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Oscillations in Difference Equations with Several Arguments Using an Iterative Method

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Abstract. This paper presents new sufficient conditions, involving lim sup, for the oscillation of all solutions of difference equations with non-monotone deviating arguments and nonnegative coefficients. We establish these criteria by applying an iterative method. We consider difference equations of both the retarded and the advanced type. We illustrate the results and the improvement over other known oscillation criteria by examples, numerically solved in MATLAB.

1. Introduction

Consider the difference equation with several variable deviating arguments of either retarded

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0, \tag{E}$$

or advanced type

$$\nabla x(n) - \sum_{i=1}^{m} q_i(n)x(\sigma_i(n)) = 0, \quad n \in \mathbb{N}, \tag{E'}$$

where $\mathbb{N}_0$, $\mathbb{N}$ are the sets of nonnegative integers and positive integers, respectively. Here, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$ and ∇ corresponds to the backward difference operator $\nabla x(n) = x(n) - x(n-1)$.

Equations (E) and (E′) are studied under the following assumptions: everywhere $(p_i(n))_{n\geq 0}$, $(q_i(n))_{n\geq 1}$, $1 \leq i \leq m$, are sequences of nonnegative real numbers, $(\tau_i(n))_{n\geq 0}$, $(\sigma_i(n))_{n\geq 1}$, $1 \leq i \leq m$, are sequences of integers such that

$$\tau_i(n) \leq n-1, \quad \forall n \in \mathbb{N}_0 \quad \text{and} \quad \lim_{n\to\infty} \tau_i(n) = \infty, \quad 1 \leq i \leq m \tag{1.1}$$

and

$$\sigma_i(n) \geq n+1, \quad \forall n \in \mathbb{N}, \quad 1 \leq i \leq m, \tag{1.1'}$$

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respectively.

Set

$$v = -\min_{\substack{n\geq 0\\ 1\leq i\leq m}} \tau_i(n).$$

Clearly, v is a finite positive integer if (1.1) holds.

By a *solution* of (E), we mean a sequence of real numbers $(x(n))_{n\geq -v}$ which satisfies (E) for all $n \geq 0$. It is clear that, for each choice of real numbers $c_{-v}, c_{-v+1}, ..., c_{-1}, c_0$, there exists a unique solution $(x(n))_{n\geq -v}$ of (E) which satisfies the initial conditions $x(-v) = c_{-v}$, $x(-v+1) = c_{-v+1}, ..., x(-1) = c_{-1}$, $x(0) = c_0$. When the initial data is given, we can obtain a unique solution to (E) by using the method of steps.

By a solution of (E′), we mean a sequence of real numbers $(x(n))_{n\geq 0}$ which satisfies (E′) for all $n \geq 1$.

A solution $(x(n))_{n\geq -v}$ (or $(x(n))_{n\geq 0}$) of (E) (or (E′)) is called *oscillatory*, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

While deviating difference equations with one argument have been studied widely and extensively in the literature by many researchers, the study for such equations, especially systems, with several arguments is scarce and relatively rare, probably due to its extreme complexity in analysis and the lack of the established theory. Recent studies in biological, physical and economical systems involving multiple feedback mechanisms have spurred a great attention to equations (E) and (E′). Hence, in the last few decades, the oscillatory behavior, stability and the existence of positive solutions of difference equations (E) and (E′) has been the subject of investigations. See, for example, [1–17] and the references cited therein. Most of these papers concerned with the special case where the arguments are nondecreasing, while a small number of these papers are dealing with the general case where the arguments are not necessarily monotone. See, for example, [2, 3, 4].

The consideration of non-monotone arguments other than the pure mathematical interest, it approximates the natural phenomena described by equation of the type (E) or (E′). That is because there are always natural disturbances (e.g. noise in communication systems) that affect all the parameters of the equation and therefore the fair (from a mathematical point of view) monotone arguments become non-monotone almost always. In view of this, an interesting question arising in the case where the arguments $\tau_i(n)$ and $\sigma_i(n)$ are non-monotone, is whether we can state further oscillation criteria which essentially improve all the known results in the literature. In the present paper a positive answer to the above question is given.

Throughout this paper, we are going to use the following notations:

$$\sum_{i=k}^{k-1} A(i) = 0 \quad \text{and} \quad \prod_{i=k}^{k-1} A(i) = 1$$

$$\alpha := \min_{1\leq i\leq m} \alpha_i \quad \text{where} \quad \alpha_i = \liminf_{n\to\infty} \sum_{j=\tau_i(n)}^{n-1} p_i(j) \tag{1.2}$$

$$\beta := \min_{1\leq i\leq m} \beta_i \quad \text{where} \quad \beta_i = \liminf_{n\to\infty} \sum_{j=n+1}^{\sigma_i(n)} q_i(j) \tag{1.3}$$

$$D(\omega) := \begin{cases} 0, & \text{if } \omega > 1/e \\ \frac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2}, & \text{if } \omega \in [0, 1/e] \end{cases} \tag{1.4}$$

$$LD := \liminf_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n-1} p_i(j) \tag{1.5}$$

$$MD := \limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=\tau(n)}^{n} p_i(j) \tag{1.6}$$

$$MA := \limsup_{n\to\infty} \sum_{i=1}^{m} \sum_{j=n}^{\sigma(n)} q_i(j) \tag{1.7}$$

where $\tau(n) = \max_{1\leq i\leq m} \tau_i(n)$, $\sigma(n) = \min_{1\leq i\leq m} \sigma_i(n)$ and $\tau_i(n)$, $\sigma_i(n)$ are nondecreasing.

1.1. Retarded difference equations

Suppose that $\tau_i(n)$, $1 \le i \le m$ are nondecreasing.

In 2006, Berezansky and Braverman [1] and in 2014, Chatzarakis, Pinelas and Stavroulakis [7] proved that, if

$$\limsup_{n\to\infty} \sum\nolimits_{i=1}^{m} p_i(n) > 0 \quad \text{and} \quad LD > \frac{1}{e}, \tag{1.8}$$

or

$$MD > 1 \tag{1.9}$$

respectively, then all solutions of (E) are oscillatory.

Assume that the arguments $\tau_i(n)$, $1 \le i \le m$ are not necessarily monotone.

Set

$$h(n) = \max_{1\le i\le m} h_i(n) \quad \text{where} \quad h_i(n) = \max_{0\le s\le n} \tau_i(s), \quad n \ge 0 \tag{1.10}$$

and

$$a_1(n,k) := \prod\nolimits_{i=k}^{n-1} \left[1 - \sum\nolimits_{\ell=1}^{m} p_\ell(i)\right] \tag{1.11}$$

$$a_{r+1}(n,k) := \prod\nolimits_{i=k}^{n-1} \left[1 - \sum\nolimits_{\ell=1}^{m} p_\ell(i) a_r^{-1}(i, \tau_\ell(i))\right], \ \ r \in \mathbb{N} \tag{1.12}$$

Clearly, $h(n)$, $h_i(n)$ are nondecreasing and $\tau_i(n) \le h_i(n) \le h(n) \le n-1$ for all $n \ge 0$.

In 2015, Braverman, Chatzarakis and Stavroulakis [2] proved that if there exists a subsequence $\theta(n)$, $n \in \mathbb{N}$ of positive integers such that

$$\sum\nolimits_{i=1}^{m} p_i(\theta(n)) \ge 1, \qquad \forall n \in \mathbb{N}, \tag{1.13}$$

then all solutions of (E) are oscillatory.

Under the assumption that

$$\sum\nolimits_{i=1}^{m} p_i(n) < 1, \qquad \forall n \ge 0, \tag{1.14}$$

the same authors proved that, if for some $r \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum\nolimits_{j=h(n)}^{n} \sum\nolimits_{i=1}^{m} p_i(j) a_r^{-1}(h(n), \tau_i(j)) > 1, \tag{1.15}$$

or

$$\limsup_{n\to\infty} \sum\nolimits_{j=h(n)}^{n} \sum\nolimits_{i=1}^{m} p_i(j) a_r^{-1}(h(n), \tau_i(j)) > 1 - D(\alpha), \tag{1.16}$$

then all solutions of (E) are oscillatory.

Recently, Chatzarakis, Horvat-Dmitrović and Pašić [4] proved that if for some $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum\nolimits_{j=h(n)}^{n} \mathcal{P}(j) \prod\nolimits_{i=\tau(j)}^{h(n)-1} \frac{1}{1-\mathcal{P}_\ell(i)} > 1, \tag{1.17}$$

or

$$\limsup_{n\to\infty} \sum\nolimits_{j=h(n)}^{n} \mathcal{P}(j) \prod\nolimits_{i=\tau(j)}^{h(n)-1} \frac{1}{1-\mathcal{P}_\ell(i)} > 1 - D(\alpha), \tag{1.18}$$

where

$$\mathcal{P}_\ell(n) = \mathcal{P}(n) \left[1 + \sum\nolimits_{i=\tau(n)}^{n-1} \mathcal{P}(i) \prod\nolimits_{j=\tau(i)}^{h(n)-1} \frac{1}{1-\mathcal{P}_{\ell-1}(j)}\right], \tag{1.19}$$

with $\mathcal{P}(n) = \sum_{i=1}^{m} p_i(n) = \mathcal{P}_0(n)$, then all solutions of (E) are oscillatory.

1.2. Advanced difference equations

Suppose that $\sigma_i(n)$, $1 \leq i \leq m$ are nondecreasing.

In 2014, Chatzarakis, Pinelas and Stavroulakis [7] proved that, if

$$MA > 1, \tag{1.20}$$

then all solutions of (E′) are oscillatory.

Assume that the arguments $\sigma_i(n)$, $1 \leq i \leq m$ are not necessarily monotone. Set

$$\rho(n) = \min_{1\leq i\leq m} \rho_i(n), \quad \text{where} \;\; \rho_i(n) = \min_{s\geq n} \sigma_i(s), \quad n \geq 0, \tag{1.21}$$

$$b_1(n,k) := \prod\nolimits_{i=n+1}^{k} \left[1 - \sum\nolimits_{\ell=1}^{m} q_\ell(i)\right], \tag{1.22}$$

$$b_{r+1}(n,k) := \prod\nolimits_{i=n+1}^{k} \left[1 - \sum\nolimits_{\ell=1}^{m} q_\ell(i) b_r^{-1}(i,\sigma_\ell(i))\right], \quad r \in \mathbb{N}. \tag{1.23}$$

Clearly, $\rho(n)$, $\rho_i(n)$ are nondecreasing and $\sigma_i(n) \geq \rho_i(n) \geq \rho(n) \geq n+1$ for all $n \geq 1$.

In 2015, Braverman, Chatzarakis and Stavroulakis [2] proved that if there exists a subsequence $\theta(n)$, $n \in \mathbb{N}$ of positive integers such that

$$\sum\nolimits_{i=1}^{m} q_i(\theta(n)) \geq 1, \quad \forall n \in \mathbb{N}, \tag{1.24}$$

then all solutions of (E′) are oscillatory.

Under the assumption that

$$\sum\nolimits_{i=1}^{m} q_i(n) < 1, \qquad \forall n \geq 1, \tag{1.25}$$

the same authors proved that, if for some $r \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum\nolimits_{j=n}^{\rho(n)} \sum\nolimits_{i=1}^{m} q_i(j) b_r^{-1}(\rho(n),\sigma_i(j)) > 1, \tag{1.26}$$

or

$$\limsup_{n\to\infty} \sum\nolimits_{j=n}^{\rho(n)} \sum\nolimits_{i=1}^{m} q_i(j) b_r^{-1}(\rho(n),\sigma_i(j)) > 1 - D(\beta), \tag{1.27}$$

then all solutions of (E′) are oscillatory.

Recently, Chatzarakis, Horvat-Dmitrović and Pašić [4] proved that if for some $\ell \in \mathbb{N}$

$$\limsup_{n\to\infty} \sum\nolimits_{j=n}^{\rho(n)} Q(j) \prod\nolimits_{i=\rho(n)+1}^{\sigma(j)} \frac{1}{1 - Q_\ell(i)} > 1, \tag{1.28}$$

or

$$\limsup_{n\to\infty} \sum\nolimits_{j=n}^{\rho(n)} Q(j) \prod\nolimits_{i=\rho(n)+1}^{\sigma(j)} \frac{1}{1 - Q_\ell(i)} > 1 - D(\beta), \tag{1.29}$$

where

$$Q_\ell(n) = Q(n)\left[1 + \sum\nolimits_{i=n+1}^{\rho(n)} Q(i) \prod\nolimits_{j=\rho(n)+1}^{\sigma(i)} \frac{1}{1 - Q_{\ell-1}(j)}\right], \tag{1.30}$$

with $Q(n) = \sum_{i=1}^{m} q_i(n) = Q_0(n)$, then all solutions of (E′) are oscillatory.

2. Main Results

2.1. Retarded difference equations

We study further (E) and derive new sufficient oscillation conditions, involving lim sup, which improve all previous known results in the literature.

Theorem 2.1. *Assume that (1.1) and (1.14) hold, and $h(n)$ is defined by (1.10). If for some $w \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum\nolimits_{\ell=h(n)}^{n} \overline{P}(\ell) \prod\nolimits_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} > 1, \tag{2.1}$$

where

$$\overline{P}_w(n) = \overline{P}(n)\left[1 + \sum\nolimits_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) \exp\left(\sum\nolimits_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod\nolimits_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}_{w-1}(i)}\right)\right], \tag{2.2}$$

with $\overline{P}(n) = \sum_{i=1}^{m} p_i(n) = \overline{P}_0(n)$, then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $(x(n))_{n\geq -v}$ of (E). Since $(-x(n))_{n\geq -v}$ is also a solution of (E), we can confine our discussion only to the case where $x(n) > 0$ for all large n. Let $n_1 \geq -v$ be an integer such that $x(n) > 0$ for all $n \geq n_1$. Then, there exists $n_2 \geq n_1$ such that $x(\tau_i(n)) > 0$, for all $n \geq n_2$. In view of this, Eq.(E) becomes

$$\Delta x(n) = -\sum\nolimits_{i=1}^{m} p_i(n) x(\tau_i(n)) \leq 0, \quad \text{for all } n \geq n_2,$$

which means that $(x(n))$ is an eventually nonincreasing sequence of positive numbers. In view of this and taking into account the fact that $\tau_i(n) < n$, (E) implies

$$\Delta x(n) + \left(\sum\nolimits_{i=1}^{m} p_i(n)\right) x(n) \leq 0,$$

or

$$\Delta x(n) + \overline{P}(n) x(n) \leq 0, \quad n \geq n_2. \tag{2.3}$$

Applying the discrete Grönwall inequality, we obtain

$$x(k) \geq x(n) \prod\nolimits_{i=k}^{n-1} \frac{1}{1-\overline{P}(i)}, \quad \text{for all } n \geq k \geq n_2. \tag{2.4}$$

Dividing (E) by $x(n)$ and summing up from k to $n-1$, we take

$$\sum\nolimits_{j=k}^{n-1} \frac{\Delta x(j)}{x(j)} = -\sum\nolimits_{j=k}^{n-1} \sum\nolimits_{i=1}^{m} p_i(j) \frac{x(\tau_i(j))}{x(j)}$$

or

$$\sum\nolimits_{j=k}^{n-1} \frac{\Delta x(j)}{x(j)} \leq -\sum\nolimits_{j=k}^{n-1} \left(\sum\nolimits_{i=1}^{m} p_i(j)\right) \frac{x(\tau(j))}{x(j)}$$

i.e.,

$$\sum\nolimits_{j=k}^{n-1} \frac{\Delta x(j)}{x(j)} \leq -\sum\nolimits_{j=k}^{n-1} \overline{P}(j) \frac{x(\tau(j))}{x(j)}. \tag{2.5}$$

Also, since $e^x \geq x+1$, $x>0$ we have

$$\begin{aligned}\sum_{j=k}^{n-1} \frac{\Delta x(j)}{x(j)} &= \sum_{j=k}^{n-1} \left(\frac{x(j+1)}{x(j)} - 1 \right) \\ &= \sum_{j=k}^{n-1} \left[\exp\left(\ln \frac{x(j+1)}{x(j)} \right) - 1 \right] \\ &\geq \sum_{j=k}^{n-1} \left[\ln \frac{x(j+1)}{x(j)} + 1 - 1 \right] \\ &= \sum_{j=k}^{n-1} \ln \frac{x(j+1)}{x(j)} = \ln \frac{x(n)}{x(k)},\end{aligned}$$

or

$$\sum_{j=k}^{n-1} \frac{\Delta x(j)}{x(j)} \geq \ln \frac{x(n)}{x(k)}. \tag{2.6}$$

Combining (2.5) and (2.6), we obtain

$$-\sum_{j=k}^{n-1} \overline{P}(j) \frac{x(\tau(j))}{x(j)} \geq \ln \frac{x(n)}{x(k)},$$

or

$$\ln \frac{x(k)}{x(n)} \geq \sum_{j=k}^{n-1} \overline{P}(j) \frac{x(\tau(j))}{x(j)}. \tag{2.7}$$

Since $\tau(j) < j$, (2.4) implies

$$x(\tau(j)) \geq x(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)}. \tag{2.8}$$

In view of (2.8), (2.7) gives

$$\ln \frac{x(k)}{x(n)} \geq \sum_{j=k}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)},$$

or

$$x(k) \geq x(n) \exp\left(\sum_{j=k}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)} \right). \tag{2.9}$$

Summing up (E) from $\tau(n)$ to $n-1$, we have

$$x(n) - x(\tau(n)) + \sum_{\ell=\tau(n)}^{n-1} \sum_{i=1}^{m} p_i(\ell) x(\tau_i(\ell)) = 0,$$

or

$$x(n) - x(\tau(n)) + \sum_{\ell=\tau(n)}^{n-1} \left(\sum_{i=1}^{m} p_i(\ell) \right) x(\tau(\ell)) \leq 0,$$

i.e.,

$$x(n) - x(\tau(n)) + \sum_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) x(\tau(\ell)) \leq 0. \tag{2.10}$$

Since $\tau(\ell) \leq h(\ell) \leq h(n) < n$, (2.9) guarantees that

$$x(\tau(\ell)) \geq x(n) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)} \right). \tag{2.11}$$

Combining (2.10) and (2.11) we have

$$x(n) - x(\tau(n)) + x(n) \sum_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)}\right) \leq 0.$$

Multiplying the last inequality by $\overline{P}(n)$, we take

$$\overline{P}(n)x(n) - \overline{P}(n)x(\tau(n)) + \overline{P}(n)x(n) \sum_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)}\right) \leq 0. \tag{2.12}$$

Furthermore,

$$\Delta x(n) = -\sum_{i=1}^{m} p_i(n) x\left(\tau_i(n)\right) \leq -x\left(\tau(n)\right) \sum_{i=1}^{m} p_i(n),$$

i.e.,

$$\Delta x(n) \leq -\overline{P}(n) x\left(\tau(n)\right).$$

In view of this, (2.12) gives

$$\Delta x(n) + \overline{P}(n)x(n) + \overline{P}(n)x(n) \sum_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)}\right) \leq 0,$$

or

$$\Delta x(n) + \overline{P}(n)\left[1 + \sum_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)}\right)\right] x(n) \leq 0.$$

Therefore

$$\Delta x(n) + \overline{P}_1(n)x(n) \leq 0, \tag{2.13}$$

where

$$\overline{P}_1(n) = \overline{P}(n)\left[1 + \sum_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}(i)}\right)\right].$$

Repeating the above argument, where (2.13) is used instead of (2.3), leads to a new estimate

$$\Delta x(n) + \overline{P}_2(n)x(n) \leq 0,$$

where

$$\overline{P}_2(n) = \overline{P}(n)\left[1 + \sum_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}_1(i)}\right)\right].$$

Continuing by induction, we get

$$\Delta x(n) + \overline{P}_w(n)x(n) \leq 0, \quad (w \in \mathbb{N}) \tag{2.13'}$$

where

$$\overline{P}_w(n) = \overline{P}(n)\left[1 + \sum_{\ell=\tau(n)}^{n-1} \overline{P}(\ell) \exp\left(\sum_{j=\tau(\ell)}^{n-1} \overline{P}(j) \prod_{i=\tau(j)}^{j-1} \frac{1}{1-\overline{P}_{w-1}(i)}\right)\right]$$

and

$$x(\tau(\ell)) \geq x(h(n)) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)}. \tag{2.14}$$

Summing up (E) from $h(n)$ to n, we have

$$x(n+1) - x(h(n)) + \sum_{\ell=h(n)}^{n} \sum_{i=1}^{m} p_i(\ell)\, x(\tau_i(\ell)) = 0,$$

or

$$x(n+1) - x(h(n)) + \sum_{\ell=h(n)}^{n} \left(\sum_{i=1}^{m} p_i(\ell) \right) x(\tau(\ell)) \leq 0,$$

i.e.,

$$x(n+1) - x(h(n)) + \sum_{\ell=h(n)}^{n} \overline{P}(\ell) x(\tau(\ell)) \leq 0.$$

Taking into account the fact that (2.14) holds, the last inequality gives

$$x(n+1) - x(h(n)) + x(h(n)) \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} \leq 0. \tag{2.15}$$

The strict inequality is valid if we omit $x(n+1) > 0$ in the left-hand side:

$$-x(h(n)) + x(h(n)) \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} < 0.$$

This implies

$$x(h(n)) \left[\sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} - 1 \right] < 0,$$

i.e.,

$$\limsup_{t\to\infty} \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} \leq 1,$$

which contradicts (2.1).

The proof of the theorem is complete. □

To establish the next theorem we need the following lemma.

Lemma 2.2. *[2, Lemma 2.2] Assume that (1.1) holds, $x(n)$ is an eventually positive solution of (E), and α is defined by (1.2). If $0 < \alpha \leq 1/e$, then*

$$\liminf_{n\to\infty} \frac{x(n+1)}{x(h(n))} \geq D(\alpha). \tag{2.16}$$

Theorem 2.3. *Assume that (1.1) and (1.14) hold, $h(n)$ is defined by (1.10) and α by (1.2) with $0 < \alpha \leq 1/e$. If for some $w \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} > 1 - D(\alpha), \tag{2.17}$$

where $\overline{P}_w(n)$ is defined by (2.2), then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\geq -v}$ is a nonoscillatory solution of (E). Since $(-x(n))_{n\geq -v}$ is also a solution of (E), we can confine our discussion only to the case where $x(n) > 0$ for all large n. Then, as in the proof of Theorem 2.1, for sufficiently large n, (2.15) is satisfied, i.e.,

$$x(n+1) - x(h(n)) + x(h(n)) \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} \leq 0.$$

That is,

$$\sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} \leq 1 - \frac{x(n+1)}{x(h(n))},$$

which gives

$$\limsup_{n\to\infty} \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} \leq 1 - \liminf_{n\to\infty} \frac{x(n+1)}{x(h(n))}.$$

Since $0 < \alpha \leq 1/e$, by Lemma 2.2 inequality (2.16) holds. So the last inequality leads to

$$\limsup_{n\to\infty} \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)} \leq 1 - D(\alpha),$$

which contradicts condition (2.17).

The proof of the theorem is complete. □

Remark 2.4. *It is clear that the left-hand sides of both conditions (2.1) and (2.17) are identical, also the right-hand side of condition (2.17) reduces to (2.1) in case that $\alpha = 0$. So it seems that Theorem 2.3 is the same as Theorem 2.1 when $\alpha = 0$. However, one may notice that condition $0 < \alpha \leq 1/e$ is required in Theorem 2.3 but not in Theorem 2.1.*

Theorem 2.5. *Assume that (1.1) and (1.14) hold, $h(n)$ is defined by (1.10) and α by (1.2) with $0 < \alpha \leq 1/e$. If for some $w \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)} > \frac{1}{D(\alpha)}, \tag{2.18}$$

where $\overline{P}_w(n)$ is defined by (2.2), then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that $(x(n))_{n\geq -v}$ is an eventually solution of (E). Then, as in the proof of Theorem 2.1, by the use of the discrete Grönwall inequality from (2.14) we take, for a sufficiently large n

$$x(\tau(\ell)) \geq x(n+1) \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)}. \tag{2.19}$$

Summing up (E) from $h(n)$ to n, we have

$$x(n+1) - x(h(n)) + \sum_{\ell=h(n)}^{n} \sum_{i=1}^{m} p_i(\ell)\, x(\tau_i(\ell)) = 0,$$

or

$$x(n+1) - x(h(n)) + \sum_{\ell=h(n)}^{n} \left(\sum_{i=1}^{m} p_i(\ell)\right) x(\tau(\ell)) \leq 0,$$

i.e.,

$$x(n+1) - x(h(n)) + \sum_{\ell=h(n)}^{n} \overline{P}(\ell) x(\tau(\ell)) \leq 0.$$

In view of (2.19), the last inequality gives

$$x(n+1) - x(h(n)) + \sum_{\ell=h(n)}^{n} \overline{P}(\ell) x(n+1) \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)} \leq 0,$$

or

$$x(n+1) - x(h(n)) + x(h(n)) \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \frac{x(n+1)}{x(h(n))} \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)} \leq 0.$$

Since $x(n+1) > 0$, the last inequality leads to

$$-x(h(n)) + x(h(n)) \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \frac{x(n+1)}{x(h(n))} \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)} < 0,$$

or

$$x(h(n)) \left[\frac{x(n+1)}{x(h(n))} \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)} - 1 \right] < 0.$$

Thus, for all sufficiently large n it holds

$$\sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)} < \frac{x(h(n))}{x(n+1)}. \tag{2.20}$$

Letting $n \to \infty$, we take

$$\limsup_{n\to\infty} \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)} \leq \limsup_{n\to\infty} \frac{x(h(n))}{x(n+1)}.$$

Since $0 < \alpha \leq 1/e$, by Lemma 2.2 inequality (2.16) holds. So the last inequality leads to

$$\limsup_{n\to\infty} \sum_{\ell=h(n)}^{n} \overline{P}(\ell) \prod_{i=\tau(\ell)}^{n} \frac{1}{1-\overline{P}_w(i)} \leq \frac{1}{D(\alpha)},$$

which contradicts condition (2.18).

The proof of the theorem is complete. □

Remark 2.6. *If $\overline{P}_w(n) \geq 1$ then (2.13′) guarantees that all solutions of (E) are oscillatory. In fact, (2.13′) gives*

$$\Delta x(n) + x(n) \leq 0$$

which means that $x(n+1) \leq 0$. This contradics $x(n) > 0$ for all $n \geq n_2$. Thus, in Theorems 2.1, 2.3 and 2.5 we consider only the case $\overline{P}_w(n) < 1$. Another conclusion, that can be drawn from the above, is that if at some point through the iterative process, we get a value of w, for which $\overline{P}_w(n) \geq 1$, then the process terminates, since in any case, all solutions of (E) will be oscillatory. The value of w, that is the number of iterations, obviously, depends on the coefficients $p_i(n)$ and the form of the non-monotone arguments $\tau_i(n)$.

2.2. Advanced difference equations

Similar oscillation theorems for the (dual) advanced difference equation (E′) can be derived easily. The proof of these theorems are omitted, since they are quite similar to the proofs for a retarded equation.

Theorem 2.7. *Assume that (1.1′) and (1.25) hold, and $\rho(n)$ is defined by (1.21). If for some $w \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{\ell=n}^{\rho(n)} \overline{Q}(\ell) \prod_{i=\rho(n)+1}^{\sigma(\ell)} \frac{1}{1-\overline{Q}_w(i)} > 1, \tag{2.21}$$

where

$$\overline{Q}_w(n) = \overline{Q}(n)\left[1+\sum_{\ell=n+1}^{\sigma(n)} \overline{Q}(\ell)\exp\left(\sum_{j=n+1}^{\sigma(\ell)} \overline{Q}(j)\prod_{i=j+1}^{\sigma(j)} \frac{1}{1-\overline{Q}_{w-1}(i)}\right)\right], \tag{2.22}$$

with $\overline{Q}(n) = \sum_{i=1}^{m} q_i(n) = \overline{Q}_0(n)$, then all solutions of (E′) are oscillatory.

Theorem 2.8. *Assume that (1.1′) and (1.25) hold, $\rho(n)$ is defined by (1.21) and β by (1.3) with $0 < \beta \le 1/e$. If for some $w \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{\ell=n}^{\rho(n)} \overline{Q}(\ell) \prod_{i=\rho(n)+1}^{\sigma(\ell)} \frac{1}{1-\overline{Q}_w(i)} > 1 - D(\beta), \tag{2.23}$$

where $\overline{Q}_w(n)$ is defined by (2.22), then all solutions of (E′) are oscillatory.

Remark 2.9. *It is clear that the left-hand sides of both conditions* (2.21) *and* (2.23) *are identical, also the right hand side of condition* (2.23) *reduces to* (2.21) *in case that $\beta = 0$. So it seems that Theorem 2.8 is the same as Theorem 2.7 when $\beta = 0$. However, one may notice that condition $0 < \beta \le 1/e$ is required in Theorem 2.8 but not in Theorem 2.7.*

Theorem 2.10. *Assume that (1.1′) and (1.25) hold, $\rho(n)$ is defined by (1.21) and β by (1.3) with $0 < \beta \le 1/e$. If for some $w \in \mathbb{N}$*

$$\limsup_{n\to\infty} \sum_{\ell=n}^{\rho(n)} \overline{Q}(\ell) \prod_{i=n}^{\sigma(\ell)} \frac{1}{1-\overline{Q}_w(i)} > \frac{1}{D(\beta)}, \tag{2.24}$$

where $\overline{Q}_w(n)$ is defined by (2.22), then all solutions of (E′) are oscillatory.

Remark 2.11. *Similar comments as those in Remark 2.6 can be made for Theorems 2.7, 2.8 and 2.10, concerning equation (E′).*

2.3. Difference inequalities

A slight modification in the proofs of Theorem 2.1, 2.3, 2.5, 2.7, 2.8 and 2.10 leads to the following results about deviating difference inequalities.

Theorem 2.12. *Assume that all conditions of Theorem 2.1 [2.7] or 2.3 [2.8] or 2.5 [2.10] hold. Then*

(i) the retarded [advanced] difference inequality

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) \le 0,\ n \in \mathbb{N}_0 \ \left[\nabla x(n) - \sum_{i=1}^{m} q_i(n)x(\sigma_i(n)) \ge 0,\ n \in \mathbb{N}\right],$$

has no eventually positive solutions;

(ii) the retarded [advanced] difference inequality

$$\Delta x(n) + \sum_{i=1}^{m} p_i(n)x(\tau_i(n)) \ge 0,\ n \in \mathbb{N}_0 \ \left[\nabla x(n) - \sum_{i=1}^{m} q_i(n)x(\sigma_i(n)) \le 0,\ n \in \mathbb{N}\right],$$

has no eventually negative solutions.

3. Examples

In this section, examples illustrate cases when the results of the present paper imply oscillation while previously known results fail.

Example 3.1. *Consider the retarded difference equation*

$$\Delta x(n) + \frac{48}{625} x(\tau_1(n)) + \frac{24}{625} x(\tau_2(n)) + \frac{19}{1250} x(\tau_3(n)) = 0, \quad n \in \mathbb{N}_0, \tag{3.1}$$

with (see Fig. 1, (a))

$$\tau_1(n) = \begin{cases} n-2, & \text{if } n = 5\mu \\ n-1, & \text{if } n = 5\mu+1 \\ n-5, & \text{if } n = 5\mu+2 \\ n-2, & \text{if } n = 5\mu+3 \\ n-4, & \text{if } n = 5\mu+4 \end{cases} \quad \text{and} \quad \begin{array}{l} \tau_2(n) = \tau_1(n) - 1 \\ \tau_3(n) = \tau_1(n) - 2 \end{array}$$

where $\mu \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of non-negative integers.

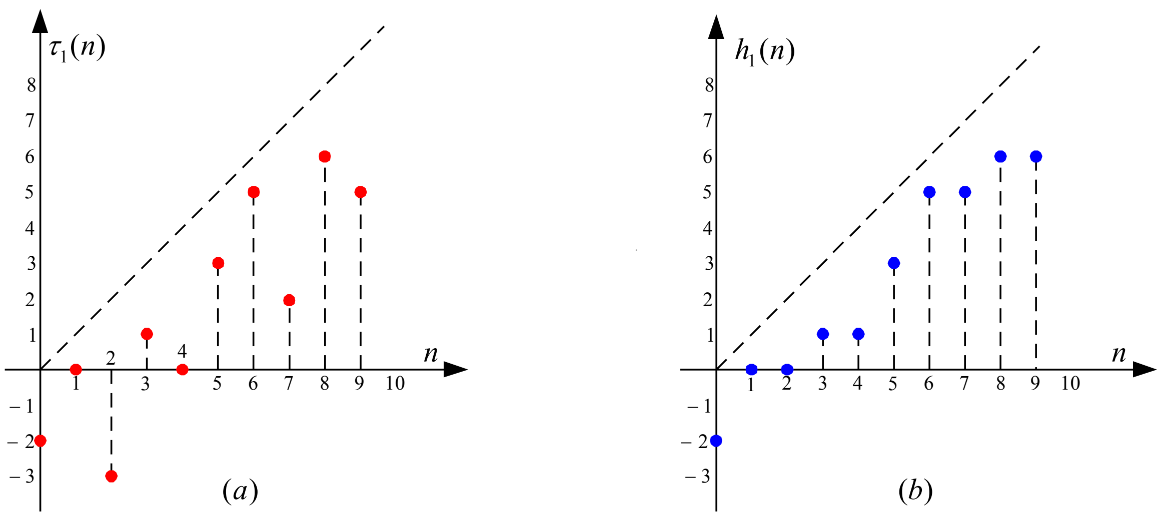

Figure 1: The graphs of $\tau_1(n)$ and $h_1(n)$

By (1.10), we see (Fig. 1, (b)) that

$$h_1(n) = \begin{cases} n-2, & \text{if } n = 5\mu \\ n-1, & \text{if } n = 5\mu+1 \\ n-2, & \text{if } n = 5\mu+2 \\ n-2, & \text{if } n = 5\mu+3 \\ n-3, & \text{if } n = 5\mu+4 \end{cases} \quad \text{and} \quad \begin{array}{l} h_2(n) = h_1(n) - 1 \\ h_3(n) = h_1(n) - 2 \end{array}$$

and consequently

$$h(n) = \max_{1 \le i \le 3} \{h_i(n)\} = h_1(n).$$

Also, it is obvious that

$$\tau(n) = \max_{1 \le i \le 3} \tau_i(n) = \tau_1(n).$$

It is easy to see that $\sum_{i=1}^{3} p_i(n) = \frac{48}{625} + \frac{24}{625} + \frac{19}{1250} = 0.1304 < 1$, *i.e., (1.14) is satisfied.*
We observe that the function $F_w : \mathbb{N}_0 \to \mathbb{R}_+$ *defined as*

$$F_w(n) = \sum\nolimits_{\ell=h(n)}^{n} \overline{P}(\ell) \prod\nolimits_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_w(i)}$$

with $\overline{P}(n) = \sum_{i=1}^{3} p_i(n) = \frac{163}{1250}$, *attains its maximum at* $n = 5\mu + 4$, $\mu \in \mathbb{N}_0$, *for every* $w \in \mathbb{N}$. *Specifically,*

$$\begin{aligned} F_1(n) &= \sum\nolimits_{\ell=h(n)}^{n} \overline{P}(\ell) \prod\nolimits_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}_1(i)} \\ &= \sum\nolimits_{\ell=h(n)}^{n} \overline{P}(\ell) \prod\nolimits_{i=\tau(\ell)}^{h(n)-1} \frac{1}{1-\overline{P}(i)\left[1+\sum_{j=\tau(i)}^{i-1} \overline{P}(j) \exp\left(\sum_{k=\tau(j)}^{i-1} \overline{P}(k) \prod_{u=\tau(k)}^{k-1} \frac{1}{1-\overline{P}_0(u)}\right)\right]}. \end{aligned}$$

Thus, by using an algorithm on MATLAB software (see Appendix), we obtain

$$F_1(5\mu + 4) \simeq 1.0094$$

and therefore

$$\limsup_{n\to\infty} F_1(n) \simeq 1.0094 > 1.$$

That is, condition (2.1) of Theorem 2.1 is satisfied for $w = 1$. *Therefore all solutions of (3.1) are oscillatory.*

Observe, however, that

$$\begin{aligned} LD &= \liminf_{\mu\to\infty} \sum\nolimits_{i=1}^{3} \sum\nolimits_{j=5\mu}^{5\mu} p_i(j) \\ &= \frac{48}{625} + \frac{24}{625} + \frac{19}{1250} = 0.1304 < \frac{1}{e}, \end{aligned}$$

$$\begin{aligned} MD &= \limsup_{\mu\to\infty} \sum\nolimits_{i=1}^{3} \sum\nolimits_{j=5\mu+1}^{5\mu+4} p_i(j) \\ &= 4 \cdot \left(\frac{48}{625} + \frac{24}{625} + \frac{19}{1250}\right) = 0.5216 < 1. \end{aligned}$$

Also, the function $\Phi_r : \mathbb{N}_0 \to \mathbb{R}_+$ *defined as*

$$\Phi_r(n) = \sum\nolimits_{j=h(n)}^{n} \sum\nolimits_{i=1}^{m} p_i(j) a_r^{-1}(h(n), \tau_i(j))$$

attains its maximum at $n = 5\mu + 4$, $\mu \in \mathbb{N}_0$, *for every* $r \in \mathbb{N}$. *Specifically,*

$$\begin{aligned}
\Phi_1(5\mu+4) &= \sum_{j=5\mu+1}^{5\mu+4} \sum_{i=1}^{3} p_i(j) a_1^{-1}(5\mu+1, \tau_i(j)) \\
&= \frac{48}{625} a_1^{-1}(5\mu+1, 5\mu) + \frac{24}{625} a_1^{-1}(5\mu+1, 5\mu-1) + \frac{19}{1250} a_1^{-1}(5\mu+1, 5\mu-2) \\
&\quad + \frac{48}{625} a_1^{-1}(5\mu+1, 5\mu-3) + \frac{24}{625} a_1^{-1}(5\mu+1, 5\mu-4) + \frac{19}{1250} a_1^{-1}(5\mu+1, 5\mu-5) \\
&\quad + \frac{48}{625} a_1^{-1}(5\mu+1, 5\mu+1) + \frac{24}{625} a_1^{-1}(5\mu+1, 5\mu) + \frac{19}{1250} a_1^{-1}(5\mu+1, 5\mu-1) \\
&\quad + \frac{48}{625} a_1^{-1}(5\mu+1, 5\mu) + \frac{24}{625} a_1^{-1}(5\mu+1, 5\mu-1) + \frac{19}{1250} a_1^{-1}(5\mu+1, 5\mu-2) \\
&= \frac{48}{625} \frac{1}{1-0.1304} + \frac{24}{625} \frac{1}{(1-0.1304)^2} + \frac{19}{1250} \frac{1}{(1-0.1304)^3} \\
&\quad + \frac{48}{625} \cdot \frac{1}{(1-0.1304)^4} + \frac{24}{625} \frac{1}{(1-0.1304)^5} + \frac{19}{1250} \frac{1}{(1-0.1304)^6} \\
&\quad + \frac{48}{625} \cdot 1 + \frac{24}{625} \frac{1}{1-0.1304} + \frac{19}{1250} \frac{1}{(1-0.1304)^2} \\
&\quad + \frac{48}{625} \frac{1}{1-0.1304} + \frac{24}{625} \frac{1}{(1-0.1304)^2} + \frac{19}{1250} \frac{1}{(1-0.1304)^3} \\
&\simeq 0.7122.
\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} \Phi_1(n) \simeq 0.7122 < 1.$$

Since

$$\alpha = \min_{1\le i\le 3} \alpha_i = \min\left\{\frac{48}{625}, \frac{48}{625}, \frac{57}{1250}\right\} = \frac{57}{1250} = 0.0456,$$

we have

$$\limsup_{n\to\infty} \Phi_1(n) \simeq 0.7122 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.9989.$$

Finally, the function $G_\ell : \mathbb{N}_0 \to \mathbb{R}_+$ *defined as*

$$G_\ell(n) = \sum_{j=h(n)}^{n} \mathcal{P}(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-\mathcal{P}_\ell(i)}$$

with $\mathcal{P}(n) = \sum_{i=1}^{3} p_i(n) = \frac{163}{1250}$, *attains its maximum at* $n = 5\mu + 4$, $\mu \in \mathbb{N}_0$, *for every* $\ell \in \mathbb{N}$. *Specifically,*

$$\begin{aligned}
G_1(n) &= \sum_{j=h(n)}^{n} \mathcal{P}(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-\mathcal{P}_1(i)} \\
&= \sum_{j=h(n)}^{n} \mathcal{P}(j) \prod_{i=\tau(j)}^{h(n)-1} \frac{1}{1-\mathcal{P}(i)\left[1+\sum_{k=\tau(i)}^{i-1} \mathcal{P}(k) \prod_{m=\tau(k)}^{h(i)-1} \frac{1}{1-\mathcal{P}(m)}\right]}
\end{aligned}$$

and therefore

$$\begin{aligned}
G_1(5\mu+4) &= \sum_{j=5\mu+1}^{5\mu+4} \mathcal{P}(j) \prod_{i=\tau(j)}^{5\mu} \frac{1}{1-\mathcal{P}(i)\left[1+\sum_{k=\tau(i)}^{i-1} \mathcal{P}(k) \prod_{m=\tau(k)}^{h(i)-1} \frac{1}{1-\mathcal{P}(m)}\right]} \\
&\simeq 0.7966.
\end{aligned}$$

Thus

$$\limsup_{n\to\infty} G_1(n) \simeq 0.7966 < 1,$$

$$\limsup_{n\to\infty} G_1(n) \simeq 0.7966 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.9989,$$

That is, none of conditions (1.8), (1.9), (1.15) *(for* $r = 1$*)*, (1.16) *(for* $r = 1$*)*, (1.17) *(for* $\ell = 1$*) and* (1.18) *(for* $\ell = 1$*) is satisfied.*

Notation. *It is worth noting that the improvement of condition (2.1) to the corresponding condition (1.9) is significant, approximately 93.52%, if we compare the values on the left-hand side of these conditions. Also, the improvement compared to conditions (1.15) and (1.17) is very satisfactory, around 41.73% and 26.71%, respectively. Also, observe that the conditions (1.15)–(1.18) do not lead to oscillation for the first iteration. On the contrary, condition (2.1) is satisfied from the first iteration. This means that our condition is better and much faster than (1.15)–(1.18).*

Example 3.2. *Consider the advanced difference equation*

$$\nabla x(n) - \frac{29}{250}x(\sigma_1(n)) - \frac{14}{375}x(\sigma_2(n)) - \frac{7}{375}x(\sigma_2(n)) = 0, \quad n \in \mathbb{N}, \tag{3.2}$$

with (see Fig. 2, (a))

$$\sigma_1(n) = \begin{cases} n+3 & \text{if } n = 5\mu+1 \\ n+1 & \text{if } n = 5\mu+2 \\ n+5 & \text{if } n = 5\mu+3 \\ n+2 & \text{if } n = 5\mu+4 \\ n+1 & \text{if } n = 5\mu+5 \end{cases} \quad \text{and} \quad \begin{array}{l} \sigma_2(n) = \sigma_1(n)+1 \\ \sigma_3(n) = \sigma_1(n)+2 \end{array}$$

where $\mu \in \mathbb{N}_0$ *and* $\mathbb{N}_0$ *is the set of non-negative integers.*

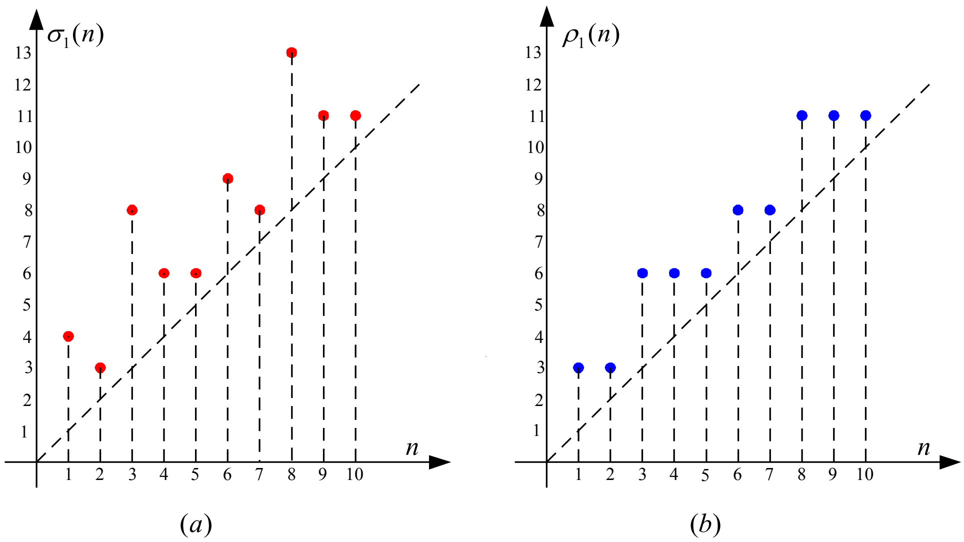

Figure 2: The graphs of $\sigma_1(n)$ and $\rho_1(n)$

By (1.21), we see (Fig. 2, (b)) that

$$\rho_1(n) = \begin{cases} n+2, & \text{if } n = 5\mu+1 \\ n+1, & \text{if } n = 5\mu+2 \\ n+3, & \text{if } n = 5\mu+3 \\ n+2, & \text{if } n = 5\mu+4 \\ n+1, & \text{if } n = 5\mu+5 \end{cases} \quad \text{and} \quad \begin{array}{l} \rho_2(n) = \rho_1(n)+1 \\ \rho_3(n) = \rho_1(n)+2 \end{array}$$

and consequently

$$\rho(n) = \min_{1\le i\le 3}\{\rho_i(n)\} = \rho_1(n).$$

Also, it is obvious that

$$\sigma(n) = \min_{1\le i\le 3}\sigma_i(n) = \sigma_1(n).$$

It is easy to see that $\sum_{i=1}^{3} q_i(n) = \frac{29}{250} + \frac{14}{375} + \frac{7}{375} = 0.172 < 1$, *i.e., (1.25) is satisfied, and*

$$\beta = \min_{1\le i\le 3}\beta_i = \min\left\{\frac{29}{250}, \frac{28}{375}, \frac{21}{375}\right\} = \frac{21}{375} = 0.056.$$

Thus

$$\frac{1}{D(\beta)} \simeq 600.9796.$$

We observe that the function $F : \mathbb{N}_0 \to \mathbb{R}_+$ *defined as*

$$F_w(n) = \sum_{\ell=n}^{\rho(n)} \overline{Q}(\ell) \prod_{i=n}^{\sigma(\ell)} \frac{1}{1-\overline{Q}_w(i)}$$

with $\overline{Q}(n) = \sum_{i=1}^{3} q_i(n) = \frac{43}{250}$, *attains its maximum at* $n = 5\mu+3$, $\mu \in \mathbb{N}_0$, *for every* $w \in \mathbb{N}$. *Specifically,*

$$\begin{aligned} F_1(n) &= \sum_{\ell=n}^{\rho(n)} \overline{Q}(\ell) \prod_{i=n}^{\sigma(\ell)} \frac{1}{1-\overline{Q}_1(i)} \\ &= \sum_{\ell=n}^{\rho(n)} \overline{Q}(\ell) \prod_{i=n}^{\sigma(\ell)} \frac{1}{1-\overline{Q}(i)\left[1+\sum_{j=i+1}^{\sigma(i)} \overline{Q}(j)\exp\left(\sum_{k=i+1}^{\sigma(j)} \overline{Q}(k) \prod_{u=k+1}^{\sigma(k)} \frac{1}{1-\overline{Q}_0(u)}\right)\right]}. \end{aligned}$$

Thus, by using an algorithm on MATLAB software, we obtain

$$F_1(5\mu+3) \simeq 602.1269.$$

Thus

$$\limsup_{n\to\infty} F_1(n) \simeq 602.1269 > \frac{1}{D(\beta)} \simeq 600.9796,$$

that is, condition (2.24) of Theorem 2.10 is satisfied for $w = 1$. *Therefore, all solutions of (3.2) are oscillatory.*

Observe, however, that

$$\begin{aligned} MA &= \limsup_{n\to\infty} \sum_{i=1}^{3} \sum_{j=5\mu+3}^{5\mu+6} q_i(j) \\ &= 4 \cdot \left(\frac{29}{250} + \frac{14}{375} + \frac{7}{375}\right) = 0.688 < 1. \end{aligned}$$

The function $G : \mathbb{N}_0 \to \mathbb{R}_+$ *defined as*

$$G_r(n) = \sum_{j=n}^{\rho(n)} \sum_{i=1}^{m} q_i(j) b_r^{-1}(\rho(n), \sigma_i(j))$$

attains its maximum at $n = 5\mu + 3$, $\mu \in \mathbb{N}_0$, *for every* $r \in \mathbb{N}$. *Specifically,*

$$\begin{aligned}
G_1(5\mu+3) &= \sum_{j=5\mu+3}^{5\mu+6} \sum_{i=1}^{3} q_i(j) b_1^{-1}(5\mu+6, \sigma_i(j)) \\
&= \frac{29}{250} b_1^{-1}(5\mu+6, \sigma_1(5\mu+3)) + \frac{14}{375} b_1^{-1}(5\mu+6, \sigma_2(5\mu+3)) + \frac{7}{375} b_1^{-1}(5\mu+6, \sigma_3(5\mu+3)) \\
&\quad + \frac{29}{250} b_1^{-1}(5\mu+6, \sigma_1(5\mu+4)) + \frac{14}{375} b_1^{-1}(5\mu+6, \sigma_2(5\mu+4)) + \frac{7}{375} b_1^{-1}(5\mu+6, \sigma_3(5\mu+4)) \\
&\quad + \frac{29}{250} b_1^{-1}(5\mu+6, \sigma_1(5\mu+5)) + \frac{14}{375} b_1^{-1}(5\mu+6, \sigma_2(5\mu+5)) + \frac{7}{375} b_1^{-1}(5\mu+6, \sigma_3(5\mu+5)) \\
&\quad + \frac{29}{250} b_1^{-1}(5\mu+6, \sigma_1(5\mu+6)) + \frac{14}{375} b_1^{-1}(5\mu+6, \sigma_2(5\mu+6)) + \frac{7}{375} b_1^{-1}(5\mu+6, \sigma_3(5\mu+6)) \\
&= \frac{29}{250} b_1^{-1}(5\mu+6, 5\mu+8) + \frac{14}{375} b_1^{-1}(5\mu+6, 5\mu+9) + \frac{7}{375} b_1^{-1}(5\mu+6, 5\mu+10) \\
&\quad + \frac{29}{250} b_1^{-1}(5\mu+6, 5\mu+6) + \frac{14}{375} b_1^{-1}(5\mu+6, 5\mu+7) + \frac{7}{375} b_1^{-1}(5\mu+6, 5\mu+8) \\
&\quad + \frac{29}{250} b_1^{-1}(5\mu+6, 5\mu+6) + \frac{14}{375} b_1^{-1}(5\mu+6, 5\mu+7) + \frac{7}{375} b_1^{-1}(5\mu+6, 5\mu+8) \\
&\quad + \frac{29}{250} b_1^{-1}(5\mu+6, 5\mu+9) + \frac{14}{375} b_1^{-1}(5\mu+6, 5\mu+10) + \frac{7}{375} b_1^{-1}(5\mu+6, 5\mu+11)
\end{aligned}$$

$$\begin{aligned}
&= \frac{29}{250}\left[\frac{1}{(1-0.172)^2} + 1 + 1 + \frac{1}{(1-0.172)^3}\right] \\
&\quad + \frac{14}{375}\left[\frac{1}{(1-0.172)^3} + 2\cdot\frac{1}{1-0.172} + \frac{1}{(1-0.172)^4}\right] \\
&\quad + \frac{7}{375}\left[\frac{1}{(1-0.172)^4} + 2\cdot\frac{1}{(1-0.172)^2} + \frac{1}{(1-0.172)^5}\right] \\
&\simeq 0.9831.
\end{aligned}$$

Therefore

$$\limsup_{n\to\infty} G_1(n) \simeq 0.9831 < 1$$

and

$$0.9831 < 1 - \frac{1-\beta-\sqrt{1-2\beta-\beta^2}}{2} \simeq 0.9983$$

That is, none of conditions (1.20), (1.26) *(for* $r = 1$*) and* (1.27) *(for* $r = 1$*) is satisfied.*

Notation. *It is worth noting that the conditions (1.26) and (1.27) do not lead to oscillation for the first iteration. On the contrary, condition (2.24) is satisfied from the first iteration. This means that our condition is better and much faster than (1.26) and (1.27).*

Remark 3.3. *Similarly, one can construct examples to illustrate the other main results.*

Appendix. In this appendix, for completeness, we give the algorithm on Matlab software used in Example 1 for calculation of $F_1(5\mu + 4)$. For Example 2, the algorithm is omitted since it is similar.

ALGORITHM for Example 1

```
clear; clc;
coeff = 0.1304;
alphas3 = 11;
betas3 = 14;
s3 = 0;
for is3 = alphas3 : 1 : betas3;
    alphap2 = TFunction(is3);
    betap2 = 10;
    p2 = 1;
    for ip2 = alphap2 : 1 : betap2;
        alphas2 = TFunction(ip2);
        betas2 = ip2-1;
        s2 = 0;
        for is2 = alphas2 : 1 : betas2;
            alphas1 = TFunction(is2);
            betas1 = ip2-1;
            s1 = 0;
            for is1 = alphas1 : 1 : betas1;
                alphap1 = TFunction(is1);
                betap1 = is1-1 ;
                p1 = 1;
                for ip1 = alphap1 : 1 : betap1;
                    if alphap1 > betap1;
                        p1 = 1;
                        else p1 = p1/(1-coeff);
                    end
                end
                if alphas1 > betas1;
                    s1 = 0;
                    else s1 = s1+coeff*p1;
                end
            end
            if alphas2 > betas2;
                s2 = 0;
                else s2 = s2+coeff*exp(s1);
            end
        end
        if alphap2 > betap2;
            p2 = 0;
            else p2 = p2/(1-coeff*(1+s2));
        end
    end
    if alphas3 > betas3;
        s3 = 0;
        else s3 = s3+coeff*p2;
    end
end
F1n = s3
```

ALGORITHMS for functions $\tau(n)$ and $h(n)$

```
function a = TFunction(x)
    r = mod(x,5);
    if(r==0);
        a = x-2;
    end
    if(r==1);
        a = x-1;
    end
    if(r==2);
        a = x-5;
    end
    if(r==3);
        a = x-2;
    end
    if(r==4);
        a = x-4;
    end
end
```

```
function a  = HFunction(x)
    r = mod(x,5);
    if(r==0);
        a = x-2;
    end
    if(r==1);
        a = x-1;
    end
    if(r==2);
        a = x-2;
    end
    if(r==3);
        a = x-2;
    end
    if(r==4);
        a = x-3;
    end
end
```

References

[1] L. Berezansky and E. Braverman, On existence of positive solutions for linear difference equations with several delays, *Advances Dyn. Syst. Appl.*, **1** (2006), 29–47.

[2] E. Braverman, G. E. Chatzarakis and I. P. Stavroulakis, Iterative oscillation tests for difference equations with several non-monotone arguments, *J. Difference Equ. Appl.*, **2015**, http://dx.doi.org/10.1080/10236198.2015.1051480, 21 pages.

[3] E. Braverman and B. Karpuz, On oscillation of differential and difference equations with non-monotone delays, *Appl. Math. Comput.*, **218** (2011), 3880–3887.

[4] Chatzarakis, Horvat-Dmitrović and Pašić, Oscillation tests for Difference Equations with several non-monotone deviating arguments, 2017 (to appear).

[5] G. E. Chatzarakis, T. Kusano and I. P. Stavroulakis, Oscillation conditions for difference equations with several variable arguments, *Math. Bohem.*, **140** (2015), No. 3, 291–311.

[6] G. E. Chatzarakis, J. Manojlovic, S. Pinelas and I. P. Stavroulakis, Oscillation criteria of difference equations with several deviating arguments, *Yokohama Math. J.*, **60** (2014), 13–31.

[7] G. E. Chatzarakis, S. Pinelas and I. P. Stavroulakis, Oscillations of difference equations with several deviated arguments, *Aequat. Math.*, **88** (2014), 105–123.

[8] L. H. Erbe, Q. K. Kong and B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 1995.

[9] X. Li and D. Zhu, Oscillation of advanced difference equations with variable coefficients, *Ann. Differential Equations*, **18** (2002), 254–263.

[10] X. N. Luo, Y. Zhou and C. F. Li, Oscillations of a nonlinear difference equation with several delays, *Mathematica Bohemica*, **128** (2003), 309–317.

[11] X. H. Tang and J. S. Yu, Oscillation of delay difference equations, *Comput. Math. Appl.*, **37** (1999), 11–20.

[12] X. H. Tang and R. Y. Zhang, New oscillation criteria for delay difference equations, *Comput. Math. Appl.*, **42** (2001), 1319–1330.

[13] X. Wang, Oscillation of delay difference equations with several delays, *J. Math. Anal. Appl.* **286** (2003), 664–674.

[14] X. P. Wu and L. Wang, Zero-Hopf bifurcation analysis in delayed differential equations with two delays, *J. Frankl. Inst.* **354** (2017), 1484–1513.

[15] W. Yan, Q. Meng and J. Yan, Oscillation criteria for difference equation of variable delays, *DCDIS Proceedings* **3** (2005), 641–647.

[16] B. G. Zhang and C. J. Tian, Nonexistence and existence of positive solutions for difference equations with unbounded delay, *Comput. Math. Appl.*, **36** (1998), 1–8.

[17] B. G. Zhang and Y. Zhou, Oscillations of difference equations with several delays, *Comput. Math. Appl.*, **44** (2002), 817–821.