\tau(y'' - \frac{u_2}{u_3} z'') = 0.$$

If $\tau = 0$, the Frenet equations (4) imply $N = constant$, which we have excluded as the possibility. Thus

$$y'' = \frac{u_2}{u_3} z''. \tag{19}$$

Differentiating the last relation with respect to x, we obtain

$$y''' = \frac{u_2}{u_3} z'''. \tag{20}$$

From (1), (19) and (20) we get $\tau = 0$, which gives a contradiction again.

If $N(x)$ is the spacelike (timelike) vector and U is the timelike (spacelike) vector, according to the Definition 2.4 we have

$$\sinh\omega = \epsilon_1 \frac{1}{\kappa}(u_2 y'' - u_3 z'') = c_0,$$

where $c_0 \in \mathbb{R}_0$, $\epsilon_1 = \pm 1$ and $u_2^2 - u_3^2 = \pm 1$. By applying the similar calculation, we obtain that $N(x)$ is a constant vector, which is a contradiction. The above results can analogously be proved for the B-slant helices. Thus we can state the following theorem.

Theorem 3.8. *There are no N-slant and B-slant helices in $\mathbb{G}_3^1$ with non-constant Frenet vectors.*

4. Darboux Helices in the Pseudo-Galilean Space $\mathbb{G}_3^1$

Darboux helices in the Euclidean 3-space and in the Minkowski 3-space are defined as the curves whose the Darboux vector makes a constant angle with some fixed axis. In this section, we show that there are no Darboux helices in pseudo-Galilean space $\mathbb{G}_3^1$. We first give the definition of such helices.

Definition 4.1. *An admissible curve α in the pseudo-Galilean space $\mathbb{G}_3^1$ is called Darboux helix, if its Darboux vector makes a constant angle with some fixed direction.*

The fixed direction in the Definition 4.1 is called an axis of the helix. We will exclude the case when the Darboux vector is constant, since it trivially makes constant angle with any fixed direction. By using the relations (2) and (7), we find that the Darboux vector of an admissible curve α is given by

$$D(x) = (\epsilon\tau, \epsilon\tau y' - z'', \epsilon\tau z' - y''). \tag{21}$$

Theorem 4.2. *There are no Darboux helices in $\mathbb{G}_3^1$ with non-constant Darboux vector.*

Proof. Assume that there exists Darboux helix α in $\mathbb{G}_3^1$ with non-constant Darboux vector. Depending on the torsion τ of α, we may consider two cases:

(a) Assume that $\tau \neq 0$. According to Definition 4.1 and relation (21), the unit Darboux vector D_0 of α given by

$$D_0 = \frac{D}{||D||} = \left(\epsilon, \epsilon y' - \frac{z''}{\tau}, \epsilon z' - \frac{y''}{\tau}\right) \tag{22}$$

makes a constant angle with some fixed axis spanned by the unit non-isotropic constant vector $U = (1, u_2, u_3)$. By Definition 2.1 it holds

$$\varphi^2 = \left|\left(\epsilon y' - \frac{z''}{\tau} - u_2\right)^2 - \left(\epsilon z' - \frac{y''}{\tau} - u_3\right)^2\right| = c^2, \quad c \in \mathbb{R}_0.$$

The previous equation implies the following two subcases.
(a.1) If $\left(\epsilon y' - \frac{z''}{\tau} - u_2\right)^2 - \left(\epsilon z' - \frac{y''}{\tau} - u_3\right)^2 = c^2$, then

$$\epsilon y' - \frac{z''}{\tau} - u_2 = c\cosh\psi, \quad \epsilon z' - \frac{y''}{\tau} - u_3 = c\sinh\psi, \tag{23}$$

where $\psi(x)$ is some differentiable function. The last two equations give

$$z'' = \epsilon\tau y' - \tau u_2 - c\tau\cosh\psi, \quad y'' = \epsilon\tau z' - \tau u_3 - c\tau\sinh\psi. \tag{24}$$

Differentiating the second equation of (24) with respect to x, we get

$$y''' = \epsilon\tau' z' + \epsilon\tau z'' - \tau' u_3 - c\tau'\sinh\psi - c\tau\psi'\cosh\psi. \tag{25}$$

Substituting the second equation of (24) and (25) in the first equation of (6), we find

$$\epsilon\tau' z' - \tau' u_3 - c\tau'\sinh\psi - c\tau\psi'\cosh\psi = \frac{\kappa'}{\kappa}(\epsilon\tau z' - \tau u_3 - c\tau\sinh\psi).$$

The last equation is satisfied if and only if $\psi' = 0$ and $\tau = c_1\kappa$, $c_1 \in \mathbb{R}_0$. By using the equation $\psi' = 0$ and relations (22) and (23), we get $D_0 = constant$, which is a contradiction.
(a.2) If $\left(\epsilon y' - \frac{z''}{\tau} - u_2\right)^2 - \left(\epsilon z' - \frac{y''}{\tau} - u_3\right)^2 = -c^2$, then

$$\epsilon y' - \frac{z''}{\tau} - u_2 = c\sinh\psi, \quad \epsilon z' - \frac{y''}{\tau} - u_3 = c\cosh\psi,$$

where $\psi(x)$ is some differentiable function. The last two equations give

$$z'' = \epsilon\tau y' - \tau u_2 - c\tau\sinh\psi, \quad y'' = \epsilon\tau z' - \tau u_3 - c\tau\cosh\psi. \tag{26}$$

Differentiating the second equation of (26) with respect to x, we get

$$y''' = \epsilon\tau' z' + \epsilon\tau z'' - \tau' u_3 - c\tau'\cosh\psi - c\tau\psi'\sinh\psi. \tag{27}$$

Substituting the second equation of (26) and (27) in the first equation of (6), we find

$$\epsilon\tau' z' - \tau' u_3 - c\tau'\cosh\psi - c\tau\psi'\sinh\psi = \frac{\kappa'}{\kappa}(\epsilon\tau z' - \tau u_3 - c\tau\cosh\psi).$$

The last equation is satisfied if and only if $\psi' = 0$ and $\tau = c_2\kappa$, $c_2 \in \mathbb{R}_0$. Then we get $D_0 = constant$, which is a contradiction.
(b) Assume that $\tau = 0$. Substituting $\tau = 0$ in the relation (7) it follows that the Darboux vector of the Darboux helix is given by $D = -\epsilon\kappa B$. Moreover, from the Frenet equations (4) it follows that the binormal vector B is constant. This implies that the Darboux vector D always has a fixed direction, which is also a contradiction. □

References

[1] A.T. Ali, *Position vectors of curves in the Galilean space* $\mathbb{G}_3$, Matematički Vesnik **64** (2012), 200–210.

[2] M. Barros, M. Caballero and M. Ortega, *Rotational surfaces in* L^3 *and solitons in the nonlinear* σ*-model*, Commun. Math. Phys. 290 (2009) 437477.

[3] M. Bektaş, *The characterizations of general helices in the 3-dimensional pseudo-Galilean space*, Soochow J. Math. **31(3)** (2005), 441–447.

[4] B. Divjak, *Geometrija pseudogalilejevih prostora*, Ph. D. Thesis, University of Zagreb, 1997.

[5] B. Divjak, *Curves in pseudo-Galilean geometry*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **41** (1998), 117–128.

[6] B. Divjak and Ž. Milin-Šipuš, *Special curves on ruled surfaces in Galilean and pseudo-Galilean spaces*, Acta Math. Hungar. **98** (2003), 203-215.

[7] Z. Erjavec, *On generalization of helices in the Galilean and the pseudo-Galilean space*, J. Math. Research **6** (2014), 39-50.

[8] S. Izymiya and N. Takeuchi, *New special curves and developable surfaces*, Turk. J. Math. **28** (2004), 153-163.

[9] L. Kula and Y. Yayli, *On slant helix and its spherical indicatrix*, Appl. Math. Comput. **169** (2005), 600-607.

[10] L. Kula, N. Ekmekci, Y. Yaylı and K. İlarslan, *Characterizations of slant helices in Euclidean 3-space*, Turk. J. Math. **34** (2010), 261-273.

[11] P. Lucas and J. A. Ortega-Yagües, *Slant helices in the Euclidean 3-space revisited*, Bull. Belg. Math. Soc. Simon Stevin, **23(1)** (2016), 133–150.

[12] E. Nešović, U. Öztürk and E. Betül Koç Öztürk, *On k-type pseudo null Darboux helices in Minkowski* 3*-space*, J. Math. Anal. Appl. **439** (2016), 690-700.

[13] E. Nešović, M. Petrović-Torgašev, L. Verstraelen, *Curves in Lorentzian spaces*, Bollettino U.M.I. **8** (2005), 685-696.

[14] E. Nešović, *Differential geometry of curves in Minkowski spaces*, Ph. D. Thesis, University of Kragujevac, 2002.

[15] A.O. Ogrenmis, M. Ergut, M. Bektas, *On the helices in the Galilean space* $\mathbb{G}_3$, Iranian J. Sci. & Tech. **31** (2007), 177-181.

[16] U. Öztürk and E. Nešović, *On pseudo null and null cartan darboux helices in Minkowski 3-space*, Kuwait J. Sci. **43** (2016), 64-82.

[17] B.J. Pavković, I. Kamenarović, *The equiform differential geometry of curves in the Galilean space* $\mathbb{G}_3$, Glasnik mat. **22(2)** (1987), 449-457.

[18] O. Röschel, *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Institut für Math. und Angew. Geometrie, Leoben, 1984.

[19] P.D. Scofield, *Curves of constant precession*, Amer. Math. Monthly, **102** (1995), 531-537.

[20] E. Ziplar, A. Senol and Y. Yayli, *On Darboux helices in Euclidean 3-space*, Global Journal of Science Frontier Research, **12** (2012), 72-80.