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Some Perturbation Results for Ascent and Descent via Measure of Non-Compactness

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Abstract. The aim of this paper is to enlarge some known results from Fredholm and perturbation theory via measure of non-compactness. As applications, we focus on the study of the essential ascent and the essential descent spectra of an operator T defined on a given Banach space. Some perturbation results are also investigated.

1. Introduction

The notion of a measure of non-compactness of operators have been successfully applied in operator theory and turns out to be very useful tools in functional analysis, for instance in the theory of operator equations in Banach spaces, in the characterizations of compact operators between Banach spaces and in the metric fixed point theory. They are also used in the studies of functional equations, ordinary and partial differential equations, fractional partial differential equations and optimal control theory, see for instance [2–4, 10] and [16]. We refer to reader to these works with references given there.

Given a Banach space X, we denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators on X and $\mathcal{K}(X)$ its ideal of compact operators. For an operator $T \in \mathcal{L}(X)$, let $N(T)$ and $R(T)$ denote the null space and the range of T, respectively. We say that T is upper semi-Fredholm (resp. lower semi-Fredholm) if $\alpha(T) := dim(N(T)) < \infty$ and $R(T)$ is closed (resp. $\beta(T) := codim(R(T)) < \infty$ and $R(T)$ is closed). The set of upper semi-Fredholm (resp. lower semi-Fredholm) operators on X will be denoted by Φ_+ (resp. Φ_-). If $T \in \Phi_+ \cup \Phi_-$, then the index of T is given by $\text{ind}(T) := \alpha(T) - \beta(T)$. If $\text{ind}(T)$ is finite, then T is called Fredholm; such class of operators will be denoted by Φ. If $T - \lambda \in \Phi$ for all $\lambda \in \mathbb{C}$, we say that T is a Riesz operator. Define

$$\mathcal{R}(X) := \left\{T \in \mathcal{L}(X) : T - \lambda \in \Phi \text{ for all } \lambda \in \mathbb{C}\right\}.$$

Let $T \in \mathcal{L}(X)$. It is well known that $(N(T^k))_k$ forms an ascending sequence of subspaces, and if $N(T^k) = N(T^{k+1})$ for some $k \in \mathbb{N}$, then $N(T^k) = N(T^r)$ for all $r \geq k$. The smallest number k such that $N(T^k) = N(T^{k+1})$

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is called the ascent of T, we denote it by $a(T)$. If no such integer exists, then $a(T)$ is taken to be ∞. For a nonnegative integer k, let $\alpha_k(T) = \dim N(T^{k+1})/N(T^k)$. Following [12], the essential ascent of $T \in \mathcal{L}(X)$ is defined by

$$a_e(T) = \inf\{k : \alpha_k(T) < \infty\}.$$

It is easy to see that $a_e(T) = 0$ for every upper semi-Fredholm operator T. Analogously, $(R(T^k))_k$ forms a descending sequence; the smallest integer k for which $R(T^k) = R(T^{k+1})$ is called the descent of T; we denote it by $d(T)$. If no such integer exists, we shall say that T has an infinite descent. For a nonnegative integer k, set $\beta_k(T) = \dim R(T^k)/R(T^{k+1})$. Following [7], the essential descent of T is defined by

$$d_e(T) = \inf\Big\{k : \beta_k(T) < \infty\Big\}.$$

Clearly $d_e(T) = 0$ for every lower semi-Fredholm operator T. For more information about the essential ascent and the essential descent, we refer to [5–8, 12, 13]. We define the hyper-kernel and the hyper-range of $T \in \mathcal{L}(X)$, respectively by

$$N^\infty(T) := \bigcup_{n=0}^{\infty} N(T^n) \text{ and } R^\infty(T) := \bigcap_{n=0}^{\infty} R(T^n).$$

For given $T \in \mathcal{L}(X)$, the quantities

$$r_+(T) := \sup\{\varepsilon \geq 0 : |\lambda| < \varepsilon \Rightarrow T - \lambda \in \Phi_+\},$$

$$r_-(T) := \sup\{\varepsilon \geq 0 : |\lambda| < \varepsilon \Rightarrow T - \lambda \in \Phi_-\}$$

are semi-Fredholm radii of T (see [19, 21]). The number

$$r_e(T) := \sup\{|\lambda| : T - \lambda \notin \Phi\}$$

is called the essential spectral radius of T. Recall that an operator $T \in \mathcal{R}(X)$ if and only if $r_e(T) = 0$ (see [15, Theorem 3.3.1]).

The essential minimum modulus and the essential surjection modulus of an operator $T \in \mathcal{L}(X)$ are defined, respectively by

$$m_e(T) = \inf\{\|T + K\| : K \notin \Phi_+\}$$

and

$$n_e(T) = \inf\{\|T + K\| : K \notin \Phi_-\}.$$

An operator F is said to be an upper (resp. lower) semi-Fredholm perturbation if $F + T \in \Phi_+$ (resp. $F + T \in \Phi_-$) whenever $T \in \Phi_+$ (resp. $T \in \Phi_-$). The sets of all upper semi-Fredholm and all lower semi-Fredholm perturbations are denoted by $P\Phi_+$ and $P\Phi_-$, respectively. Obviously $\mathcal{K}(X) \subset P\Phi_+ \cap P\Phi_-$.

Let us consider the following functions

$$\|T\|_{\mathcal{K}} := \inf\{\|T + K\| : K \in \mathcal{K}(X)\},$$

$$\|T\|_{P\Phi_+} := \inf\{\|T + K\| : K \in P\Phi_+\},$$

$$\|T\|_{P\Phi_-} := \inf\{\|T + K\| : K \in P\Phi_-\}.$$

Clearly $\|T\|_{P\Phi_+}$ and $\|T\|_{P\Phi_-}$ are two semi-norms on X.

In this paper, we are interested to the stability of the class of semi-Fredholm, finite ascent, finite essential ascent, finite descent and finite essential descent operators under perturbations via measure of non-compactness. The paper is organized as follows. Section 2 is devoted to some semi-Fredholm perturbation results, and to the stability of the essential spectra of bounded operators on a Banach space. In particular, we give conditions under which a polynomial $P(T)$ of an operator T is Fredholm. As applications, we study in Section 3, the stability of the ascent, the essential ascent, the descent and the essential descent of a bounded operator under perturbations.

We end this introduction by recalling some preliminary results needed in the sequel.

Lemma 1.1. *([17, 21]) Let $T, S \in \mathcal{L}(X)$. Then*

(*i*) $m_e(T + S) = m_e(T)$ *whenever* $S \in P\Phi_+$;
(*ii*) $m_e(T + S) \leq m_e(T) + \|S\|_{P\Phi_+}$;
(*iii*) $T \in \Phi_+$ *if and only if* $m_e(T) > 0$;
(*iv*) *if* $\|S\|_{P\Phi_+} < m_e(T)$ *then* $T, T + S \in \Phi_+$ *and* $ind(T + S) = ind(T)$;
(*v*) *if* $\|S\| < m_e(T)$ *then* $T, T + S \in \Phi_+$ *and* $ind(T + S) = ind(T)$;
(*vi*) $m_e(\lambda T) = |\lambda| m_e(T)$ *for every* $\lambda \in \mathbb{C}$;
(*vii*) $m_e(T) \leq r_+(T)$;
(*viii*) $0 < m_e(I) \leq 1$, *where I denotes the identity operator on X.*

Lemma 1.2. *([17, 21]). Let $T, S \in \mathcal{L}(X)$. Then*

(*i*) $n_e(T + S) = n_e(T)$ *whenever* $S \in P\Phi_-$;
(*ii*) $n_e(T + S) \leq n_e(T) + \|S\|_{P\Phi_-}$;
(*iii*) $T \in \Phi_-$ *if and only if* $n_e(T) > 0$;
(*iv*) *if* $\|S\|_{P\Phi_-} < n_e(T)$ *then* $T, T + S \in \Phi_-$ *and* $ind(T + S) = ind(T)$;
(*v*) *if* $\|S\| < n_e(T)$ *then* $T, T + S \in \Phi_-$ *and* $ind(T + S) = ind(T)$;
(*vi*) $n_e(\lambda T) = |\lambda| n_e(T)$ *for every* $\lambda \in \mathbb{C}$;
(*vii*) $n_e(T) \leq r_-(T)$.

Lemma 1.3. *([1, Theorems 1.42, 1.46, 1.47]) Let $S, T \in \mathcal{L}(X)$. Then*

(*i*) *If* $ST \in \Phi_-$ *then* $S \in \Phi_-$;
(*ii*) *if* $ST \in \Phi_+$ *then* $T \in \Phi_+$;
(*iii*) *if* $ST \in \Phi$ *then* $S \in \Phi_-$ *and* $T \in \Phi_+$;
(*iv*) *if* $S, T \in \Phi_+$ *then* $ST \in \Phi_+$ *with* $ind(ST) = ind(S) + ind(T)$;
(*v*) *if* $S, T \in \Phi_-$ *then* $ST \in \Phi_-$ *with* $ind(ST) = ind(S) + ind(T)$;
(*vi*) *if* $S, T \in \Phi$ *then* $ST \in \Phi$ *with* $ind(ST) = ind(S) + ind(T)$.

2. Stability of Semi-Fredholm Operators

Let us introduce the following quantities for an operator $T \in \mathcal{L}(X)$:

$$\delta(T) := \sup\{m_e(T + K) : m_e(K) = 0\}$$

and

$$\gamma(T) := \sup\{n_e(T + K) : n_e(K) = 0\}.$$

We have the following properties

Proposition 2.1. *Let $T \in \mathcal{L}(X)$. Then*

(*i*) $\delta(T) = 0$ *if and only if* $T \in P\Phi_+$;
(*ii*) $\delta(T + S) \leq \delta(T) + \|S\|_{P\Phi_+} \leq \delta(T) + \|S\|_{\mathcal{K}}$;
(*iii*) $\delta(T + S) = \delta(T)$ *for all* $S \in P\Phi_+$;
(*iv*) $\delta(\lambda T) = |\lambda| \delta(T)$ *for all* $\lambda \in \mathbb{C}$;
(*v*) $m_e(T) \leq \delta(T)$.

Proof. (*i*) Suppose that $\delta(T) = 0$ and let $K \in \phi_+$. Then $m_e(K) > 0$ (Lemma 1.1, Part (*iii*)). Assume that $T + K \notin \phi_+$ then from the definition of δ we have that $\delta(T) \geq m_e(T - (T - K)) = m_e(K) > 0$ which is Contradiction. Thus $T + K \in \phi_+$ and consequently $T \in P\phi_+$. Conversely, suppose that $T \in P\phi_+$. Then, by using Lemma 1.1 (*ii*) and (*vi*), one can deduce that $m_e(T + T + S) = m_e(T + S)$ for all $S \in \mathcal{L}(X)$. It follows that

$$\begin{aligned} 2\delta(T) &= 2\sup\{m_e(T + K) : m_e(K) = 0\} \\ &= \sup\{m_e(T + T + 2K) : m_e(K) = 0\} \\ &= \sup\{m_e(T + T + S) : m_e(S) = 0\} \\ &= \sup\{m_e(T + S) : m_e(S) = 0\} \\ &= \delta(T). \end{aligned}$$

Consequently, $\delta(T) = 0$.

(*ii*) We have

$$\begin{aligned} \delta(T + S) &= \sup\{m_e(T + S + K) : m_e(K) = 0\} \\ &\leq \sup\{m_e(T + K) + \|S\|_{P\Phi_+} : m_e(K) = 0\} \quad \text{(by Lemma 1.1, Part } (ii)) \\ &= \sup\{m_e(T + K) : m_e(K) = 0\} + \|S\|_{P\Phi_+} \\ &= \delta(T) + \|S\|_{P\Phi_+} \\ &\leq \delta(T) + \|S\|_{\mathcal{K}} \quad (\text{as } \mathcal{K}(X) \subset P\Phi_+). \end{aligned}$$

(*iii*) Let $S \in P\Phi_+$. From Part (*ii*) we have $\delta(T + S) \leq \delta(T) + \|S\|_{P\Phi_+} = \delta(T)$ and $\delta(T) = \delta(T + S - S) \leq \delta(T + S) + \|S\|_{P\Phi_+} = \delta(T + S)$. Thus $\delta(T + S) = \delta(T)$.

(*iv*) The result is trivial if $\lambda = 0$. Suppose that λ is nonzero. Then, since $m_e(\lambda K) = |\lambda| m_e(K)$, it follows that $m_e(K) = 0$ if and only if $m_e(\lambda K) = 0$ for any $K \in \mathcal{L}(X)$.
Thus,

$$\begin{aligned} \delta(\lambda T) &= \sup\{m_e(\lambda T + K) : m_e(K) = 0\} \\ &= \sup\{m_e(\lambda T + \lambda K) : m_e(K) = 0\} \\ &= |\lambda| \sup\{m_e(T + K) : m_e(K) = 0\} \\ &= |\lambda| \delta(T). \end{aligned}$$

(*v*) Clear.
□

In the following theorem we establish stability property in the semi-Fredholm operators set and the Fredholm operators set. This theorem provides an extension of the important results [21, Theorem 6.1] and [11, Proposition 1]

Theorem 2.2. *Let $S, T \in \mathcal{L}(X)$. Then*

(*i*) *If $\delta(T) < m_e(S)$, then $S, T + S \in \Phi_+$ and $ind(T + S) = ind(S)$;*
(*ii*) *if $T \in \Phi_+$, then $\delta(T) > 0$;*
(*iii*) *if $\delta(T) < |\lambda| m_e(I)$ for $\lambda \in \mathbb{C}$, then $T - \lambda \in \Phi$ and $ind(T - \lambda) = 0$;*
(*iv*) *$r_e(T) = \lim\limits_{n\to\infty} (\delta(T^n))^{\frac{1}{n}}$;*
(*v*) *$\delta(T) \leq r_+(T) + r_0$, where $r_0 := \sup\{\|S\|_{P\Phi_+} : m_e(S) = 0\}$.*

Proof. (*i*) Trivially by Lemma 1.1, Part (*iii*), we have $S \in \Phi_+$. On the other hand, let $\lambda \geq 1$ and suppose that $T + \lambda S \notin \Phi_+$. By Proposition 2.1, we get $m_e(T + \lambda S) = 0$ which implies that

$$\delta(T) \geq m_e(T - (T + \lambda S)) = m_e(\lambda S) = |\lambda| m_e(S) \geq m_e(S)$$

leading to a contraction. Thus $T + \lambda S \in \Phi_+$. For $\lambda = 1$, we get in particular $T + S \in \Phi_+$. Now, if $t \in [0, 1]$, then $\delta(tT) \leq t\delta(T) < m_e(S)$. This implies that $tT + S \in \Phi_+$. By the continuity of the index on $[0, 1]$, we obtain

$$\text{ind}(T + S) = \text{ind}(1.T + S) = \text{ind}(0.T + S) = \text{ind}(S).$$

(*ii*) This follows immediately from Lemma 1.1, Part (*iii*) and Proposition 2.1, Part (*v*).

(*iii*) Replacing S by λI in Part (*i*), we obtain $T - \lambda \in \Phi_+$ with $\text{ind}(T - \lambda) = \text{ind}(\lambda I) = 0$, which implies that $T - \lambda \in \Phi$.

(*iv*) From Proposition 2.1, Part (*ii*), we have $\delta(T) \leq \|T\|_K$. Hence

$$\overline{\lim_{n\to\infty}}(\delta(T^n))^{\frac{1}{n}} \leq r_e(T).$$

Now let us consider $|\lambda| > (\delta(\frac{1}{m_e(I)}T^n))^{\frac{1}{n}}$ for some $n \in \mathbb{N}$. Then $\delta(T^n) < |\lambda|^n m_e(I)$. By Part (*iii*), it follows that $T^n - \lambda^n \in \Phi$ with $\text{ind}(T^n - \lambda^n) = 0$. Since

$$\begin{aligned} T^n - \lambda^n &= (T - \lambda)(T^{n-1} + \lambda T^{n-2} + ... + \lambda^{n-2}T + \lambda^{n-1}) \\ &= (T^{n-1} + \lambda T^{n-2} + ... + \lambda^{n-2}T + \lambda^{n-1})(T - \lambda), \end{aligned}$$

then by Lemma 1.3, Part (*iii*), $T - \lambda \in \Phi$. Hence $r_e(T)(m_e(I))^{\frac{1}{n}}) \leq (\delta(T^n))^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Thus

$$r_e(T) \leq \underline{\lim_{n\to\infty}}(\delta(T^n))^{\frac{1}{n}} \leq \overline{\lim_{n\to\infty}}(\delta(T^n))^{\frac{1}{n}} \leq r_e(T).$$

This proves (*iv*).

(*v*) By Proposition 2.1, $m_e(T + S) \leq m_e(T) + \|S\|_{P\Phi_+} \leq r_+(T) + r_0$ for all S satisfying $m_e(S) = 0$. Consequently,

$$\delta(T) \leq r_+(T) + r_0.$$

□

As a consequence of the above theorem we have the following results.

Corollary 2.3. *Let $T \in \mathcal{L}(X)$ and let $P(T)$ and $Q(T)$ be polynomials in T such that $Q(0) \neq 0$. Let $\lambda_0 := min\{|z| : P(z) = 0\}$ and $\lambda \in \mathbb{C}\backslash\{0\}$.*

(*i*) *If $\delta(T) < |\lambda_0| m_e(I)$, then $P(T) \in \Phi$ with $ind(P(T)) = 0$;*

(*ii*) *if $Q(z)$ divides $P(z) - \lambda$ and $\delta(P(T)) < |\lambda| m_e(I)$, then $Q(T) \in \Phi$. Moreover if $\delta(P(tT)) < |\lambda| m_e(I)$ for all $t \in [0, 1]$, then $ind(Q(T)) = 0$;*

(*iii*) *if $\delta(T^n) < m_e(I)$ for some $n \in \mathbb{N}$, then $I - T, T^{n-1} + ... + T + I \in \Phi$ and $ind(I - T) = ind(T^{n-1} + ... + T + I) = 0$.*

Proof. (*i*) Let $P(T) := \prod_{i=1}^{p}(T - \lambda_i)^{m_i}$ and assume that $\delta(T) < |\lambda_0| m_e(I)$. From Theorem 2.2 it follows that $T - \lambda_i \in \Phi$ with $\text{ind}(T - \lambda_i) = 0$ for $1 \leq i \leq p$; then $P(T) \in \Phi$. Moreover,

$$\text{ind}(P(T)) = \sum_{i=1}^{p} \text{ind}(T - \lambda_i)^{m_i} = \sum_{i=1}^{p} m_i \text{ind}(T - \lambda_i) = 0.$$

(*ii*) Let $P(z) - \lambda = Q(z)H(z)$ for some polynomial $H(z)$, then $P(T) - \lambda = Q(T)H(T)$. Since $\delta(P(T)) < |\lambda|$, it follows from Theorem 2.2, Part (*iii*), that $Q(T)H(T) \in \Phi$. By using Lemma 1.3 (*iii*) and the fact that $Q(T)H(T) = H(T)Q(T)$, we obtain $Q(T) \in \phi$. Suppose moreover that $\delta(P(tT)) < |\lambda| m_e(I)$ for all $t \in [0, 1]$. We prove in the same way as above by interchanging T and tT and since $Q(0) \neq 0$, that $Q(tT) \in \Phi$ for all $t \in [0, 1]$. Now, by the stability of the index in each connected component of Φ and the compactness of $[0, 1]$, we get

$$\text{ind}(Q(T)) = \text{ind}(Q(1.T)) = \text{ind}(Q(0.T)) = 0.$$

(iii) This follows from Part (ii) by taking $P(z) = z^n$, $Q(z) = z^{n-1} + ... + z + 1$ and $\lambda = 1$.
$\square$

For $T \in \mathcal{L}(X)$, let $\sigma(T)$, $\sigma_e(T)$, $\rho(T)$ and T^* denote the spectrum, the essential spectrum, the resolvent and the adjoint operator of T. For $r > 0$ and $x \in X$, let $B(x, r) = \{y \in X : \|x - y\| < r\}$. If $L \subset X$, then we denote its closure by $\overline{L}$.

Corollary 2.4. *If $T \in \mathcal{L}(X)$, then*

$$\sigma_e(T) = \bigcap_{S \in \mathcal{S}} \sigma(T + S) \subset \overline{B}(0, r_e(T)),$$

where $\mathcal{S} := \{S \in \mathcal{L}(X) : \delta(S(T + S - \lambda)^{-1}) < m_e(I)$ for all $\lambda \in \rho(T + S)\}$.

Proof. We first claim that $\sigma_e(T) \subset \bigcap_{S \in \mathcal{S}} \sigma(T + S)$. Indeed, if $\lambda \notin \bigcap_{S \in \mathcal{S}} \sigma(T + S)$, then there exists $S \in \mathcal{S}$ such that $\lambda \in \rho(T+S)$, so $T+S-\lambda \in \Phi$ with $\mathrm{ind}(T+S-\lambda) = 0$ and $\delta(S(T+S-\lambda)^{-1}) < m_e(I)$. From Theorem 2.2, Part (iii), we have $I-S(T+S-\lambda)^{-1} \in \Phi_+$ and $\mathrm{ind}(I-(T+S-\lambda)^{-1}) = 0$. Using the fact that $T-\lambda = (I-S(T+S-\lambda)^{-1})(T+S-\lambda)$ and Lemma 1.3, we derive that $T - \lambda \in \Phi$. Thus $\sigma_e(T) \subset \bigcap_{S \in \mathcal{S}} \sigma(T + S)$.

Conversely, let $S \in \mathcal{K}(X)$, then $S(T + S - \lambda)^{-1} \in \mathcal{K}(X)$ for all $\lambda \in \rho(T + S)$. It follows from Proposition 2.1, Part (i), that $\delta(S(T + S - \lambda)^{-1}) = 0 < m_e(I)$ and hence $S \in \mathcal{S}$. Consequently,

$$\bigcap_{S \in \mathcal{S}} \sigma(T + S) \subset \bigcap_{S \in \mathcal{K}(X)} \sigma(T + S) = \sigma_e(T).$$

On the other hand, let $|\lambda| > r_e(T) = \lim_{n\to\infty} (\delta(T^n))^{\frac{1}{n}} = \lim_{n\to\infty} (\frac{1}{m_e(I)} \delta(T^n))^{\frac{1}{n}}$. Then, there exists k for which $|\lambda| > (\frac{1}{m_e(I)} \delta(T^k))^{\frac{1}{k}}$, that is, $|\lambda|^k m_e(I) > \delta(T^k)$. From Theorem 2.2, Part (iii), we have $T^k - \lambda^k \in \Phi$. Now, since

$$\begin{aligned} T^n - \lambda^n &= (T - \lambda)(T^{n-1} + \lambda T^{n-2} + ... + \lambda^{n-2} T + \lambda^{n-1}) \\ &= (T^{n-1} + \lambda T^{n-2} + ... + \lambda^{n-2} T + \lambda^{n-1})(T - \lambda), \end{aligned}$$

it follows from Lemma 1.3 that $T - \lambda \in \Phi$. Consequently, $\lambda \notin \sigma_e(T)$. $\square$

The next proposition is obtained by duality from Proposition 2.1, so we omit its proof.

Proposition 2.5. *Let $T \in \mathcal{L}(X)$. Then*

- (i) *$\gamma(T) = 0$ if and only if $T \in P\Phi_-$;*
- (ii) *$\gamma(T + S) \le \gamma(T) + \|S\|_{P\Phi_-} \le \gamma(T) + \|S\|_{\mathcal{K}}$;*
- (iii) *$\gamma(T + S) = \gamma(T)$ for all $S \in P\Phi_-$;*
- (iv) *$\gamma(\lambda T) = |\lambda|\gamma(T)$ for all $\lambda \in \mathbb{C}$;*
- (v) *$n_e(T) \le \gamma(T)$.*

Using the same arguments as in the proof of Theorem 2.2, we get the following results.

Theorem 2.6. *Let $S, T \in \mathcal{L}(X)$. Then*

- (i) *If $\gamma(T) < n_e(S)$, then $S, T + S \in \Phi_-$ and $ind(T + S) = ind(S)$;*
- (ii) *if $T \in \Phi_-$, then $\gamma(T) > 0$;*
- (iii) *if $\gamma(T) < |\lambda| n_e(I)$ for $\lambda \in \mathbb{C}$, then $T - \lambda \in \Phi$ and $ind(T - \lambda) = 0$;*
- (iv) *$r_e(T) = \lim_{n\to\infty} (\gamma(T^n))^{\frac{1}{n}}$;*
- (v) *$\gamma(T) \le r_+(T) + r_0$, where $r_0 := \sup\{\|S\|_{P\Phi_-} : n_e(S) = 0\}$.*

Corollary 2.7. *Let $T \in \mathcal{L}(X)$ and let $P(T)$ and $Q(T)$ be polynomials in T such that $Q(0) \neq 0$. Let $\lambda_0 := min\{|z| : P(z) = 0\}$ and $\lambda \in \mathbb{C}\backslash\{0\}$.*

(*i*) *If $\gamma(T) < |\lambda_0| n_e(I)$, then $P(T) \in \Phi$ with $ind(P(T)) = 0$;*
(*ii*) *if $Q(z)$ divides $P(z) - \lambda$ and $\gamma(P(T)) < |\lambda| n_e(I)$, then $Q(T) \in \Phi$ and $ind(Q(T)) = 0$;*
(*iii*) *if $\gamma(T^n) < n_e(I)$ for some $n \in \mathbb{N}$, then $I - T, T^{n-1} + ... + T + I \in \Phi$ and $ind(I - T) = ind(T^{n-1} + ... + T + I) = 0$.*

Corollary 2.8. *If $T \in \mathcal{L}(X)$, then*

$$\sigma_e(T) = \bigcap_{S \in \mathcal{S}} \sigma(T + S) \subset \overline{B}(0, r_e(T)),$$

where $\mathcal{S} := \left\{ S \in \mathcal{L}(X) : \gamma(S(T + S - \lambda)^{-1}) < n_e(I) \text{ for all } \lambda \in \rho(T + S) \right\}$.

3. The framework of the ascent and the descent

In [5, Proposition 3.1], O. Bel Hadj Fredj has shown that the ascent spectrum and the essential ascent spectrum are invariant under commuting perturbation F such that a power of F is of finite rank. Also in [6, 9] O. Bel Hadj Fredj and M. A. Kaashoek, D. C. Lay established that if F is a bounded operator for which there exists some positive integer n such that F^n is of finite rank, then for every bounded operator commuting with F, T has finite descent (resp. finite essential descent) if and only if $T + F$ does. In this section we focus on to study the stability of the above spectrums under perturbations via measure of non-compactness as a generalization of results proved in [5, 6] and [9]. We also extend the well known results [14, Corollary 2 and Corollary 3]. We begin with the following lemmas which are used to prove the main result of this section. The next lemma is an improvement of [18, Proposition 1.6].

Lemma 3.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$.*

(*i*) *If $a(T) < \infty$, then $N^\infty(T) \cap R^\infty(T) = \{0\}$;*
(*ii*) *if $N^\infty(T) \cap R^\infty(T) = \{0\}$ and $a_e(T) < \infty$, then $a(T) < \infty$.*

Proof. (*i*) Suppose that $p := a(T) < \infty$. Then $N(T^p) = N(T^{p+1})$, and therefore $N(T^p) = N(T^n)$ for all $n \geq p$. It follows that $N^\infty(T) = N(T^p)$. Let $y \in N^\infty(T) \cap R^\infty(T) = N(T^p) \cap R^\infty(T)$. Then $y = T^p x$ for some $x \in X$ and $T^p y = 0$. It follows that $T^{2p} x = 0$, so $x \in N(T^{2p}) = N(T^p)$. Thus $y = T^p x = 0$, and hence $N^\infty(T) \cap R^\infty(T) = \{0\}$.

(*ii*) Suppose that $N^\infty(T) \cap R^\infty(T) = \{0\}$ and $p := a_e(T) < \infty$. Then $N(T) \cap R(T^p) = N(T) \cap R(T^n)$ for all $n \geq p$. It follows that $N(T) \cap R(T^p) = N(T) \cap R^\infty(T) \subset N^\infty(T) \cap R^\infty(T) = \{0\}$. Since $N(T) \cap R(T^p) \simeq N(T^{p+1})/N(T^p)$ then, $N(T^p) = N(T^{p+1})$. Consequently $a(T) < \infty$.
□

Lemma 3.2. ([*1, Corollary 1.43*]) *Let $T \in \Phi_- \cup \Phi_+$. Then $a(T) = d(T^*)$ and $d(T) = a(T^*)$.*

For $S, T \in \mathcal{L}(X)$, $ST = TS$ and $n \in \mathbb{N}$, let $S_n := S_{|R(T^n)}$.

Theorem 3.3. *Let $S, T \in \mathcal{L}(X)$. Then*

(*i*) *If $\delta(S) < m_e(T)$, then $a(T) < \infty$ if and only if $a(T + S) < \infty$;*
(*ii*) *if $ST = TS, R(S) \subset R(T) \cap R(T + S), R(T^n)$ is closed and $S_n \in P\Phi_+$ for all $n \in \mathbb{N}$, then $a_e(T) < \infty$ if and only if $a_e(T + S) < \infty$;*
(*iii*) *if $\gamma(T) < n_e(S)$, then $d(T) < \infty$ if and only if $d(T + S) < \infty$;*
(*iv*) *if $ST = TS, R(S) \subset R(T) \cap R(T + S)$ and $R(T^n)$ is closed for all $n \in \mathbb{N}$, then $d_e(T) < \infty$ if and only if $d_e(T + S) < \infty$.*

Proof. (i) Suppose that $a(T) < \infty$. According to Lemma 3.1, we have $N^\infty(T) \cap R^\infty(T) = \{0\}$. If $\lambda \in [0,1]$, then $T + \lambda S \in \Phi_+$ by Theorem 2.2. Thus, there exists $\varepsilon(\lambda) > 0$ such that

$$\overline{N^\infty(T+\lambda S)} \cap R^\infty(T+\lambda S) = \overline{N^\infty(T+\eta S)} \cap R^\infty(T+\eta S)$$

for all η in $B(\lambda, \varepsilon(\lambda))$. This shows that $\overline{N^\infty(T+\lambda S)} \cap R^\infty(T+\lambda S)$ is a locally constant function in the connected set $[0,1]$, and so it is constant. Since $N^\infty(T) \cap R^\infty(T) = \overline{N^\infty(T)} \cap R^\infty(T) = \{0\}$, we conclude that $\overline{N^\infty(T+S)} \cap R^\infty(T+S) = \{0\}$. Hence $a(T+S) < \infty$.

Conversely, since $\delta(S) < m_e(T)$, then by Theorem 2.2, $T \in \Phi_+$ and $T - S \in \Phi_+$. Now, if we consider $T+S$ instead to T and follow a similar reasoning as in the above, we get $a(T) = a(T+S-S) < \infty$.

(ii) Note that, since S and T commute and $R(S) \subset R(T)$, then $S(R(T^k)) \subset R(T^k)$ and $R(T+S)^k \subset R(T^k)$ for all $k \in \mathbb{N}$. Suppose that $n = a_e(T) < \infty$. Then the operator T_n defined on the Banach space $R(T^n)$ is upper semi-Fredholm. Since $S_n \in P\Phi_+$, then $T_n + S_n \in \Phi_+$. Hence $dimN(T+S) \cap R(T^n) < \infty$. It follows that $dimN(T+S) \cap R(T+S)^n \leq dimN(T+S) \cap R(T^n) < \infty$. Consequently, $a_e(T+S) < \infty$.

Conversely, suppose that $n := a_e(T+S) < \infty$. Then $dimN(T+S) \cap R(T+S)^n < \infty$. Since T and S commute then so is $T+S$ and S. On the other hand the fact that $R(S) \subset R(T+S)$ implies that $R(T^n) = R(T+S-S)^n \subset R(T+S)^n$. It follows that $dimN(T+S) \cap R(T^n) \leq dimN(T+S) \cap R(T+S)^n < \infty$. This means that $T_n + S_n = (T+S)_n \in \Phi_+$ and hence $T_n = T_n + S_n - S_n \in \Phi_+$. Thus implies that $a_e(T) < \infty$.

(iii) Assume that $d(T) < \infty$. Since $\gamma(T) < n_e(S)$, then $T, T+S \in \Phi_-$. By Lemma 3.2, we have $T^* \in \Phi_+$ with $a(T^*) < \infty$. Now by using Part (i), it follows that $a(T^* + S^*) < \infty$, that is, $d(T+S) < \infty$. We prove the reverse implication by using the same argument as above.

(iv) Likewise in Part (ii), we have $S(R(T^k)) \subset R(T^k), k \geq 1$. Since $n := d_e(T) < \infty$, then $dim\dfrac{R(T^n)}{R(T^{n+1})} < \infty$. On the other hand, the fact that $ST = TS$ and $R(S) \subset R(T) \cap R(T+S)$ implies that $R(T^n) = R(T+S)^n$. It follows that $dim\dfrac{R(T+S)^n}{R(T+S)^{n+1}} = dim\dfrac{R(T^n)}{R(T^{n+1})} < \infty$. Thus $d_e(T+S) < \infty$. Conversely, it suffices to interchange T and $T+S$.

□

Corollary 3.4. *Let $S, T \in \mathcal{L}(X)$. Then*

(i) *If $T \in \Phi_+$ and $S \in P\Phi_+$, then $a(T) < \infty$ if and only if $a(T+S) < \infty$;*

(ii) *if $T \in \Phi_-$ and $S \in P\Phi_-$, then $d(T) < \infty$ if and only if $d(T+S) < \infty$.*

Proof. (i) Since $S \in P\Phi_+$, then by Proposition 2.1, we have $\delta(S) = 0$. Moreover $m_e(T) > 0$ by Lemma 1.1. Hence the equivalence follows immediately from Theorem 3.3 (i).

(ii) Similarly, since according to Proposition 2.5, $\gamma(S) = 0$, and $n_e(T) > 0$, then the equivalence follows directly from Theorem 3.3 (iii).

□

Corollary 3.5. *For $\lambda \in \mathbb{C}$ put $T_\lambda := T - \lambda$.*

(a) *If $T \in \mathcal{R}(X)$ and $S \in P\Phi_+$, then*

  (i) $\sigma_{asc}(T) = \sigma_{asc}(T+S)$;

  (ii) $\sigma_{des}(T) = \sigma_{des}(T+S)$;

(b) *If $ST = TS, R(S) \subset R(T_\lambda) \cap R(T_\lambda + S)$ and $R(T_\lambda)^n$ is closed for all $n \in \mathbb{N}$, then*

  (i) $\sigma^e_{asc}(T) = \sigma^e_{asc}(T+S)$ *whenever* $S_{|R(T_\lambda)^n} \in P\phi_+, \lambda \in \mathbb{C}$;

  (ii) $\sigma^e_{des}(T) = \sigma^e_{des}(T+S)$.

Proof. (*a*) (*i*) If $\lambda \notin \sigma_{asc}(T)$, then $a(T-\lambda) < \infty$, and since $T-\lambda \in \Phi_+$ as $T \in \mathcal{R}(X)$, then $\delta(S) = 0 < m_e(T-\lambda)$. It follows, from Theorem 3.3(*i*), that $a(T+S-\lambda) < \infty$; so $\lambda \notin \sigma_{asc}(T+S)$.

Conversely, since $S \in P\Phi_+$ and $T \in \mathcal{R}(X)$ then $T+S-\lambda \in \phi_+$ for all $\lambda \in \mathbb{C}$. By using the same reasoning as in the above and by writing $T+S$ instead to T and $-S$ instead to S we show that $\lambda \notin \sigma_{asc}(T)$.

(*ii*) The proof is similar to that of Part (*i*).

(*b*) Now suppose that $ST = TS, R(S) \subset R(T_\lambda) \cap R(T_\lambda + S)$ and $R(T_\lambda)^n$ is closed for all $n \in \mathbb{N}$

(*i*) Let $\lambda \notin \sigma^e_{asc}(T)$, then $n := a_e(T-\lambda) < \infty$. Now, by interchanging T and T_λ in Theorem 3.3 (*ii*), it follows that $a_e(T+S-\lambda) < \infty$. Consequently, $\lambda \notin \sigma^e_{asc}(T)$. In the same way we prove the converse inclusion.

(*ii*) The proof goes along the same lines as that of (*iii*).

□

Let $\sigma_a(T)$ and $\sigma_d(T)$ denote the approximate point spectrum and the approximate defect spectrum of $T \in \mathcal{L}(X)$, respectively. We say that $T \in \mathcal{L}(X)$ is upper semi-Browder if $T \in \Phi_+$, $\mathrm{ind}(T) \leq 0$ and $a(T) < \infty$. Such class of operators will be denoted by $B_+(X)$. We call T lower semi-Browder if $T \in \Phi_-$, $\mathrm{ind}(T) \geq 0$ and $d(T) < \infty$. We denote this class of operators by $B_-(X)$. Set

$$\sigma_{ab}(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not upper semi-Browder}\},$$

$$\sigma_{ad}(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not lower semi-Browder}\}.$$

Recall that (see [15]) $\sigma_{ab}(T) = \bigcap_{TK=KT, K\in\mathcal{K}(X)} \sigma_a(T+K)$ and $\sigma_{ad}(T) = \bigcap_{TK=KTK\in\mathcal{K}(X)} \sigma_d(T+K)$. We call $\sigma_{ab}(T)$ and $\sigma_{db}(T)$, respectively the Browder essential approximate point spectrum and the Browder essential approximate defect spectrum of T. For $T \in \mathcal{L}(X)$, set

$$\mathcal{F}^+(T) := \Big\{S \in \mathcal{L}(X) : \delta(S) < m_e(T-\lambda) \text{ for all } \lambda \in \rho_{asc}(T)\Big\}$$

and

$$\mathcal{F}^-(T) := \Big\{S \in \mathcal{L}(X) : \delta(S) < n_e(T-\lambda) \text{ for all } \lambda \in \rho_{des}(T)\Big\}.$$

Proposition 3.6. *Let $T \in \mathcal{L}(X)$. Then*

(*i*) *$\sigma_{ab}(T+S) = \sigma_{ab}(T)$ for all $S \in \mathcal{F}^+(T)$;*
(*ii*) *$\sigma_{db}(T+S) = \sigma_{db}(T)$ for all $S \in \mathcal{F}^-(T)$.*

Proof. (*i*) We first claim that $\sigma_{ab}(T+S) \subset \sigma_{ab}(T)$. Indeed, if $\lambda \notin \sigma_{ab}(T)$, then $T-\lambda \in \Phi_+$, $\mathrm{ind}(T-\lambda) \leq 0$ and $a(T-\lambda) < \infty$. Since $S \in \mathcal{F}^+(T)$, then $\delta(S) < m_e(T-\lambda)$, and hence by Theorem 3.3, Part (*i*), we have $a(T+S-\lambda) < \infty$. Now, by using Theorem 2.2 one can deduce that $T+S-\lambda \in \Phi_+$ and $\mathrm{ind}(T+S-\lambda) = \mathrm{ind}(T-\lambda) \leq 0$. This means that $\lambda \notin \sigma_{ab}(T+S)$. Similarly, we prove that $\sigma_{ab}(T) \subset \sigma_{ab}(T+S)$.

(*ii*) In the same way as in Part (*i*) we prove the equality $\sigma_{db}(T+S) = \sigma_{db}(T)$ for all $S \in \mathcal{F}^-(T)$.

□

Corollary 3.7. *Let $T \in \mathcal{L}(X)$. Then*

$\sigma_{ab}(T) \subset \bigcap_{S\in\mathcal{F}^+(T)} \sigma_a(T+S)$;

(*i*) $\sigma_{db}(T) \subset \bigcap_{S\in\mathcal{F}^-(T)} \sigma_d(T+S)$.

Proof. (*i*) If $\lambda \notin \bigcap_{S\in\mathcal{F}^+(T)} \sigma_a(T+S)$, then $\lambda \notin \sigma_a(T+S_0)$ for some $S_0 \in \mathcal{F}^+(T)$. Hence $\inf_{\|x\|=1} \|Tx + S_0x - \lambda x\| > 0$ which implies that $T+S_0-\lambda$ is bounded from below. This shows that $a(T+S_0-\lambda) = 0$, $\mathrm{ind}(T+S_0-\lambda) \leq 0$ and $R(T+S_0-\lambda)$ is closed. Therefore $T+S_0-\lambda \in B_+(X)$. Now using Proposition 3.6, Part (*i*), one can conclude that $\lambda \notin \sigma_{ab}(T+S_0) = \sigma_{ab}(T)$.

(*ii*) In the same way let $\lambda \notin \bigcap_{S\in\mathcal{F}^-(T)} \sigma_d(T+S)$. Then there exists $S_0 \in \mathcal{F}^-(T)$ such that $T+S_0-\lambda$ is surjective which implies that $T+S_0 \in \Phi_-$ with $d(T+S_0) = 0$ and $\mathrm{ind}(T+S_0) \geq 0$. Therefore $T+S_0-\lambda \in B_-(X)$. From Proposition 3.6 we get $\lambda \notin \sigma_{db}(T+S_0) = \sigma_{db}(T)$.

□

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