



On a Riemannian Manifold with a Circulant Structure whose Third Power is the Identity

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Abstract. It is studied a 3-dimensional Riemannian manifold equipped with a tensor structure of type $(1, 1)$, whose third power is the identity. This structure has a circulant matrix with respect to some basis, i.e. the structure is circulant. On such a manifold a fundamental tensor by the metric and by the covariant derivative of the circulant structure is defined. An important characteristic identity for this tensor is obtained. It is established that the image of the fundamental tensor with respect to the usual conformal transformation satisfies the same identity. A Lie group as a manifold of the considered type is constructed and some of its geometrical characteristics are found.

1. Introduction

In differential geometry of the Riemannian manifolds with additional structures, the covariant derivative of the corresponding structure plays an important role. In this connection, for example, the classifications in [4], [6] and [7] are made. One of the basic classes in these classifications is the largest class, which is invariant under the conformal transformations of the Riemannian metric.

In [1], [2] and [3], problems in differential geometry of a 3-dimensional Riemannian manifold (M, g, Q) with a tensor structure Q , whose matrix in some basis is circulant, are considered. This structure satisfies $Q^3 = \text{id}$, $Q \neq \text{id}$, and it is compatible with the metric g , so that an isometry is induced in any tangent space on (M, g, Q) .

In the present work we continue studying such a manifold (M, g, Q) . In Section 2, we give some necessary facts about (M, g, Q) . In Section 3, we define the fundamental tensor F by the metric g and by the covariant derivative of Q . We obtain the important characteristic identity (8) for F . We establish that the image of the fundamental tensor with respect to the usual conformal transformation satisfies the same identity, i.e. the conformal manifold (M, \bar{g}, Q) belongs to the same class. In Section 4, we get some curvature properties of (M, g, Q) . In Section 5, we construct a manifold (G, g, Q) of the type of (M, g, Q) , where G is a Lie group. In Section 6, we consider a subgroup G' of G , for which the manifold (G', g, Q) is of invariant sectional curvatures under Q . We study some particular cases.

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2. Preliminaries

We consider the manifold (M, g, Q) , introduced in [1], i.e. M is a 3-dimensional Riemannian manifold equipped with an additional tensor structure Q of type $(1, 1)$, which satisfies

$$Q^3 = \text{id}, \quad Q \neq \text{id}, \quad (1)$$

and Q has a circulant matrix with respect to some basis, as follows:

$$(Q_j^s) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (2)$$

The metric g and the structure Q satisfy

$$g(Qx, Qy) = g(x, y), \quad x, y \in \mathfrak{X}(M). \quad (3)$$

Necessary, the matrix of g with respect to the same basis has the form

$$(g_{ij}) = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix}, \quad (4)$$

where A and B are smooth functions of an arbitrary point $p(x^1, x^2, x^3)$ on M . It is supposed that $A > B > 0$ in order g to be positive definite.

Moreover, in [1] it is defined another metric \tilde{g} associated to g by Q , as follows:

$$\tilde{g}(x, y) = g(x, Qy) + g(Qx, y). \quad (5)$$

Here and anywhere in this work, x, y, z, u will stand for arbitrary elements of the algebra on the smooth vector fields on M or vectors in the tangent space $T_pM, p \in M$. The Einstein summation convention is used, the range of the summation indices being always $\{1, 2, 3\}$.

3. The fundamental tensor F on (M, g, Q)

Let ∇ be the Levi-Civita connection of g and $\{e_i\}$ be an arbitrary basis of T_pM . We consider the tensor F of type $(0, 3)$ and the Lee forms θ and θ^* , defined by

$$F(x, y, z) = (\nabla_x \tilde{g})(y, z), \quad \theta(x) = g^{ij}F(e_i, e_j, x), \quad \theta^*(x) = g^{ij}F(e_i, Qe_j, x). \quad (6)$$

Obviously, the tensor F has the property

$$F(x, z, y) = F(x, y, z). \quad (7)$$

Theorem 3.1. For the tensor F it is valid the identity

$$F(x, y, z) = \frac{1}{3} \{g(x, y)\theta(z) + g(x, z)\theta(y) + \tilde{g}(x, y)\theta^*(z) + \tilde{g}(x, z)\theta^*(y)\}. \quad (8)$$

Proof. The components of the geometric quantities of (M, g, Q) , given in the formulas (9), (10) (11) and (12), are obtained in [1] and [2]. The inverse matrix of (g_{ij}) is

$$(g^{ij}) = \frac{1}{D} \begin{pmatrix} A+B & -B & -B \\ -B & A+B & -B \\ -B & -B & A+B \end{pmatrix}, \quad (9)$$

where $D = (A - B)(A + 2B)$. The Christoffel symbols of g are as follows:

$$\begin{aligned} \Gamma_{ij}^k &= \frac{1}{2D} \left((A + B)(-B_k + B_i + B_j) - B(A_i + A_j) \right), \\ \Gamma_{ii}^i &= \frac{1}{2D} \left((A + B)A_i - B(4B_i - A_j - A_k) \right), \\ \Gamma_{ii}^k &= \frac{1}{2D} \left((A + B)(2B_i - A_k) - B(2B_i - A_j + A_i) \right), \\ \Gamma_{ij}^i &= \frac{1}{2D} \left((A + B)A_j - B(-B_k + B_i + B_j + A_i) \right), \end{aligned} \tag{10}$$

where $i \neq j, j \neq k, i \neq k$ and $A_i = \frac{\partial A}{\partial x^i}, B_i = \frac{\partial B}{\partial x^i}$. The matrix of the associated metric \tilde{g} , determined by (5), is of the type:

$$(\tilde{g}_{ij}) = \begin{pmatrix} 2B & A + B & A + B \\ A + B & 2B & A + B \\ A + B & A + B & 2B \end{pmatrix}. \tag{11}$$

The inverse matrix of (\tilde{g}_{ij}) has the form

$$(\tilde{g}^{ij}) = \frac{1}{2D} \begin{pmatrix} -A - 3B & A + B & A + B \\ A + B & -A - 3B & A + B \\ A + B & A + B & -A - 3B \end{pmatrix}. \tag{12}$$

Now, we calculate the components of F, θ and θ^* , defined by (6).

Using the well-known identities for a Riemannian metric:

$$\nabla_k \tilde{g}_{ij} = \partial_k \tilde{g}_{ij} - \Gamma_{ki}^s \tilde{g}_{sj} - \Gamma_{kj}^s \tilde{g}_{si}, \tag{13}$$

and due to (7), (11) and (10), we find the following components $F_{ijk} = F(e_i, e_j, e_k)$ of F :

$$\begin{aligned} F_{111} &= -2B_1 + A_2 + A_3, & F_{211} &= F_{311} = -B_1 + B_2 + B_3 - A_1, \\ F_{112} &= F_{221} = \frac{1}{2}(A_3 - B_1 - B_2 + B_3), & F_{312} &= B_3 - \frac{1}{2}(A_1 + A_2), \\ F_{222} &= -2B_2 + A_1 + A_3, & F_{322} &= F_{122} = B_1 - B_2 + B_3 - A_2, \\ F_{113} &= F_{331} = \frac{1}{2}(A_2 - B_1 + B_2 - B_3), & F_{213} &= B_2 - \frac{1}{2}(A_1 + A_3), \\ F_{123} &= B_1 - \frac{1}{2}(A_2 + A_3), & F_{223} &= F_{332} = \frac{1}{2}(A_1 + B_1 - B_2 - B_3), \\ F_{133} &= F_{233} = B_1 + B_2 - B_3 - A_3, & F_{333} &= -2B_3 + A_1 + A_2. \end{aligned} \tag{14}$$

For the components of θ and θ^* , from (2), (6), (9) and (14), we have

$$\begin{aligned} \theta_1 &= \frac{3}{2D} \left((A + B)(A_2 + A_3) + 2B(A_1 - B_2 - B_3) - 2AB_1 \right), \\ \theta_2 &= \frac{3}{2D} \left((A + B)(A_1 + A_3) + 2B(A_2 - B_1 - B_3) - 2AB_2 \right), \\ \theta_3 &= \frac{3}{2D} \left((A + B)(A_1 + A_2) + 2B(A_3 - B_1 - B_2) - 2AB_3 \right), \end{aligned} \tag{15}$$

$$\begin{aligned} \theta_1^* &= -\frac{3}{2D} \left((A - 2B)B_1 - A(B_2 + B_3 - A_1) + B(A_2 + A_3) \right), \\ \theta_2^* &= -\frac{3}{2D} \left((A - 2B)B_2 - A(B_1 + B_3 - A_2) + B(A_1 + A_3) \right), \\ \theta_3^* &= -\frac{3}{2D} \left((A - 2B)B_3 - A(B_1 + B_2 - A_3) + B(A_1 + A_2) \right). \end{aligned} \tag{16}$$

From (4), (11), (14), (15) and (16) it follows

$$F_{kij} = \frac{1}{3} \left(g_{kj}\theta_i + g_{ki}\theta_j + \tilde{g}_{kj}\theta_i^* + \tilde{g}_{ki}\theta_j^* \right), \tag{17}$$

which is equivalent to (8). \square

Theorem 3.2. Under the conformal transformation

$$\bar{g}(x, y) = \alpha g(x, y), \tag{18}$$

where α is a smooth positive function, the tensor F is transformed into the tensor

$$\bar{F}(x, y, z) = \frac{1}{3} \left\{ \bar{g}(x, y)\bar{\theta}(z) + \bar{g}(x, z)\bar{\theta}(y) + \bar{g}(x, y)\bar{\theta}^*(z) + \bar{g}(x, z)\bar{\theta}^*(y) \right\} \tag{19}$$

with $\bar{\theta} = \theta + \frac{3}{2\alpha} d\alpha \circ (Q + Q^2)$ and $\bar{\theta}^* = \theta^* - \frac{3}{2\alpha} d\alpha$.

Proof. We denote the following products

$$\tilde{g}_{ij}g^{is} = \Phi_j^s, \quad g_{ij}\tilde{g}^{is} = \frac{1}{2}S_j^s, \tag{20}$$

where Φ_j^s and $\frac{1}{2}S_j^s$ are mutually inverse matrices. Because of (4), (9), (11) and (12) we have

$$(\Phi_j^s) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad (S_j^s) = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}. \tag{21}$$

Having in mind (2) and (21), we get $\Phi = Q + Q^2$.

From (15), (16) and the second matrix of (21), we obtain

$$\theta_i^* = -\frac{1}{2}S_i^s\theta_s. \tag{22}$$

According to the transformation (18), the components of the tensor \bar{F} are $\bar{F}_{ijk} = \bar{\nabla}_i\bar{g}_{jk}$, where $\bar{\nabla}$ is the Levi-Civita connection of \bar{g} .

Bearing in mind (5) and (18), we have that $\bar{g} = \alpha\tilde{g}$. Then

$$\bar{\nabla}\bar{g} = \alpha\bar{\nabla}\tilde{g} + \tilde{g}\bar{\nabla}\alpha. \tag{23}$$

From the Christoffel formulas

$$2\Gamma_{ij}^k = g^{ks}(\partial_i g_{sj} + \partial_j g_{si} - \partial_s g_{ij}), \quad 2\bar{\Gamma}_{ij}^k = \tilde{g}^{ks}(\partial_i \tilde{g}_{sj} + \partial_j \tilde{g}_{si} - \partial_s \tilde{g}_{ij}) \tag{24}$$

and (18) we get

$$\bar{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2\alpha}(\delta_j^k\alpha_i + \delta_i^k\alpha_j - g_{ij}g^{ks}\alpha_s), \quad \alpha_s = \frac{\partial\alpha}{\partial x^s}.$$

Then, using (13) for $\bar{\nabla}\tilde{g}$, we obtain

$$\bar{\nabla}_k\tilde{g}_{ji} = \nabla_k\tilde{g}_{ji} - \frac{1}{2\alpha}(\tilde{g}_{ji}\alpha_k + \tilde{g}_{ik}\alpha_j - g_{kj}\Phi_i^s\alpha_s) - \frac{1}{2\alpha}(\tilde{g}_{kj}\alpha_i + \tilde{g}_{ij}\alpha_k - g_{ik}\Phi_j^s\alpha_s). \tag{25}$$

Substituting (17) and (25) into (23), we get

$$\bar{F}_{kij} = \frac{1}{3}(\bar{g}_{kj}\bar{\theta}_i + \bar{g}_{ki}\bar{\theta}_j + \bar{g}_{kj}\bar{\theta}_i^* + \bar{g}_{ki}\bar{\theta}_j^*), \quad \bar{\theta}_i = \theta_i + \frac{3}{2\alpha}\Phi_i^s\alpha_s, \quad \bar{\theta}_i^* = -\frac{1}{2}S_i^s\bar{\theta}_s. \tag{26}$$

Then, due to (2), (21) and (22), for (M, \bar{g}, Q) it is valid the identity (19). \square

Note. According to Theorem 3.2, we can say that (M, g, Q) and (M, \bar{g}, Q) belong to classes of the same type, defined by the equality (8) for the corresponding metric.

Immediately, from (6) and (19), we have the following

Corollary 3.3. *If $F = 0$ holds, then it is valid*

$$\bar{F}(x, y, z) = \frac{1}{2}\{\bar{g}(x, y)\alpha(\Phi z) + \bar{g}(x, z)\alpha(\Phi y) - \bar{g}(x, y)\alpha(z) - \bar{g}(x, z)\alpha(y)\}. \tag{27}$$

Next, we obtain

Corollary 3.4. *If $F = 0$ holds, then \bar{F} vanishes if and only if α is a constant function.*

Proof. In a local form (27) is

$$\bar{F}_{kij} = \frac{1}{2}(\bar{g}_{kj}\Phi_i^s\alpha_s + \bar{g}_{ki}\Phi_j^s\alpha_s - \bar{g}_{kj}\alpha_i - \bar{g}_{ki}\alpha_j). \tag{28}$$

Let $\bar{F} = 0$ be valid. Then, from (18) and (28), it follows $g_{kj}\Phi_i^s\alpha_s + g_{ki}\Phi_j^s\alpha_s - \tilde{g}_{kj}\alpha_i - \tilde{g}_{ki}\alpha_j = 0$. Contracting by g^{kj} in the latter equality, and using (20) and (21), we get $\Phi_i^s\alpha_s = 0$, which implies $\alpha_1 = \alpha_2 = \alpha_3 = 0$, i.e. α is a constant.

Vice versa. If α is a constant, then its partial derivatives are $\alpha_1 = \alpha_2 = \alpha_3 = 0$, and having in mind (28), we get $\bar{F} = 0$. \square

4. Some curvature properties of (M, g, Q)

It is well-known, that the curvature tensor R of ∇ is defined by

$$R(x, y)z = \nabla_x\nabla_y z - \nabla_y\nabla_x z - \nabla_{[x, y]}z. \tag{29}$$

Also, we consider the tensor of type $(0, 4)$ associated with R , defined as follows:

$$R(x, y, z, u) = g(R(x, y)z, u). \tag{30}$$

The Ricci tensor ρ and the scalar curvature τ with respect to g are as usually

$$\rho(y, z) = g^{ij}R(e_i, y, z, e_j), \quad \tau = g^{ij}\rho(e_i, e_j). \tag{31}$$

Let $\tilde{\Gamma}$ and $\tilde{\nabla}$ be the Christoffel symbols and the Levi-Civita connection of \tilde{g} , respectively. Let \tilde{R} be the curvature tensor of $\tilde{\nabla}$. The Ricci tensor $\tilde{\rho}$ and the scalar curvature $\tilde{\tau}$ with respect to \tilde{g} are

$$\tilde{\rho}(y, z) = \tilde{g}^{ij}\tilde{R}(e_i, y, z, e_j), \quad \tilde{\tau} = \tilde{g}^{ij}\tilde{\rho}(e_i, e_j). \tag{32}$$

From (13), using the Christoffel formulas (24) for Γ and $\tilde{\Gamma}$, we obtain the following relation

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2}g^{ks}(\nabla_i\tilde{g}_{js} + \nabla_j\tilde{g}_{is} - \nabla_s\tilde{g}_{ij}). \tag{33}$$

We substitute (17) into (33) and we get

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{3}\tilde{g}^{ks}(g_{ij}\theta_s + \tilde{g}_{ij}\theta_s^*). \tag{34}$$

From (20), (21) and (22), we find

$$\tilde{g}^{sk}\theta_s = -\theta^{*k}, \quad \tilde{g}_{sk}\theta^s = \theta_k - 2\theta_k^*, \quad \tilde{g}_{sk}\theta^{*s} = -\theta_k, \quad \tilde{g}^{sk}\theta_s^* = -\frac{1}{2}(\theta^k + \theta^{*k}). \tag{35}$$

We apply (35) into (34) and we obtain that the tensor $T = \tilde{\Gamma} - \Gamma$ of the affine deformation of ∇ has components

$$T_{ij}^k = -\frac{1}{6}(2g_{ij}\theta^{*k} + \tilde{g}_{ij}(\theta^k + \theta^{*k})). \tag{36}$$

It is well-known the relation $\tilde{R}_{ijs}^k = R_{ijs}^k + \nabla_j T_{is}^k - \nabla_s T_{ij}^k + T_{is}^a T_{aj}^k - T_{ij}^a T_{as}^k$. Then, using (22), (35) and (36), we calculate

$$\begin{aligned} \tilde{R}_{ijs}^k &= R_{ijs}^k - \frac{1}{3}g_{is}(\nabla_j \theta^{*k} - \frac{1}{3}\theta_j^* \theta^{*k}) + \frac{1}{3}g_{ij}(\nabla_s \theta^{*k} - \frac{1}{3}\theta_s^* \theta^{*k}) \\ &\quad - \frac{1}{6}\tilde{g}_{is}(\nabla_j \theta^{*k} + \nabla_j \theta^k - \frac{1}{3}\theta_j \theta^{*k} - \frac{1}{3}\theta_j^* \theta^{*k}) + \frac{1}{6}\tilde{g}_{ij}(\nabla_s \theta^{*k} + \nabla_s \theta^k - \frac{1}{3}\theta_s \theta^{*k} - \frac{1}{3}\theta_s^* \theta^{*k}). \end{aligned}$$

By contracting $k = s$ in the latter equality, and having in mind (17), (20), (22), (31), (32) and (35), we get

$$\tilde{\rho}_{ij} = \rho_{ij} + \frac{1}{3}g_{ij}\nabla_s \theta^{*s} + \frac{1}{6}\tilde{g}_{ij}(\nabla_s \theta^s + \nabla_s \theta^{*s}). \tag{37}$$

Let us denote

$$\tau^* = \tilde{g}^{ij}\rho(e_i, e_j), \quad \tilde{\tau}^* = g^{ij}\tilde{\rho}(e_i, e_j). \tag{38}$$

Due to (31), (32), (37) and (38) we obtain

$$\tilde{\rho}_{ij} = \rho_{ij} + \frac{1}{3}(\tilde{\tau}^* - \tau)g_{ij} + \frac{1}{6}(2\tilde{\tau} - 2\tau^* + \tilde{\tau}^* - \tau)\tilde{g}_{ij},$$

Therefore we establish the following

Theorem 4.1. For the Ricci tensors ρ and $\tilde{\rho}$ and for the scalar curvatures τ , τ^* , $\tilde{\tau}$ and $\tilde{\tau}^*$ the following relation is valid:

$$\tilde{\rho}(x, y) = \rho(x, y) + \frac{1}{3}(\tilde{\tau}^* - \tau)g(x, y) + \frac{1}{6}(2\tilde{\tau} - 2\tau^* + \tilde{\tau}^* - \tau)\tilde{g}(x, y). \tag{39}$$

In [3], a Riemannian manifold (M, g, Q) is called almost Einstein if the metrics g and \tilde{g} satisfy

$$\rho(x, y) = \beta g(x, y) + \gamma \tilde{g}(x, y), \tag{40}$$

where β and γ are smooth functions on M .

Corollary 4.2. If the Levi-Civita connection $\tilde{\nabla}$ of \tilde{g} is a locally flat connection, then (M, g, Q) is an almost Einstein manifold and the Ricci tensor ρ has the form

$$\rho(x, y) = \frac{\tau}{3}g(x, y) + \frac{2\tau^* + \tau}{6}\tilde{g}(x, y). \tag{41}$$

Proof. If $\tilde{\nabla}$ is a locally flat connection, then we have $\tilde{R} = 0$. From (32) and (38) it follows $\tilde{\rho} = 0$ and $\tilde{\tau} = \tilde{\tau}^* = 0$. Then (39) implies (41) and according to (40) we have that (M, g, Q) is an almost Einstein manifold. \square

Note. Examples of manifolds (M, g, Q) satisfying (41) are considered in [3].

In an analogous way we prove

Corollary 4.3. If the Levi-Civita connection ∇ of g is a locally flat connection, then (M, \tilde{g}, Q) is an almost Einstein manifold and the Ricci tensor $\tilde{\rho}$ has the form

$$\tilde{\rho}(x, y) = \frac{\tilde{\tau}^*}{3}g(x, y) + \frac{\tilde{\tau}^* + 2\tilde{\tau}}{6}\tilde{g}(x, y).$$

5. Lie groups as manifolds of the considered type

Let G be a 3-dimensional real connected Lie group and \mathfrak{g} be its Lie algebra with a basis $\{x_1, x_2, x_3\}$ of left invariant vector fields. We introduce a circulant structure Q and a Riemannian metric g as follows:

$$Qx_1 = x_2, \quad Qx_2 = x_3, \quad Qx_3 = x_1, \quad (42)$$

$$g(x_i, x_j) = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases} \quad (43)$$

Obviously, for the manifold (G, g, Q) the equalities (1) and (3) are valid, i.e. (G, g, Q) is a Riemannian manifold of the same type as (M, g, Q) .

For the associated metric \tilde{g} , due to (5), we get

$$\tilde{g}(x_i, x_j) = \begin{cases} 1, & i \neq j; \\ 0, & i = j. \end{cases} \quad (44)$$

The corresponding Lie algebra \mathfrak{g} is determined as follows:

$$[x_i, x_j] = C_{ij}^k x_k, \quad (45)$$

where $C_{ij}^k = -C_{ji}^k$. The Jacobi identity implies [5]

$$\begin{aligned} C_{23}^1(C_{31}^3 - C_{12}^2) &= C_{23}^2 C_{21}^1 + C_{23}^3 C_{31}^1, \\ C_{12}^3(C_{31}^1 - C_{23}^2) &= C_{23}^3 C_{21}^2 + C_{31}^3 C_{12}^2, \\ C_{13}^2(C_{23}^3 - C_{12}^2) &= C_{23}^2 C_{13}^3 + C_{12}^2 C_{31}^1. \end{aligned} \quad (46)$$

Let us denote

$$\lambda_1 = C_{12}^1, \quad \lambda_2 = C_{12}^2, \quad \lambda_3 = C_{12}^3, \quad \mu_1 = C_{13}^1, \quad \mu_2 = C_{13}^2, \quad \mu_3 = C_{13}^3, \quad \nu_1 = C_{23}^1, \quad \nu_2 = C_{23}^2, \quad \nu_3 = C_{23}^3. \quad (47)$$

Then (45), applying (47), takes the form

$$[x_1, x_2] = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3, \quad [x_2, x_3] = \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3, \quad [x_1, x_3] = \mu_1 x_1 + \mu_2 x_2 + \mu_3 x_3. \quad (48)$$

According to (46) and (47), we obtain the following conditions:

$$\nu_1(\mu_3 - \lambda_2) = \nu_2 \lambda_1 + \mu_1 \nu_3, \quad \lambda_3(\mu_1 - \nu_2) = \nu_3 \lambda_2 + \mu_3 \lambda_1, \quad \mu_2(\nu_3 - \lambda_1) = \nu_2 \mu_3 + \lambda_2 \mu_1. \quad (49)$$

The well-known Koszul formula implies

$$2g(\nabla_{x_i} x_j, x_k) = g([x_i, x_j], x_k) + g([x_k, x_i], x_j) + g([x_k, x_j], x_i),$$

and using (43) and (48), we obtain

$$\begin{aligned} \nabla_{x_1} x_1 &= -\lambda_1 x_2 - \mu_1 x_3, & \nabla_{x_1} x_2 &= \lambda_1 x_1 + \frac{1}{2}(\lambda_3 - \mu_2 - \nu_1)x_3, \\ \nabla_{x_1} x_3 &= \mu_1 x_1 + \frac{1}{2}(\mu_2 - \lambda_3 + \nu_1)x_2, & \nabla_{x_2} x_1 &= -\lambda_2 x_2 - \frac{1}{2}(\lambda_3 + \nu_1 + \mu_2)x_3, \\ \nabla_{x_2} x_2 &= \lambda_2 x_1 - \nu_2 x_3, & \nabla_{x_2} x_3 &= \frac{1}{2}(\nu_1 + \mu_2 + \lambda_3)x_1 + \nu_2 x_2, & \nabla_{x_3} x_3 &= \mu_3 x_1 + \nu_3 x_2, \\ \nabla_{x_3} x_1 &= \frac{1}{2}(\nu_1 - \mu_2 - \lambda_3)x_2 - \mu_3 x_3, & \nabla_{x_3} x_2 &= \frac{1}{2}(-\nu_1 + \mu_2 + \lambda_3)x_1 - \nu_3 x_3. \end{aligned} \quad (50)$$

Further, bearing in mind (6), (43) and (50), for the components of F , θ and θ^* we get

$$\begin{aligned}
 F_{111} &= 2\lambda_1 + 2\mu_1, & F_{112} &= \frac{1}{2}(2\mu_1 + \mu_2 - \lambda_3 + \nu_1), & F_{113} &= \frac{1}{2}(2\lambda_1 + \lambda_3 - \mu_2 - \nu_1), \\
 F_{122} &= -2\lambda_1 + \nu_1 + \mu_2 - \lambda_3, & F_{123} &= -\lambda_1 - \mu_1, & F_{221} &= \frac{1}{2}(\lambda_3 + \nu_1 + \mu_2 + 2\nu_2), \\
 F_{222} &= -2\lambda_2 + 2\nu_2, & F_{223} &= -\frac{1}{2}(2\lambda_2 + \nu_1 + \mu_2 + \lambda_3), & F_{211} &= 2\lambda_2 + \nu_1 + \mu_2 + \lambda_3, \\
 F_{213} &= \lambda_2 - \nu_2, & F_{133} &= -2\mu_1 - \nu_1 - \mu_2 + \lambda_3, & F_{233} &= -\lambda_3 - \nu_1 - 2\nu_2 - \mu_2, \\
 F_{311} &= \lambda_3 - \nu_1 + \mu_2 + 2\mu_3, & F_{312} &= \nu_3 + \mu_3, & F_{313} &= -\frac{1}{2}(-\lambda_1 + \nu_1 - \mu_2 + 2\nu_3), \\
 F_{322} &= -\lambda_3 + \nu_1 - \mu_2 + 2\nu_3, & F_{332} &= -\frac{1}{2}(\lambda_3 - \nu_1 + \mu_2 + 2\mu_3), & F_{333} &= -2\mu_3 - 2\nu_3,
 \end{aligned}
 \tag{51}$$

$$\begin{aligned}
 \theta_1 &= 2\lambda_1 + \lambda_3 + 2\mu_1 + \mu_2 + \nu_2 - \nu_3, \\
 \theta_2 &= -2\lambda_2 - \lambda_3 + \mu_1 - \mu_3 + \nu_1 + 2\nu_2, \\
 \theta_3 &= \lambda_1 - \lambda_2 - \mu_2 - 2\mu_3 - \nu_1 - 2\nu_3,
 \end{aligned}
 \tag{52}$$

$$\begin{aligned}
 \theta_1^* &= -\frac{1}{2}(-\lambda_1 - 3\lambda_2 - 2\lambda_3 - \mu_1 - 2\mu_2 - 3\mu_3 + \nu_2 - \nu_3), \\
 \theta_2^* &= -\frac{1}{2}(3\lambda_1 + \lambda_2 + 2\lambda_3 + \mu_1 - \mu_3 - 2\nu_1 - \nu_2 - 3\nu_3), \\
 \theta_3^* &= -\frac{1}{2}(\lambda_1 - \lambda_2 + 3\mu_1 + 2\mu_2 + \mu_3 + 2\nu_1 + 3\nu_2 + \nu_3).
 \end{aligned}
 \tag{53}$$

Theorem 5.1. *The manifold (G, g, Q) has a corresponding Lie algebra determined by*

$$\begin{aligned}
 [x_1, x_2] &= \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3, \\
 [x_2, x_3] &= \nu_1 x_1 + \nu_2 x_2 + \nu_3 x_3, \\
 [x_1, x_3] &= (\nu_2 + \lambda_3)x_1 + (\nu_3 + \lambda_1)x_2 + (\nu_1 + \lambda_2)x_3,
 \end{aligned}
 \tag{54}$$

where the structure constants satisfy the following conditions:

$$\begin{aligned}
 2\nu_1\lambda_2 + \nu_1^2 - \lambda_1\nu_2 - \nu_2\nu_3 - \lambda_3\nu_3 &= 0, \\
 2\nu_2\lambda_3 + \lambda_3^2 - \lambda_2\nu_3 - \lambda_1\lambda_2 - \lambda_1\nu_1 &= 0, \\
 -\lambda_1^2 + \nu_3^2 + \lambda_2\lambda_3 - \nu_1\nu_2 &= 0.
 \end{aligned}
 \tag{55}$$

Proof. Since (G, g, Q) is of the type of the manifold (M, g, Q) , the equality (8) is valid. Having in mind (8), (43) and (44), we get

$$\begin{aligned}
 F_{111} &= \frac{2}{3}\theta_1, & F_{112} &= \frac{1}{3}(\theta_2 + \theta_1^*), & F_{113} &= \frac{1}{3}(\theta_3 + \theta_1^*), & F_{122} &= F_{322} = \frac{2}{3}\theta_2^*, \\
 F_{123} &= \frac{1}{3}(\theta_3^* + \theta_2^*), & F_{221} &= \frac{1}{3}(\theta_1 + \theta_2^*), & F_{222} &= \frac{2}{3}\theta_2, & F_{223} &= \frac{1}{3}(\theta_3 + \theta_2^*), \\
 F_{211} &= F_{311} = \frac{2}{3}\theta_1^*, & F_{213} &= \frac{1}{3}(\theta_3^* + \theta_1^*), & F_{133} &= F_{233} = \frac{2}{3}\theta_3^*, & F_{312} &= \frac{1}{3}(\theta_2^* + \theta_1^*), \\
 F_{313} &= \frac{1}{3}(\theta_1 + \theta_3^*), & F_{332} &= \frac{1}{3}(\theta_2 + \theta_3^*), & F_{333} &= \frac{2}{3}\theta_3.
 \end{aligned}
 \tag{56}$$

Then, using (51), (52) and (53), we find

$$\mu_1 = \nu_2 + \lambda_3, \quad \mu_2 = \nu_3 + \lambda_1, \quad \mu_3 = \nu_1 + \lambda_2.
 \tag{57}$$

From (48), (49) and (57) we obtain (54) and (55). \square

Proposition 5.2. *The components of R, F, θ and θ^* with respect to the basis $\{x_i\}$ for (G, g, Q) are*

$$\begin{aligned} R_{1212} &= \frac{3}{4}(\lambda_1^2 + \lambda_3^2) - \frac{1}{4}(v_1^2 + v_3^2) + \lambda_2^2 + v_2^2 + \frac{1}{2}(\lambda_1(\lambda_3 - v_3) - v_1(\lambda_1 + v_3) + \lambda_3(2v_2 - v_1 + v_3)), \\ R_{1313} &= \frac{3}{4}(\lambda_1^2 + \lambda_3^2 + v_1^2 + v_3^2) + \lambda_2^2 + v_2^2 + \frac{1}{2}(\lambda_1(\lambda_3 + v_3) + v_1(\lambda_1 + 4\lambda_2 + v_3) + \lambda_3(v_1 + 4v_2 + v_3)), \\ R_{2323} &= -\frac{1}{4}(\lambda_1^2 + \lambda_3^2) + \frac{3}{4}(v_1^2 + v_3^2) + \lambda_2^2 + v_2^2 - \frac{1}{2}(\lambda_1(\lambda_3 + v_3) - v_1(\lambda_1 + 2\lambda_2 + v_3) + \lambda_3(v_1 + v_3)), \\ R_{1223} &= -\lambda_2\lambda_3 + \lambda_1v_1 + \lambda_1^2 + \lambda_1v_3 + \lambda_3v_3, \\ R_{1213} &= \lambda_2(\lambda_1 + v_1 + v_3) + \lambda_3(\lambda_1 + \lambda_2 + v_1), \\ R_{1332} &= -v_2(\lambda_1 + v_3 + \lambda_3) - v_1(\lambda_3 + v_3 + v_2), \end{aligned} \quad (58)$$

$$\begin{aligned} F_{111} &= 2\lambda_1 + 2\lambda_3 + 2v_2, \quad F_{112} = F_{221} = \frac{1}{2}(\lambda_1 + \lambda_3 + v_1 + 2v_2 + v_3), \quad F_{123} = -\lambda_1 - \lambda_3 - v_2, \\ F_{113} = F_{331} &= \frac{1}{2}(\lambda_1 + \lambda_3 - v_1 - v_3), \quad F_{122} = F_{322} = -\lambda_1 - \lambda_3 + v_1 + v_3, \\ F_{222} &= -2\lambda_2 + 2v_2, \quad F_{211} = F_{311} = \lambda_1 + 2\lambda_2 + \lambda_3 + v_1 + v_3, \quad F_{213} = \lambda_2 - v_2, \\ F_{223} = F_{332} &= -\frac{1}{2}(\lambda_1 + 2\lambda_2 + \lambda_3 + v_1 + v_3), \quad F_{233} = F_{133} = -\lambda_1 - \lambda_3 - v_1 - 2v_2 - v_3, \\ F_{312} &= \lambda_2 + v_1 + v_3, \quad F_{133} = -\lambda_1 - \lambda_3 - v_1 - 2v_2 - v_3, \quad F_{333} = -2\lambda_2 - 2v_1 - 2v_3, \end{aligned} \quad (59)$$

$$\begin{aligned} \theta_1 &= 3(\lambda_1 + \lambda_3 + v_2), \quad \theta_2 = 3(v_2 - \lambda_2), \quad \theta_3 = -3(\lambda_2 + v_1 + v_3), \\ \theta_1^* &= \frac{3}{2}(\lambda_1 + 2\lambda_2 + \lambda_3 + v_1 + v_3), \quad \theta_2^* = -\frac{3}{2}(\lambda_1 + \lambda_3 - v_1 - v_3), \\ \theta_3^* &= -\frac{3}{2}(\lambda_1 + \lambda_3 + v_1 + 2v_2 + v_3). \end{aligned} \quad (60)$$

Proof. The statement follows from (29), (30), (51), (52), (53) and (57). \square

6. Manifolds (G, g, Q) with sectional curvatures invariant under Q

If $\{x, y\}$ is a non-degenerate 2-plane spanned by vectors x, y in the tangent space of the manifold, then its sectional curvature is

$$k(x, y) = \frac{R(x, y, x, y)}{g(x, x)g(y, y) - g^2(x, y)}. \quad (61)$$

Let G' be a subgroup of G and (G', g, Q) be a manifold with sectional curvatures which are invariant under Q , i.e. according to (43) and (61) we have

$$R(Qx, Qy, Qx, Qy) = R(x, y, x, y). \quad (62)$$

Because of (42), the identity (62) is equivalent to

$$R_{1212} = R_{1313} = R_{2323}. \quad (63)$$

From (58) and (63), we find

$$\begin{aligned} \lambda_1^2 + \lambda_3^2 + \lambda_1v_3 + \lambda_3v_3 + \lambda_1\lambda_3 + \lambda_2v_1 + \lambda_3v_1 + 2\lambda_3v_2 &= 0, \\ v_1^2 + v_3^2 + \lambda_3v_2 + \lambda_1v_3 + \lambda_3v_1 + \lambda_1v_1 + v_3v_1 + 2\lambda_2v_1 &= 0. \end{aligned} \quad (64)$$

We will consider the following instances where the system of equations (55) and (64) is executed:

Case (A) $\lambda_3 = 0, v_1 = 0, v_3 = -\lambda_1,$

Case (B) $\lambda_3 = -\lambda_1 - \lambda_2, v_1 = -\lambda_1 - \lambda_2, v_2 = \lambda_1, v_3 = \lambda_2,$

Case (C) $\lambda_3 = -\lambda_1 - \lambda_2, v_1 = \lambda_1, v_2 = \lambda_2, v_3 = -\lambda_1 - \lambda_2.$

Let us consider Case (A). From (54) and (55) it follows that the Lie algebra is determined by the commutators:

$$[x_1, x_2] = \lambda_1 x_1 + \lambda_2 x_2, \quad [x_2, x_3] = v_2 x_2 - \lambda_1 x_3, \quad [x_1, x_3] = v_2 x_1 + \lambda_2 x_3. \quad (65)$$

Then, using (59) and (60), for the components of F, θ and θ^* we get

$$\begin{aligned} F_{111} &= 2\lambda_1 + 2v_2, & F_{112} &= v_2, & F_{113} &= \lambda_1, & F_{122} &= F_{322} = -2\lambda_1, \\ F_{123} &= -\lambda_1 - v_2, & F_{221} &= v_2, & F_{222} &= -2\lambda_2 + 2v_2, & F_{223} &= -\lambda_2, \\ F_{211} &= F_{311} = 2\lambda_2, & F_{213} &= \lambda_2 - v_2, & F_{133} &= F_{233} = -2v_2, & F_{312} &= \lambda_2 - \lambda_1, \\ F_{313} &= \lambda_1, & F_{332} &= -\lambda_2, & F_{333} &= -2\lambda_2 + 2\lambda_1, \end{aligned} \quad (66)$$

$$\theta_1 = 3(\lambda_1 + v_2), \quad \theta_2 = 3(v_2 - \lambda_2), \quad \theta_3 = 3(\lambda_1 - \lambda_2), \quad \theta_1^* = 3\lambda_2, \quad \theta_2^* = -3\lambda_1, \quad \theta_3^* = -3v_2. \quad (67)$$

From (31) and (58) we obtain all nonzero components of R and ρ on (G', g, Q) :

$$R_{1212} = R_{2323} = R_{1313} = \lambda_1^2 + \lambda_2^2 + v_2^2, \quad \rho_{11} = \rho_{22} = \rho_{33} = -2(\lambda_1^2 + \lambda_2^2 + v_2^2). \quad (68)$$

Then the scalar curvature of (G', g, Q) is

$$\tau = -6(\lambda_1^2 + \lambda_2^2 + v_2^2). \quad (69)$$

From (43), (68) and (69) it follows $\rho = \frac{\tau}{3}g$, i.e. (G', g, Q) is an Einstein manifold.

Using (61) and (63) we get that the sectional curvatures k_{ij} of the basic 2-planes $\{x_i, x_j\}$ are equal to

$$k = \lambda_1^2 + \lambda_2^2 + v_2^2, \quad (70)$$

i.e. (G', g, Q) is conformally flat.

Therefore, we establish the truthfulness of the following

Proposition 6.1. *In Case (A) the following properties of (G', g, Q) are valid:*

- 1) the components of F, θ and θ^* are (66) and (67);
- 2) the nonzero components of R and ρ are (68);
- 3) the manifold is an Einstein manifold and the scalar curvature is (69);
- 4) the manifold is of constant sectional curvature (70).

With similar calculations we obtain the following statements.

Proposition 6.2. *In Case (B) the Lie algebra is determined by the commutators*

$$\begin{aligned} [x_1, x_2] &= \lambda_1 x_1 + \lambda_2 x_2 - (\lambda_1 + \lambda_2)x_3, \\ [x_2, x_3] &= -(\lambda_1 + \lambda_2)x_1 + \lambda_1 x_2 + \lambda_2 x_3, \\ [x_1, x_3] &= -\lambda_2 x_1 + (\lambda_1 + \lambda_2)x_2 - \lambda_1 x_3. \end{aligned}$$

In this case

1) the components of F , θ and θ^* are

$$\begin{aligned} F_{111} &= F_{222} = F_{333} = 2(\lambda_1 - \lambda_2), \\ F_{112} &= F_{221} = F_{223} = F_{313} = F_{332} = F_{113} = \frac{1}{2}(\lambda_1 - \lambda_2), \\ F_{122} &= F_{123} = F_{211} = F_{213} = F_{133} = F_{233} = F_{311} = F_{312} = F_{322} = \lambda_2 - \lambda_1, \\ \theta_1 &= \theta_2 = \theta_3 = 3(\lambda_1 - \lambda_2), \quad \theta_1^* = \theta_2^* = \theta_3^* = \frac{3}{2}(\lambda_2 - \lambda_1); \end{aligned}$$

2) the nonzero components of R and ρ are

$$R_{1212} = R_{2323} = R_{1313} = \frac{3}{4}(\lambda_1 - \lambda_2)^2, \quad \rho_{11} = \rho_{22} = \rho_{33} = -\frac{3}{2}(\lambda_1 - \lambda_2)^2;$$

3) the manifold is an Einstein manifold and the scalar curvature is $\tau = -\frac{9}{2}(\lambda_1 - \lambda_2)^2$;

4) the manifold is of constant sectional curvature $k = \frac{3}{4}(\lambda_1 - \lambda_2)^2$.

Proposition 6.3. In Case (C) the Lie algebra is determined by the commutators

$$\begin{aligned} [x_1, x_2] &= \lambda_1 x_1 + \lambda_2 x_2 - (\lambda_1 + \lambda_2)x_3, \\ [x_2, x_3] &= \lambda_1 x_1 + \lambda_2 x_2 - (\lambda_1 + \lambda_2)x_3, \\ [x_1, x_3] &= -\lambda_1 x_1 - \lambda_2 x_2 + (\lambda_1 + \lambda_2)x_3. \end{aligned}$$

In this case

1) the components of F , θ and θ^* are zero;

2) the nonzero components of R and ρ are

$$\begin{aligned} R_{1212} &= R_{2323} = R_{1313} = R_{1332} = R_{1223} = R_{1321} = 2(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2), \\ \rho_{11} &= \rho_{22} = \rho_{33} = \rho_{12} = \rho_{13} = \rho_{23} = -4(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2); \end{aligned}$$

3) the manifold is an Einstein manifold and the scalar curvature is $\tau = -12(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)$;

4) the manifold is of constant sectional curvature $k = 2(\lambda_1^2 + \lambda_1 \lambda_2 + \lambda_2^2)$;

5) it is valid $[Qx_i, Qx_j] = [x_i, x_j]$, i.e. Q is an abelian structure.

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