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A New Estimate for the Spectral Radius of Nonnegative Tensors

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Abstract. In this paper, we are concerned with the spectral radius of nonnegative tensors. By estimating the ratio of the smallest component and the largest component of a Perron vector, a new bound for the spectral radius of nonnegative tensors is obtained. It is proved that the new bound improves some existing ones. Finally, a numerical example is implemented to show the effectiveness of the proposed bound.

1. Introduction

Let $\mathbb{C}(\mathbb{R})$ denote the set of all complex (real) field. We consider an m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ consisting of n^m entries, denoted by $\mathcal{A} \in \mathbb{C}^{[m,n]}(\mathbb{R}^{[m,n]})$, if

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{C}(\mathbb{R}),$$

where $i_j = 1, 2, \cdots, n$ for $j = 1, 2, \cdots, m$ [6, 23]. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of 2. An m-order n-dimensional tensor $\mathcal{A}$ is called nonnegative (or, respectively, positive), if $a_{i_1 \cdots i_m} \geq 0$ (or, respectively, $a_{i_1 \cdots i_m} > 0$) for all $i_1, \cdots, i_m$. Moreover, if there are a complex number λ and a nonzero complex vector $x = (x_1, \cdots, x_n)^T$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

then λ is called an eigenvalue of $\mathcal{A}$ and x an eigenvector of $\mathcal{A}$ associated with λ, where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are vectors, whose ith components are

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \cdots, i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}$$

and

$$(x^{[m-1]})_i = x_i^{m-1}, \ 1 \leq i \leq n,$$

respectively. This definition was introduced by Qi in [23] where he assumed that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is symmetric and m is even. Independently, in [17], Lim gave such a definition but restricted x to be a real vector and λ to be a real number. That is, if λ and x are restricted to the real field, then we call λ an H-eigenvalue of $\mathcal{A}$ and x an H-eigenvector of $\mathcal{A}$ associated with λ.

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Definition 1.1. *A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is called reducible if there exists a nonempty proper index subset $I \subset \{1, \cdots, n\}$ such that*

$$a_{i_1 \cdots i_m} = 0, \forall i_1 \in I, i_2 \cdots i_m \notin I.$$

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

In [5], Friedland et al. introduced nonnegative weakly irreducible tensor by considering the graph associated to tensors. Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an order m dimension n nonnegative tensor. The graph associated to $\mathcal{A}$, $G(\mathcal{A})$, is the directed graph with vertices $1, \cdots, n$ and an edge from i to j if and only if $a_{i i_2 \cdots i_m} > 0$ for some $i_l = j, l = 2, \cdots m$.

Definition 1.2. *A tensor $\mathcal{A} \in \mathbb{C}^{[m,n]}$ is called weakly irreducible if $G(\mathcal{A})$ is strongly connected.*

Subsequently, Yang and Yang [26] gave equivalent definition of weakly irreducible tensors as follows.

Definition 1.3. *A tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{C}^{[m,n]}$ is called weakly reducible if there exists a nonempty proper index subset $I \subset \{1, \cdots, n\}$ such that*

$$a_{i_1 \cdots i_m} = 0, \forall i_1 \in I, \exists i_j \notin I, j = 2, \cdots, m.$$

If $\mathcal{A}$ is not weakly reducible, then we call $\mathcal{A}$ weakly irreducible.

Obviously, if $\mathcal{A}$ is irreducible, then $\mathcal{A}$ is weakly irreducible, but not vice versa. And when the order of $\mathcal{A}$ is 2, $\mathcal{A}$ is irreducible if and only if $\mathcal{A}$ is weakly irreducible [5].

In [3], Chang et al. generalized the Perron-Frobenius theorem from nonnegative irreducible matrices to nonnegative irreducible tensors.

Theorem 1.4. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be an irreducible nonnegative tensor, and the spectral radius $\rho(\mathcal{A})$ of $\mathcal{A}$ is defined as*

$$\rho(\mathcal{A}) = \max\{|\lambda| | \lambda \in \sigma(\mathcal{A})\},$$

where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$, that is, the set contains all eigenvalues of $\mathcal{A}$. Then $\rho(\mathcal{A})$ is an eigenvalue of $\mathcal{A}$ with a positive eigenvector x corresponding to it.

Note that $\rho(\mathcal{A})$ and x are called the Perron root and the Perron vector of $\mathcal{A}$, respectively, and $(\rho(\mathcal{A}), x)$ is regarded as a Perron eigenpair. A perron vector can be used for co-ranking schemes [13, 19] for objects and relations in multi-relational or tensor data, and higher-order Markov chains [4, 14, 15].

Hereafter, Friedland et al. generalized the result in Theorem 1.4 to weakly irreducible nonnegative tensors in [5].

Theorem 1.5. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly irreducible nonnegative tensor. Then $\rho(\mathcal{A}) > 0$ is an eigenvalue $\mathcal{A}$ with a positive eigenvector x corresponding to it.*

Eigenvalue problems of higher-order tensors have become an important topic of study in a new applied mathematics branch, numerical multilinear algebra, and they have a wide range of practical applications [2, 18, 20–22]. By Theorem 1.5, it is known that the spectral radius of the weakly irreducible nonnegative tensor is also an eigenvalue. In recent years, much literature has focused on the bounds for the spectral radius of nonnegative tensors, and given various bounds to estimate the spectral radius, or algorithms to find the spectral radius, for more details, see [1, 7–12, 16, 18, 20, 24–26]. In [25], Yang and Yang extended the classical spectral radius bound for nonnegative matrices to nonnegative tensors and obtained the following result.

Lemma 1.6. [25] *Let* $\mathcal{A} \in \mathbb{R}^{[m,n]}$ *be a nonnegative tensor, then*

$$r \leq \rho(\mathcal{A}) \leq R, \tag{1}$$

where

$$r_i(\mathcal{A}) = \sum_{i_2,\cdots,i_m=1}^{n} a_{ii_2\cdots i_m}, \quad r = \min_i r_i(\mathcal{A}), \quad R = \max_i r_i(\mathcal{A}).$$

In the sequel, to obtain sharper bounds for the spectral radius of nonnegative tensors, Li and Ng [16] estimated the ratio of the smallest component and the largest component of a Perron vector and gave the following bound for the spectral radius of a nonnegative tensor and proved that it is better than the bound in (1).

Lemma 1.7. [16] *Let* $\mathcal{A} \in \mathbb{R}^{[m,n]}$ *be a nonnegative tensor, then*

$$\nu(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \omega(\mathcal{A}), \tag{2}$$

where

$$\begin{aligned}
\nu(\mathcal{A}) &= \min_{i,j} \left\{ a_{ij\cdots j} \left(\frac{1}{\tau(\mathcal{A})^{m-1}} - 1 \right) + r_i(\mathcal{A}) \right\}, \\
\omega(\mathcal{A}) &= \max_{i,j} \{ r_i(\mathcal{A}) - a_{ij\cdots j}(1 - \tau(\mathcal{A})^{m-1}) \}, \\
\tau(\mathcal{A}) &= \left(\frac{r - \beta_0(\mathcal{A})}{R - \beta_0(\mathcal{A})} \right)^{\frac{1}{2(m-1)}}, \\
\beta_0(\mathcal{A}) &= \min_{i,j} \{ a_{ij\cdots j} \}.
\end{aligned}$$

Furthermore, $r \leq \nu(\mathcal{A}) \leq \omega(\mathcal{A}) \leq R$.

In addition, recently, authors in [8] presented a new lower bound and a new upper bound for the spectral radius of a nonnegative tensor by giving a new ratio of the smallest component and the largest component of a Perron vector. It is proved that this bound is better than the one in (2).

Lemma 1.8. [8] *Let* $\mathcal{A} \in \mathbb{R}^{[m,n]}$ *be a weakly irreducible nonnegative tensor. Then*

$$\mathcal{L}(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \mathcal{U}(\mathcal{A}), \tag{3}$$

where

$$\begin{aligned}
\mathcal{L}(\mathcal{A}) &= \min_{i,j} \left\{ a_{ij\cdots j} \left[\frac{1}{\zeta(\mathcal{A})^{m-1}} - 1 \right] + r_i(\mathcal{A}) \right\}, \\
\mathcal{U}(\mathcal{A}) &= \max_{i,j} \{ r_i(\mathcal{A}) + \sum_{(i_2,\cdots,i_m) \in \cup_{k=1}^{m-1} \Delta(j;k)} a_{ij\cdots j}(\zeta(\mathcal{A})^{m-1} - 1) \}, \\
\zeta(\mathcal{A}) &= \left\{ \frac{r - \beta_0(\mathcal{A}) - \sum_{k=1}^{m-2} [\binom{m-1}{k} (n-1)^k \beta_k(\mathcal{A}) (1 - (\frac{r-\beta_0(\mathcal{A})}{R-\beta_0(\mathcal{A})})^{\frac{m-k-1}{2(m-1)}})]}{R - \beta_0(\mathcal{A})} \right\}^{\frac{1}{2(m-1)}}, \\
\beta_t(\mathcal{A}) &= \min_{i,j} \{ a_{ii_2\cdots i_m} \in \Delta(j; m-t-1) \}, t = 0, 1, \cdots, m-2, \\
\Delta(j; u) &= \bigcup_{\substack{S \subseteq \{2,\cdots,m\}, \\ |S| = u}} \{ (i_2, \cdots, i_m) : i_v = j, \forall v \in S, \text{and } i_v \neq j, \forall v \notin S \}, u = 0, 1, \cdots, m-1.
\end{aligned}$$

In the current work, we continue this research on estimates of the spectral radius for nonnegative tensors; inspired by the ideas of [27], we estimate a new ratio of the smallest component and the largest component of a Perron vector. Afterward, we present a new lower bound and a new upper bound for the spectral radius and compare this bound with some known bounds.

The remainder of the paper is organized as follows. In Section 2, we focus on the bound of the spectral radius and establish a new ratio of the smallest component and the largest component of a Perron vector. Based on the result, a new lower bound and a new upper bound for the spectral radius of a nonnegative tensor are obtained. In addition, we propose a comparison theorem for nonnegative tensors which indicates our bound improves some existing results. In Section 3, numerical example is reported to illustrate the effectiveness of the new bound. Finally, some conclusions are given to end this paper in Section 4.

2. A New Bound for the Spectral Radius of Nonnegative Tensors

In this section, we first establish a lemma to estimate the ratio of the smallest component and the largest component of a Perron vector. Then based on the result of the lemma, we investigate the bound for the spectral radius of nonnegative tensors and derive a tighter bound for that. This bound is proved to be superior to those in Lemma 1.6 and Lemma 1.7.

For convenience's sake, throughout this paper, we denote $N = \{0, 1, \cdots, n, \cdots\}, \langle p \rangle = \{1, 2, \cdots, p\}$, where p is positive integer number; $i_k(s) = (i_{k_1}, i_{k_2}, \cdots, i_{k_s})$ with $k_1 < k_2 < \cdots < k_s$; $\Delta_s = \{t_1, t_2, \cdots, t_s\}, \nabla_s = \{k_1, k_2, \cdots, k_s\}$ and

$$\Lambda_j = \{(i_{t_1}, i_{t_2}, \cdots, i_{t_s}) \backslash (j, j, \cdots, j) | i_{t_k} \in \langle n \rangle, k \in \langle s \rangle\}.$$

i_{t_s} and Λ_v are same as above whenever they occurs.

Lemma 2.1. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly irreducible nonnegative tensor with a Perron vector x, and let $x_v = \min_{i \in \langle n \rangle}\{x_i\}, x_l = \max_{i \in \langle n \rangle}\{x_i\}$ and s be a nonnegative integer. Then*

$$\frac{x_v}{x_l} \leq \phi_s(\mathcal{A}), \tag{4}$$

where

$$\phi_s(\mathcal{A}) = \left\{ \frac{r - \min\limits_{i,j} a_{ij\cdots j} - k_s(1 - \beta^{\frac{m-1-s}{2(m-1)}})}{R - \min\limits_{i,j} a_{ij\cdots j}} \right\}^{\frac{1}{2(m-1)}}, \quad \beta = \frac{r - \min\limits_{i,j} a_{ij\cdots j}}{R - \min\limits_{i,j} a_{ij\cdots j}},$$

$$k_s = \begin{cases} \min\limits_{2 \leq t_1 < t_2 < \cdots < t_s \leq m} (\min\limits_{i,j} \sum\limits_{(i_{t_1} i_{t_2}, \cdots, i_{t_s}) \in \wedge_j} a_{ij\cdots j i_{t_1} j \cdots j i_{t_2} j \cdots j i_{t_s} j \cdots j}) & s \geq 1, \\ 0 \quad s = 0. \end{cases}$$

Proof. Since $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in \mathbb{R}^{[m,n]}$ is a weakly irreducible nonnegative tensor, by Theorem 1.5, we have $\mathcal{A}x^{m-1} = \rho(\mathcal{A})x^{[m-1]}$ and a positive eigenvector $x = (x_1, x_2, \cdots, x_m)^T$. When $s = 0$, our lemma reduces to Lemma 2.16 in [16]. Hence we only prove that Inequality (4) holds for $s \geq 1$. Let $r_p = R, r_q = r$, then for any $i \in \langle n \rangle$, we have

$$\begin{aligned}
\rho(\mathcal{A})x_i^{m-1} &= \sum_{i_2, \cdots, i_m \in \langle n \rangle} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \\
&= a_{il \cdots l} x_l^{m-1} + \sum_{\substack{(i_2, \cdots, i_m) \in \langle n \rangle, \\ \delta_{li_2 \cdots i_m = 0}}} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m} \\
&\geq a_{il \cdots l} x_l^{m-1} + \sum_{\substack{(i_2, \cdots, i_m) \in \langle n \rangle, \\ \delta_{li_2 \cdots i_m = 0}}} a_{ii_2 \cdots i_m} x_v^{m-1} \\
&= a_{il \cdots l}(x_l^{m-1} - x_v^{m-1}) + r_i(\mathcal{A}) x_v^{m-1}.
\end{aligned} \tag{5}$$

Taking $i = p$ in Inequality (5), we derive that

$$\rho(\mathcal{A})x_p^{m-1} \geq a_{pl\cdots l}(x_l^{m-1} - x_v^{m-1}) + Rx_v^{m-1}. \tag{6}$$

Multiply Inequality (6) with $x_p^{-(m-1)}$ and note that $x_l = \max_{i\in\langle n\rangle}\{x_i\}$, then

$$\begin{aligned} \rho(\mathcal{A}) &\geq a_{pl\cdots l}\frac{x_l^{m-1} - x_v^{m-1}}{x_p^{m-1}} + R(\frac{x_v}{x_p})^{m-1} \\ &\geq a_{pl\cdots l}(1 - (\frac{x_v}{x_l})^{m-1}) + R(\frac{x_v}{x_l})^{m-1} \\ &\geq \min_{i,j} a_{ij\cdots j}(1 - (\frac{x_v}{x_l})^{m-1}) + R(\frac{x_v}{x_l})^{m-1} \\ &= \min_{i,j} a_{ij\cdots j} + (R - \min_{i,j} a_{ij\cdots j})(\frac{x_v}{x_l})^{m-1}. \end{aligned} \tag{7}$$

On the other hand, for any $i \in \langle n\rangle$ and any $t_1, t_2, \cdots, t_s$ satisfying $2 \leq t_1 < t_2 < \cdots < t_s \leq m$,

$$\begin{aligned} \rho(\mathcal{A})x_i^{m-1} &= \sum_{i_2,\cdots,i_m\in\langle n\rangle} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m} \\ &= \sum_{i_k=1,k\in\langle m\rangle\backslash\{1,\triangle_s\}}^{n} \sum_{i_k=1,k\in\triangle_s}^{n} a_{ii_2\cdots i_{t_1}\cdots i_{t_2}\cdots i_{t_s}\cdots i_m}x_{i_2}\cdots x_{i_m} \\ &\leq a_{iv\cdots v}x_v^{m-1} + \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_v} a_{iv\cdots vi_{t_1}v\cdots vi_{t_2}v\cdots vi_{t_s}v\cdots v}x_v^{m-1-s}x_l^s \\ &\quad + \sum_{\substack{i_k=1,k\in\langle m\rangle\backslash\{1,\triangle_s\},\\ \text{and at least one } i_k\neq v}}^{n} \sum_{i_k=1,k\in\triangle_s}^{n} a_{ii_2\cdots i_{t_1}\cdots i_{t_2}\cdots i_{t_s}\cdots i_m}x_l^{m-1} \\ &= a_{iv\cdots v}x_v^{m-1} + \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_v} a_{iv\cdots vi_{t_1}v\cdots vi_{t_2}v\cdots vi_{t_s}v\cdots v}x_v^{m-1-s}x_l^s \\ &\quad +(r_i(\mathcal{A}) - a_{iv\cdots v} - \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_v} a_{iv\cdots vi_{t_1}v\cdots vi_{t_2}v\cdots vi_{t_s}v\cdots v})x_l^{m-1} \\ &= r_i(\mathcal{A})x_l^{m-1} + a_{iv\cdots v}(x_v^{m-1} - x_l^{m-1}) \\ &\quad +x_l^s \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_v} a_{iv\cdots vi_{t_1}v\cdots vi_{t_2}v\cdots vi_{t_s}v\cdots v}(x_v^{m-1-s} - x_l^{m-1-s}). \end{aligned} \tag{8}$$

Taking $i = q$ in the above inequality gives

$$\begin{aligned} \rho(\mathcal{A})x_q^{m-1} &\leq rx_l^{m-1} + a_{qv\cdots v}(x_v^{m-1} - x_l^{m-1}) \\ &\quad +x_l^s \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_v} a_{qv\cdots vi_{t_1}v\cdots vi_{t_2}v\cdots vi_{t_s}v\cdots v}(x_v^{m-1-s} - x_l^{m-1-s}) \\ &\leq rx_l^{m-1} + \min_{i,j} a_{ij\cdots j}(x_v^{m-1} - x_l^{m-1}) + k_s(x_v^{m-1-s}x_l^s - x_l^{m-1}) \\ &= \min_{i,j} a_{ij\cdots j}x_v^{m-1} + k_s x_v^{m-1-s}x_l^s + (r - \min_{i,j} a_{ij\cdots j} - k_s)x_l^{m-1}, \end{aligned} \tag{9}$$

where

$$k_s = \min_{2\leq t_1<t_2<\cdots<t_s\leq m}(\min_{i,j} \sum_{(i_{t_1}i_{t_2},\cdots,i_{t_s})\in\wedge_j} a_{ij\cdots ji_{t_1}j\cdots ji_{t_2}j\cdots ji_{t_s}j\cdots j}).$$

Dividing by x_q^{m-1} on the both sides of (9) and $x_v = \min_{i\in\langle n\rangle}\{x_i\}$, we have

$$\rho(\mathcal{A}) \le \min_{i,j} a_{ij\cdots j} + k_s(\frac{x_l}{x_v})^s + (r - \min_{i,j} a_{ij\cdots j} - k_s)(\frac{x_l}{x_v})^{m-1}. \tag{10}$$

Combining Inequality (7) and Inequality (10), one derives

$$(R - \min_{i,j} a_{ij\cdots j})(\frac{x_v}{x_l})^{m-1} \le k_s(\frac{x_l}{x_v})^s + (r - \min_{i,j} a_{ij\cdots j} - k_s)(\frac{x_l}{x_v})^{m-1}.$$

Multiplying by $(\frac{x_v}{x_l})^{m-1}$ on the both sides of the above inequality yields

$$(R - \min_{i,j} a_{ij\cdots j})(\frac{x_v}{x_l})^{2(m-1)} \le k_s(\frac{x_v}{x_l})^{m-1-s} + (r - \min_{i,j} a_{ij\cdots j} - k_s), \tag{11}$$

Note that it is not easy to get the bound of $\frac{x_v}{x_l}$ simply from (11). However, we can overcome this difficulty by using the fact that $0 < \frac{x_v}{x_l} \le 1$ for the right-hand side of (11). Hence, one get

$$(R - \min_{i,j} a_{ij\cdots j})(\frac{x_v}{x_l})^{2(m-1)} \le r - \min_{i,j} a_{ij\cdots j}.$$

It follows from the above inequality that

$$\frac{x_v}{x_l} \le \beta^{\frac{1}{2(m-1)}},$$

where

$$\beta = \frac{r - \min_{i,j} a_{ij\cdots j}}{R - \min_{i,j} a_{ij\cdots j}},$$

which together with Inequality (11) results in

$$\begin{aligned}(\frac{x_v}{x_l})^{2(m-1)} &\le \frac{k_s\beta^{\frac{m-1-s}{2(m-1)}} + (r - \min_{i,j} a_{ij\cdots j} - k_s)}{R - \min_{i,j} a_{ij\cdots j}}\\ &= \phi_s(\mathcal{A})^{2(m-1)}\end{aligned}$$

with

$$\phi_s(\mathcal{A}) = \left\{\frac{r - \min_{i,j} a_{ij\cdots j} - k_s(1 - \beta^{\frac{m-s-1}{2(m-1)}})}{R - \min_{i,j} a_{ij\cdots j}}\right\}^{\frac{1}{2(m-1)}}.$$

This completes the proof. □

Next, we present a new bound for weakly irreducible nonnegative tensors based on Lemma 2.1.

Theorem 2.2. *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly irreducible nonnegative tensor. Then*

$$\mathcal{F}(\mathcal{A}) \le \rho(\mathcal{A}) \le \mathcal{H}(\mathcal{A}), \tag{12}$$

where

$$\begin{aligned}\mathcal{F}(\mathcal{A}) = &\max_{2\le t_1<t_2<\cdots<t_m\le m}\left\{\min_{i,j}\left\{r_i(\mathcal{A}) + a_{ij\cdots j}(\frac{1}{\phi_s(\mathcal{A})^{m-1}} - 1)\right.\right.\\ &+ \left.\left.\sum_{(i_{t_1} i_{t_2},\cdots,i_{t_s})\in\wedge_j} a_{ij\cdots j i_{t_1} j\cdots j i_{t_2} j\cdots j i_{t_s} j\cdots j}(\frac{1}{\phi_s(\mathcal{A})^{m-1-s}} - 1)\right\}\right\}\end{aligned}$$

and

$$\mathcal{H}(\mathcal{A}) = \min_{2\le t_1<t_2<\cdots<t_m\le m}\left\{\max_{i,j}\left\{r_i(\mathcal{A}) + a_{ij\cdots j}(\phi_s(\mathcal{A})^{m-1}-1) + \sum_{(i_{t_1}i_{t_2},\cdots,i_{t_s})\in\wedge_j} a_{ij\cdots ji_{t_1}j\cdots ji_{t_2}j\cdots ji_{t_s}j\cdots j}(\phi_s(\mathcal{A})^{m-1-s}-1)\right\}\right\}.$$

Proof. Since $\mathcal{A}$ is a weakly irreducible nonnegative tensor, there is a positive Perron vector $x=(x_1,x_2,\cdots,x_n)^T$ such that $\mathcal{A}x^{m-1}=\rho(\mathcal{A})x^{[m-1]}$. Suppose that $x_v=\min_{i\in\langle n\rangle}\{x_i\}$ and $x_l=\max_{i\in\langle n\rangle}\{x_i\}$, we have that for any $t_1,t_2,\cdots,t_s$ satisfying $2\le t_1<t_2<\cdots<t_s\le m$ and each $i\in\langle n\rangle$,

$$\begin{aligned}\rho(\mathcal{A})x_i^{m-1} &= \sum_{i_k=1,k\in\langle m\rangle\backslash\{1,\triangle_s\}}^{n}\ \sum_{i_k=1,k\in\triangle_s}^{n} a_{ii_2\cdots i_{t_1}\cdots i_{t_2}\cdots i_{t_s}\cdots i_m}x_{i_2}\cdots x_{i_m}\\ &\ge a_{il\cdots l}x_l^{m-1} + \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_l} a_{il\cdots li_{t_1}l\cdots li_{t_2}l\cdots li_{t_s}l\cdots l}x_l^{m-1-s}x_v^s\\ &\quad + \sum_{\substack{i_k=1,k\in\langle m\rangle\backslash\{1,\triangle_s\},\\ \text{and at least one } i_k\ne l}}^{n}\ \sum_{i_k=1,k\in\triangle_s}^{n} a_{ii_2\cdots i_{t_1}\cdots i_{t_2}\cdots i_{t_s}\cdots i_m}x_v^{m-1}\\ &= r_i(\mathcal{A})x_v^{m-1} + a_{il\cdots l}(x_l^{m-1}-x_v^{m-1}) + x_v^s\sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_l} a_{il\cdots li_{t_1}l\cdots li_{t_2}l\cdots li_{t_s}l\cdots l}(x_l^{m-1-s}-x_v^{m-1-s}).\end{aligned}$$

Taking $i=v$ and multiplying by $x_v^{-(m-1)}$ on both sides of the above inequality give

$$\rho(\mathcal{A}) \ge r_v(\mathcal{A}) + a_{vl\cdots l}((\frac{x_l}{x_v})^{m-1}-1) + \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_l} a_{vl\cdots li_{t_1}l\cdots li_{t_2}l\cdots li_{t_s}l\cdots l}((\frac{x_l}{x_v})^{m-1-s}-1)$$

Combining the above inequality with Lemma 2.1, we obtain

$$\begin{aligned}\rho(\mathcal{A}) &\ge r_v(\mathcal{A}) + a_{vl\cdots l}(\frac{1}{\phi_s(\mathcal{A})^{m-1}}-1) + \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_l} a_{vl\cdots li_{t_1}l\cdots li_{t_2}l\cdots li_{t_s}l\cdots l}(\frac{1}{\phi_s(\mathcal{A})^{m-1-s}}-1)\\ &\ge \min_{i,j}\left\{r_i(\mathcal{A}) + a_{ij\cdots j}(\frac{1}{\phi_s(\mathcal{A})^{m-1}}-1) + \sum_{(i_{t_1}i_{t_2},\cdots,i_{t_s})\in\wedge_j} a_{ij\cdots ji_{t_1}j\cdots ji_{t_2}j\cdots ji_{t_s}j\cdots j}(\frac{1}{\phi_s(\mathcal{A})^{m-1-s}}-1)\right\}.\end{aligned}$$

Since this could be ture for any $t_1,t_2,\cdots,t_s$ satisfying $2\le t_1<t_2<\cdots<t_s\le m$, we can obtain the first inequality of (12).

Next, we prove the right inequality of (12). From (8), we have

$$\rho(\mathcal{A})x_i^{m-1} \le r_i(\mathcal{A})x_l^{m-1} + a_{iv\cdots v}(x_v^{m-1}-x_l^{m-1}) + x_l^s\sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_v} a_{iv\cdots vi_{t_1}v\cdots vi_{t_2}v\cdots vi_{t_s}v\cdots v}(x_v^{m-1-s}-x_l^{m-1-s})$$

Taking $i=l$ and multiplying by $x_l^{-(m-1)}$ on both sides of the above inequality, together with Lemma 2.1, lead to

$$\begin{aligned}\rho(\mathcal{A}) &\le r_l(\mathcal{A}) + a_{lv\cdots v}(\phi_s(\mathcal{A})^{m-1}-1) + \sum_{(i_{t_1}i_{t_2}\cdots i_{t_s})\in\wedge_v} a_{lv\cdots vi_{t_1}v\cdots vi_{t_2}v\cdots vi_{t_s}v\cdots v}(\phi_s(\mathcal{A})^{m-1-s}-1)\\ &\le \max_{i,j}\left\{r_i(\mathcal{A}) + a_{ij\cdots j}(\phi_s(\mathcal{A})^{m-1}-1) + \sum_{(i_{t_1}i_{t_2},\cdots,i_{t_s})\in\wedge_j} a_{ij\cdots ji_{t_1}j\cdots ji_{t_2}j\cdots ji_{t_s}j\cdots j}(\phi_s(\mathcal{A})^{m-1-s}-1)\right\}.\end{aligned}$$

Because of the arbitrariness of $t_1, t_2, \cdots, t_s$, we finally have

$$\rho(\mathcal{A}) \leq \min_{2\leq t_1<t_2<\cdots<t_m\leq m} \left\{ \max_{i,j} \left\{ r_i(\mathcal{A}) + a_{ij\cdots j}(\phi_s(\mathcal{A})^{m-1} - 1) + \sum_{(i_{t_1} i_{t_2}, \cdots, i_{t_s})\in\wedge_j} a_{ij\cdots ji_{t_1} j\cdots ji_{t_2} j\cdots ji_{t_s} j\cdots j}(\phi_s(\mathcal{A})^{m-1-s} - 1) \right\} \right\},$$

The proof is completed. □

Remark 2.3. *As discussed in Remark 1 in [8], the bound in Theorem 2.2 still holds for general nonnegative tensors without the assumption that $\mathcal{A}$ is a weakly irreducible nonnegative tensor.*

Next, we compare the bound in Theorem 2.2 with those of Lemma 1.6 and Lemma 1.7, respectively.

Theorem 2.4. *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in \mathbb{R}^{[m,n]}$ be a nonnegative tensor. Then*

$$r \leq \nu(\mathcal{A}) \leq \mathcal{F}(\mathcal{A}) \leq \mathcal{H}(\mathcal{A}) \leq \omega(\mathcal{A}) \leq R.$$

Proof. From Lemma 1.7, we have $r \leq \nu(\mathcal{A}) \leq \omega(\mathcal{A}) \leq R$. So we only prove $\nu(\mathcal{A}) \leq \mathcal{F}(\mathcal{A})$ ($\mathcal{H}(\mathcal{A}) \leq \omega(\mathcal{A})$ can be similarly proved). Recall that

$$\tau(\mathcal{A}) = \left(\frac{r - \beta_0(\mathcal{A})}{R - \beta_0(\mathcal{A})}\right)^{\frac{1}{2(m-1)}}$$

and

$$\phi_s(\mathcal{A}) = \left\{\frac{r - \min\limits_{i,j} a_{ij\cdots j} - k_s(1 - \beta^{\frac{m-1-s}{2(m-1)}})}{R - \min\limits_{i,j} a_{ij\cdots j}}\right\}^{\frac{1}{2(m-1)}}.$$

Since $\beta = \frac{r - \min_{i,j} a_{ij\cdots j}}{R - \min_{i,j} a_{ij\cdots j}} \leq 1$ and $k_s \geq 0$, it holds that

$$k_s(1 - \beta^{\frac{m-1-s}{2(m-1)}}) \geq 0.$$

It follows from the above inequality that $\phi_s(\mathcal{A}) \leq \tau(\mathcal{A}) \leq 1$, consequently, for $\forall i, j \in \langle n\rangle$,

$$a_{ij\cdots j}\left(\frac{1}{\tau(\mathcal{A})^{m-1}} - 1\right) + r_i(\mathcal{A}) \leq r_i(\mathcal{A}) + a_{ij\cdots j}(\frac{1}{\phi_s(\mathcal{A})^{m-1}} - 1),$$

one may deduce the following result

$$\begin{aligned} \min_{i,j}\left\{a_{ij\cdots j}\left(\frac{1}{\tau(\mathcal{A})^{m-1}} - 1\right) + r_i(\mathcal{A})\right\} \leq & \min_{i,j}\left\{r_i(\mathcal{A}) + a_{ij\cdots j}(\frac{1}{\phi_s(\mathcal{A})^{m-1}} - 1)\right. \\ & \left. + \sum_{(i_{t_1} i_{t_2}, \cdots, i_{t_s})\in\wedge_j} a_{ij\cdots ji_{t_1} j\cdots ji_{t_2} j\cdots ji_{t_s} j\cdots j}(\frac{1}{\phi_s(\mathcal{A})^{m-1-s}} - 1)\right\}. \end{aligned}$$

It follows from the above inequality that $\nu(\mathcal{A}) \leq \mathcal{F}(\mathcal{A})$. With a strategy quite similar to the one utilized in the above proof, we can obtain $\mathcal{H}(\mathcal{A}) \leq \omega(\mathcal{A})$. Therefore, the conclusion follows from the above discussions. □

3. Numerical Example

For the bound in Theorem 2.2, we have showed that our bound is better than those in Lemmas 1.6 and 1.7. Now we provide a simple example to show the efficiency of the new bound.

Example 3.1. *Let $\mathcal{A} = (a_{i_1i_2i_3i_4})$ ba an order 4 dimension 3 tensor with entries defined as follows:*

$$A(1,1,:,:) = \begin{pmatrix} 0.9403 & 0.8011 & 0.7633 \\ 0.0058 & 0.2330 & 0.8264 \\ 0.6103 & 0.9325 & 0.5735 \end{pmatrix}, A(1,2,:,:) = \begin{pmatrix} 0.7926 & 0.3124 & 0.2905 \\ 0.3290 & 0.5845 & 0.4026 \\ 0.2235 & 0.8299 & 0.8621 \end{pmatrix},$$

$$A(1,3,:,:) = \begin{pmatrix} 0.6147 & 0.8272 & 0.4758 \\ 0.9912 & 0.6759 & 0.3991 \\ 0.2037 & 0.2489 & 0.5994 \end{pmatrix}, A(2,1,:,:) = \begin{pmatrix} 0.8005 & 0.8411 & 0.5722 \\ 0.1051 & 0.3545 & 0.7008 \\ 0.8214 & 0.4301 & 0.7425 \end{pmatrix},$$

$$A(2,2,:,:) = \begin{pmatrix} 0.7579 & 0.9563 & 0.2763 \\ 0.3891 & 0.5730 & 0.6223 \\ 0.4293 & 0.8497 & 0.5884 \end{pmatrix}, A(2,3,:,:) = \begin{pmatrix} 0.9635 & 0.5216 & 0.8844 \\ 0.0859 & 0.0902 & 0.4390 \\ 0.5005 & 0.9047 & 0.7817 \end{pmatrix},$$

$$A(3,1,:,:) = \begin{pmatrix} 0.1485 & 0.4457 & 0.3039 \\ 0.6198 & 0.8440 & 0.4833 \\ 0.2606 & 0.1962 & 0.3378 \end{pmatrix}, A(3,2,:,:) = \begin{pmatrix} 0.7985 & 2369 & 0.9737 \\ 0.9875 & 0.7022 & 0.9723 \\ 0.1590 & 0.3755 & 0.6437 \end{pmatrix},$$

$$A(3,3,:,:) = \begin{pmatrix} 0.8601 & 0.9852 & 0.7203 \\ 0.4019 & 0.5595 & 0.4840 \\ 0.6319 & 0.9336 & 0.6390 \end{pmatrix}.$$

We compare the bound in Theorem 2.2 with those in Lemmas 1.6-1.8. By Lemma 1.6, we have

$$15.3492 \leq \rho(\mathcal{A}) \leq 15.9820.$$

By Lemma 1.7, we get

$$15.3612 \leq \rho(\mathcal{A}) \leq 15.9704.$$

By Lemma 1.8, we have

$$15.3613 \leq \rho(\mathcal{A}) \leq 15.8850.$$

Now, by Theorem 2.2, we obtain

$$15.3942 \leq \rho(\mathcal{A}) \leq 15.9361.$$

The example shows that the bound in Theorem 2.2 is tighter than those in Lemma 1.6 and Lemma 1.7, which consists with Theorem 2.4. Moreover, the example also illustrates that in some cases the lower bound of Theorem 2.2 is better than that of Lemma 1.8, but our upper bound is not good as the one of Lemma 1.8. Until now, it is hard to theoretically compare our result with Lemma 1.8, which will be studied in future.

4. Conclusions

In this paper, we present a new ratio of the smallest component and the largest component of a Perron vector, which is obtained in a different approach by Li et al. [8]. Then based on the new ratio, a new bound for the spectral radius of a nonnegative tensor is derived. It is proved that the new bound is tighter than those in Lemmas 1.6-1.7, but it is hard to theoretically compare our result with Lemma 1.8 so for, which will be studied in future. Finally, a numerical example is implemented to validate the effectiveness of the proposed bounds.

Competing interests

The authors declare that they have no competing interests.

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References

[1] H. R. Afshin and A. R. Shojaeifard, A max version of Perron-Frobenius theorem for nonnegative tensor, Ann. Funct. Anal. 6 (2015) 145–154.

[2] K. C. Chang and K. Pearson and T. Zhang, On eigenvalue problems of real symmetric tensors, J. Math. Anal. Appl. 350 (2009) 416–422.

[3] K. C. Chang and K. Pearson and T. Zhang, Perron-Frobenius theorem for nonnegative tensors, Commun. Math. Sci. 6 (2008) 507–520.

[4] K. C. Chang and T. Zhang, On the uniqueness and non-uniqueness of the positive $\mathcal{Z}$-eigenvector for transition probability tensor, J. Math. Anal. Appl. 408 (2013) 525–540.

[5] S. Friedland and S. Gaubert and L. Han, Perron-Frobenius theorem for nonnegative multilinear forms and extensions, SIAM J. Matrix Anal. Appl. 438 (2013) 738–749.

[6] M. R. Kannan and N. S. Monderer and A. Berman, Some properties of strong $\mathcal{H}$-tensors and general $\mathcal{H}$-tensors, Linear Algebra Appl. 476 (2015) 42–55.

[7] T. G. Kolda and J. R. Mayo, Shifted power method for computing tensor eigenpairs, SIAM J. Matrix Anal. Appl. 32 (2011) 1095–1124.

[8] S. H. Li and C. Q. Li and Y. T. Li, A new bound for the spectral radius of nonnegative tensors, J. Inequal. Appl. 88 (2017) 2017.

[9] C. Q. Li and Z. Chen and Y. T. Li, A new eigenvalue inclusion set for tensors and its applications, Linear Algebra Appl. 481 (2015) 36–53.

[10] C. Q. Li and Y. T. Li and X. Kong, New eigenvalue inclusion sets for tensors, Numer. Linear Algebra Appl. 21 (2014) 39–50.

[11] C. Q. Li and Y. Q. Wang and J. Y. Yi and Y. T. Li, Bounds for the spectral radius of nonnegative tensors, J. Ind. Manag. Optim. 12 (2016) 975–990.

[12] C. Q. Li and J. J. Zhao and Y. T. Li, A new Brauer-type eigenvalue localization set for tensors, Linear Multilin. Algebra 64 (2016) 727–736.

[13] X. T. Li and M. K. Ng and Y. M. Ye, HAR: hub, authority and relevance scores in multi-relational data for query search, The SIAM international conference on data mining (2012).

[14] W. Li and L. B. Cui and M. K. Ng, The perturbation bound for the Perron vector of a transition probability tensor, Numer. Linear Algebra Appl. 20 (2013) 985–1000.

[15] W. Li and M. K. Ng, On the limiting probability distribution of a transition probability tensor, Linear Multilin. Algebra 62 (2014) 362–385.

[16] W. Li and M. K. Ng, Some bounds for the spectral radius of nonnegative tensors, Numer. Math. 130 (2015) 315–335.

[17] L. H. Lim, Singular values and eigenvalues of tensors: a variational approach, In CAMSAP'05: proceeding of the IEEE international workshop on computational advances in multisensor adaptive processing (2015) 129–132.

[18] Y. J. Liu and G. L. Zhou and N. F. Ibrahim, An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor, J. Comput. Appl. Math. 235 (2010) 286–292.

[19] M. K. Ng and X. T. Li and Y. M. Ye, MultiRank: co-ranking for objects and relations in multi-relational data, The 17th ACM SIGKDD conference on knowledge discovery and data mining (KDD-2011), Aug 21-24, San Diego, CA (2011).

[20] M. Ng and L. Q. Qi and G. L. Zhou, Finding the largest eigenvalue of a non-negative tensor, SIAM J. Matrix Anal. Appl. 31 (2009) 1090–1099.

[21] L. Q. Qi, Symmetric nonnegative tensors and copositive tensors, Linear Algebra Appl. 439 (2013) 228–238.

[22] L. Q. Qi, Eigenvalues and invariants of tensor, J. Math. Anal. Appl. 325 (2007) 1363–1377.

[23] L. Q. Qi, Eigenvalues of a real supersymetric tensor, J. Symb. Comput. 40 (2005) 1302–1324.

[24] Z. Wang and W. Wu, Bounds for the greatest eigenvalue of positive tensor, J. Ind. Manag. Optim. 10 (2014) 1031–1039.

[25] Y. N. Yang and Q. Z. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors, SIAM J. Matrix Anal. Appl. 31 (2010) 2517–2530.

[26] Y. N. Yang and Q. Z. Yang, Further results for Perron-Frobenius theorem for nonnegative tensors II, SIAM J. Matrix Anal. Appl. 32 (2011) 1236–1250.

[27] R. J. Zhao and B. Zheng and M. L. Liang,On the estimates of the $\mathcal{Z}$-eigenpair for an irreducible nonnegative tensor, J. Math. Anal. Appl. 450 (2017) 1157–1179.