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Existence and Convergence Theorem for Fixed Point Problem of Various Nonlinear Mappings and Variational Inequality Problems without Some Assumptions

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Abstract. The purpose of this article, we give a necessary and sufficient condition for the modified Mann iterative process in order to obtain a strong convergence theorem for finding a common element of the set of fixed point of a finite family of nonexpansive mappings and variational inequality problem in Hilbert space without the conditions $\bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A) \neq \emptyset$. Moreover, we utilize our main result to fixed point problems of strictly pseudocontractive mappings and the set of solutions of variational inequality problem.

1. Introduction

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a nonlinear mapping. A point $x \in C$ is called a *fixed point* of T if $Tx = x$. The set of fixed points of T is the set $Fix(T) := \{x \in C : Tx = x\}$.
A mapping T of C into itself is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

Mann's iteration process [8] is often used to approximate a fixed point of a nonexpansive mapping. But Mann's iteration process has only weak convergence. To obtain strong convergence theorems, the Mann's iteration is modified by many researchers; see for instance [7], [12], and the references therein.

Let $A : C \to H$. *The variational inequality problem* is to find a point $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0 \tag{1}$$

for all $y \in C$. The set of solution of (1) is denoted by $VI(C, A)$. In 1964, Stampacchia [13] introduced and investigated the variational inequality problem. It is well known that the application of the variational inequality problem has been expanded to problems from economics, finance, optimization and game theory; see [15]. Several authors have studied the variational inequality problem; see [16], [3], [4], and references cited therein.

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In 2003, Takahashi and Toyoda [5] introduce an iterative scheme of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem for an inverse strongly-monotone mapping as follows:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)TP_C(x_n - \lambda_n A x_n), \forall n \geq 1,$$

where $T : C \to C$ is a nonexpansive mapping and A is an a inverse strongly-monotone mapping of C into H. Then, they proved a weak convergence theorem of the sequence $\{x_n\}$ under suitable conditions of parameter $\{\alpha_n\}$ and $\{\lambda_n\}$.

In 2013, Kangtunyakarn [6] proved a strong convergence theorem for finding a common element of the set of fixed point problem of a nonexpansive mapping and the set of solution of (1) without assumption $Fix(T) \cap VI(C, A) \neq \emptyset$. He defined the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha T x_n + (1-\alpha)P_C(I - \rho A)x_n, \forall n \geq 1, \tag{2}$$

where $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ and positive real numbers α, ρ. In the last few decades many authors have studied strong convergence theorems for finding a common element of the set of fixed point of a finite family of nonexpansive mappings and the set of variational inequality problem by using condition $\bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A) \neq \emptyset$; see for instance [11] and references therein.

In this paper, motivated and inspired by [5] and [6], we give a necessary and sufficient condition for the modified Mann iterative process in order to obtain a strong convergence theorem for finding a common element of the set of fixed point of a finite family of nonexpansive mappings and the set of solutions of variational inequality problem in Hilbert space without the conditions $\bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A) \neq \emptyset$. Moreover, we utilize our main result to fixed point problems of strictly pseudocontractive mappings and the set of solutions of variational inequality problem.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

Lemma 2.1 ([9]). *Let A be a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma}$ and $0 < \rho < \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.2 (See [14]). *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let A be a mapping of C into H. Let $u \in C$. Then, for $\lambda > 0$,*

$$u = P_C(I - \lambda A)u \Leftrightarrow u \in VI(C, A),$$

where P_C is the metric projection of H onto C.

Lemma 2.3 ([1]). *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and S: $C \to C$ be a nonexpansive mapping. Then $I - S$ is demi-closed at zero.*

3. Main Result

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself. Let the sequence $\{x_n\}$ be generated by $x_0 \in H$ and*

$$\begin{cases} y_n^i = \beta T_i x_n + (1-\beta)x_n, \\ x_{n+1} = \alpha x_n + (1-\alpha)P_C(I - \rho A)\sum_{i=1}^{N} a^i y_n^i, \forall n \geq 0, \end{cases} \tag{3}$$

where $0 < \alpha, \beta < 1$, $0 < \rho < \|A\|^{-1}$, *and* $\sum_{i=1}^{N} a^i = 1$. *Then the following are equivalent.*

(i) *The sequence* $\{x_n\}$ *defined by (3) converges strongly to* $x^* \in \bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A)$;

(ii) $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, *for all* $i = 1, 2, \ldots, N$.

Proof. $(i) \Rightarrow (ii)$. *Let condition* (i) *hold. Since* $x^* \in \bigcap_{i=1}^{N} Fix(T_i) \cap VI(C, A)$, *we have*

$$\|T_i x_n - x_n\| \le \|T_i x_n - T_i x^*\| + \|x^* - x_n\| \le 2\|x^* - x_n\|,$$

which implies that $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$.
Next we claim that $(ii) \Rightarrow (i)$, *let condition* (ii) *hold. Let* $x, y \in C$. *Since* A *is a strongly positive linear bounded operator and Lemma 2.1 , we have*

$$\begin{aligned}\|(I - \rho A)x - (I - \rho A)y\| &= \|(I - \rho A)(x - y)\| \\ &\le (1 - \rho\bar{\gamma})\|x - y\|.\end{aligned}$$

We have $I - \rho A$ *is a contractive mapping with coefficient* $1 - \rho\bar{\gamma}$.
For every $n \in \mathbb{N}$, $i = 1, 2, ..., N$, *and the definition of* $\{y_n^i\}$, *we have*

$$\begin{aligned}\|y_{n+1}^i - y_n^i\| &= \|\beta T_i x_n + (1 - \beta)x_n - \beta T_i x_{n-1} - (1 - \beta)x_{n-1}\| \\ &= \|\beta(T_i x_n - T_i x_{n-1}) + (1 - \beta)(x_n - x_{n-1})\| \\ &\le \beta\|T_i x_n - T_i x_{n-1}\| + (1 - \beta)\|x_n - x_{n-1}\| \\ &\le \beta\|x_n - x_{n-1}\| + (1 - \beta)\|x_n - x_{n-1}\| \\ &= \|x_n - x_{n-1}\|. \end{aligned} \tag{4}$$

From the definition of $\{x_n\}$ *and (4), we have*

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\alpha x_n + (1 - \alpha)P_C(I - \rho A)\sum_{i=1}^{N} a^i y_n^i - \alpha x_{n-1} - (1 - \alpha)P_C(I - \rho A)\sum_{i=1}^{N} a^i y_{n-1}^i\| \\ &= \|\alpha(x_n - x_{n-1}) + (1 - \alpha)\left(P_C(I - \rho A)\sum_{i=1}^{N} a^i y_n^i - P_C(I - \rho A)\sum_{i=1}^{N} a^i y_{n-1}^i\right)\| \\ &\le \alpha\|x_n - x_{n-1}\| + (1 - \alpha)\|P_C(I - \rho A)\sum_{i=1}^{N} a^i y_n^i - P_C(I - \rho A)\sum_{i=1}^{N} a^i y_{n-1}^i\| \\ &\le \alpha\|x_n - x_{n-1}\| + (1 - \alpha)\|(I - \rho A)\sum_{i=1}^{N} a^i y_n^i - (I - \rho A)\sum_{i=1}^{N} a^i y_{n-1}^i\| \\ &\le \alpha\|x_n - x_{n-1}\| + (1 - \alpha)(1 - \rho\bar{\gamma})\sum_{i=1}^{N} a^i\|y_n^i - y_{n-1}^i\| \\ &\le \alpha\|x_n - x_{n-1}\| + (1 - \alpha)(1 - \rho\bar{\gamma})\|x_n - x_{n-1}\| \\ &= (1 - \rho\bar{\gamma}(1 - \alpha))\|x_n - x_{n-1}\| \\ &= a\|x_n - x_{n-1}\| \\ &\le a^2\|x_{n-1} - x_{n-2}\| \\ &\vdots \\ &\le a^n\|x_1 - x_0\|, \end{aligned} \tag{5}$$

where $a = (1 - \rho\bar{\gamma}(1-\alpha)) \in (0, 1)$.
For any number $n, m \in \mathbb{N}$ *and (5), we have*

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \sum_{j=n}^{n+m-1} \|x_{j+1} - x_j\| \\
&\leq \sum_{j=n}^{n+m-1} a^j \|x_1 - x_0\| \\
&\leq \left(\frac{a^n}{1-a}\right)\|x_1 - x_0\|. \qquad (6)
\end{aligned}$$

Since $a^n \to 0$ *as* $n \to \infty$, *and (6), we have* $\{x_n\}$ *is a Cauchy sequence. Since* H *is a Hilbert space, we get* $\{x_n\}$ *converges to* x^*, *i.e.,*

$$\lim_{n\to\infty} x_n = x^*. \qquad (7)$$

Next, we will show that $x^* \in \bigcap_{i=1}^N Fix(T_i) \cap VI(C, A)$. *Since* C *is closed, so we get* $x^* \in C$. *By* $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, *(7), and Lemma 2.3, we have* $x^* \in Fix(T_i)$ *for all* $i = 1, 2, \ldots, N$. *It implies that* $x^* \in \bigcap_{i=1}^N Fix(T_i)$. *From the definition of* y_n^i, $\lim_{n\to\infty} x_n = x^*$, *and* $x^* \in \bigcap_{i=1}^N Fix(T_i)$, *we have*

$$\lim_{n\to\infty} y_n^i = x^*. \qquad (8)$$

From the definition of x_n, *(7), and (8), we have*

$$x^* = \alpha x^* + (1-\alpha)P_C(I - \rho A)x^*.$$

It implies that $x^* \in Fix(P_C(I - \rho A))$. *From Lemma 2.2, we have* $x^* \in VI(C, A)$. *Hence, the sequence* $\{x_n\}$ *defined by (3) converges strongly to* $x^* \in \bigcap_{i=1}^N Fix(T_i) \cap VI(C, A)$. □

As direct proof of Theorem 3.1, we obtain the following results.

Corollary 3.2. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $A : C \to H$ *be a strongly positive linear bounded operator on* H *with coefficient* $\bar{\gamma} > 0$. *Let* T *be a nonexpansive mappings of* C *into itself. Let the sequence* $\{x_n\}$ *be generated by* $x_0 \in H$ *and*

$$\begin{cases} y_n = \beta T x_n + (1-\beta)x_n, \\ x_{n+1} = \alpha x_n + (1-\alpha)P_C(I - \rho A)y_n, \forall n \geq 0, \end{cases} \qquad (9)$$

where $0 < \alpha, \beta < 1$ *and* $0 < \rho < \|A\|^{-1}$. *Then the following are equivalent.*

(i) *The sequence* $\{x_n\}$ *defined by (9) converges strongly to* $x^* \in Fix(T) \cap VI(C, A)$;

(ii) $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.

Next, in order to prove a strong convergence theorem for κ-strictly pseudo-contractive mappings and variational inequality problem, we need Lemma 3.3. A mapping $T : C \to C$ is said to be κ-strictly pseudo-contractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\left\|Tx - Ty\right\|^2 \leq \left\|x - y\right\|^2 + \kappa \left\|(I - T)x - (I - T)y\right\|^2,$$

for all $x, y \in C$. Note that the class of strictly pseudo-contractions strictly includes the class of nonexpansive mapping.

Lemma 3.3 (See [2]). *Let $T : C \to H$ be a κ-strict pseudo-contraction. Define $S : C \to H$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in C$. Then, as $\lambda \in [k, 1)$, S is a nonexpansive mapping such that $Fix(S) = Fix(T)$.*

Theorem 3.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a strongly positive linear bounded operator on H with coefficient $\bar{\gamma} > 0$. Let $\{T_i\}_{i=1}^N$ be a finite family of κ_i-strictly pseudo-contractive mappings of C into itself with $\kappa = \max_{i=1,2,\ldots,N} \kappa_i$. Define the mapping $S_i = C \to C$ by $S_i x = \sigma x + (1-\sigma)T_i x$ for every $i = 1, 2, ..., N$, $x \in C$ and $\sigma \in (k, 1)$. Let the sequence $\{x_n\}$ be generated by $x_0 \in H$ and*

$$\begin{cases} y_n^i = \beta S_i x_n + (1-\beta)x_n, \\ x_{n+1} = \alpha x_n + (1-\alpha)P_C(I-\rho A)\sum_{i=1}^{N} a^i y_n^i, \forall n \geq 0, \end{cases} \tag{10}$$

where $0 < \alpha < 1$, $\kappa \leq \beta < 1$, $0 < \rho < \|A\|^{-1}$, and $\sum_{i=1}^N a^i = 1$. Then the following are equivalent.

(i) *The sequence $\{x_n\}$ defined by (10) converges strongly to $x^* \in \bigcap_{i=1}^N Fix(T_i) \cap VI(C, A)$;*

(ii) $\lim_{n\to\infty} \|T_i x_n - x_n\| = 0$, *for all $i = 1, 2, \ldots, N$.*

Proof. *From Lemma 3.3 and Theorem 3.1, we obtain the desired result.* □

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