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Effects on Rough $\mathcal{I}$-Lacunary Statistical Convergence to Induce the Weighted Sequence

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Abstract. Two classes of sets are introduced: rough weighted $\mathcal{I}$-lacunary statistical limit set and weighted $\mathcal{I}$-lacunary statistical cluster points set which are natural generalizations of rough $\mathcal{I}$-limit set and $\mathcal{I}$-cluster points set respectively. To highlight the variation from basic results we place into some new examples. So our aim is to analyze the different behaviors of the new convergences and characterize both the sets with topological approach like closedness, boundedness, compactness etc.

1. Introduction

The idea of convergence of a real sequence was extended to statistical convergence by Fast [14] and Steinhaus [35] (see also [34]) as follows: If $\mathbb{N}$ denotes the set of all natural numbers and $K \subseteq \mathbb{N}$ then $K(m,n)$ (where $m,n \in \mathbb{N}$) denotes the cardinality of the set $K \cap [m,n]$. The upper and lower natural (or, asymptotic) densities of the set K is defined by

$$\overline{d}(K) = \limsup_{n\to\infty} \frac{K(1,n)}{n} \text{ and } \underline{d}(K) = \liminf_{n\to\infty} \frac{K(1,n)}{n}.$$

If $\overline{d}(K) = \underline{d}(K)$ then, we say that the natural density of K exists and it is denoted by $d(K)$ and clearly $d(K) = \lim_{n\to\infty} \frac{K(1,n)}{n}$.

A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to a real number ℓ if for any $\varepsilon > 0$, the set $K(\varepsilon) = \{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\}$ has natural density zero. In this case we write $S - \lim x = \ell$. We shall also use S to denote the set of all statistically convergent sequences.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be statistically bounded [36] if there exists a positive real number M, such that the natural density of the set $\{n \in \mathbb{N} : |x_n| \geq M\}$ is zero.

A real number ζ is said to be a statistical cluster point [15, 22, 28] of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ provided for each $\varepsilon > 0$, the natural density of the set $\{n \in \mathbb{N} : |x_n - \zeta| < \varepsilon\}$ is different from zero. We denote the set of all statistical cluster points of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ by Γ_x.

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Statistical convergence turned out to be one of the most active areas of research in summability theory after works of Fridy [16] and Šalát [32].

The notion of ideal is an extension of natural density which depends on the structure of subsets of the set of natural numbers as follows:

Definition 1.1. [23, 24]. A family $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if
(i) $\emptyset \in \mathcal{I}$,
(ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$,
(iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$.

The ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\emptyset\}$ and $\mathbb{N} \notin \mathcal{I}$.

Definition 1.2. [23, 24]. A non-empty family $\mathcal{F} \subset 2^{\mathbb{N}}$ is called a filter if
(i) $\emptyset \notin \mathcal{F}$,
(ii) $A, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$,
(iii) $A \in \mathcal{F}, A \subset B$ implies $B \in \mathcal{F}$.

Clearly $\mathcal{I} \subset 2^{\mathbb{N}}$ is a non-trivial ideal of $\mathbb{N}$ iff $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{K \subset \mathbb{N} : \mathbb{N} \setminus K \in \mathcal{I}\}$ is a filter on $\mathbb{N}$, called the filter associated with $\mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is called admissible if $\mathcal{I}$ contains all the singleton set.

Using this concept of ideal, the notion of statistical convergence of a real sequence had been extended to $\mathcal{I}$-convergence by Kostyrko et al. [24] as follows: A sequence $\{x_n\}_{n\in\mathbb{N}}$ of points in a metric space (X, ρ) is said to be $\mathcal{I}$-convergent to ℓ if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{n \in \mathbb{N} : \rho(x_n, \ell) \geq \varepsilon\} \in \mathcal{I}$.

An element $\zeta \in X$ is said to be an $\mathcal{I}$-cluster point [5, 23, 24] of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in X if for each $\varepsilon > 0$, we have $\{n \in \mathbb{N} : \rho(x_n, \zeta) < \varepsilon\} \notin \mathcal{I}$ and the set of all $\mathcal{I}$-cluster points of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is denoted by $\Gamma_x(\mathcal{I})$.

Consequently, a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in a normed linear space with norm $\|.\|$ is said to be $\mathcal{I}$-bounded [26, 37] if there exists a positive real number G such that the set $\{n \in \mathbb{N} : \|x_n\| \geq G\} \in \mathcal{I}$.

In another direction, a new type of convergence called lacunary statistical convergence was introduced by Fridy et al. [17] as follows: A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ such that $k_0 = 0$, $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$, $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$ for all $r \in \mathbb{N}$. Then the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be lacunary statistically convergent to a real number ℓ if for any $\varepsilon > 0$, $\lim\limits_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |x_k - \ell| \geq \varepsilon\}| = 0$. In this case we write $S_\theta - \lim x = \ell$. We shall also use S_θ to denote the set of all statistically convergent sequences. In [17] the relation between lacunary statistical convergence and statistical convergence was established among the other thinks. In the year 2017 weighted lacunary statistical convergence is a generalization of lacunary statistical convergence was introduce by Ghosal et al. [21].

Definition 1.3. [21]. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence and $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers such that $t_n > \alpha$, $\forall\, n \in \mathbb{N}$ (where α is a positive real number) and $T_n = \sum\limits_{k=1}^{n} t_k$ (where $n \in \mathbb{N}$ and $T_0 = 0$). A sequence of real numbers $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be weighted lacunary statistically convergent (or, weighted S_θ-convergent) to a real number ℓ if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{(T_{k_r} - T_{k_{r-1}})} |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k |x_k - \ell| \geq \varepsilon\}| = 0.$$

We write $x_n \xrightarrow{WS_\theta} \ell$. The class of all weighted lacunary statistically convergent (or, weighted S_θ-convergent) sequences is denoted by WS_θ. More investigation in this direction and many applications are found in [8, 11, 18, 19, 25] where some important references are present.

More frequently in the year 2011, $\mathcal{I}$-statistical convergence [33] and $\mathcal{I}$-lacunary statistical convergence [9, 10] were improved by Das et al. as follows:

Definition 1.4. [33]. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-statistically convergent to ℓ if for arbitrary $\varepsilon, \delta > 0$, the set $\{n \in \mathbb{N} : \frac{1}{n}|\{k \leq n : |x_k - \ell| \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}$.

Definition 1.5. [9, 10]. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-lacunary statistically convergent to ℓ if for arbitrary $\varepsilon, \delta > 0$, the set $\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - \ell| \geq \varepsilon\}| \geq \delta\} \in \mathcal{I}$.

However, an important question remains unanswered in $\mathcal{I}$-lacunary statistical convergence i.e., to construct an example of a sequence which is $\mathcal{I}$-lacunary statistical convergence but not lacunary statistical convergence.

On the other hand in [1], a different direction was given to study of statistical convergence. Pal et al. [26] and Dündar et al. [13] independently extended the result given in [1] to rough $\mathcal{I}$-convergence as follows:

Definition 1.6. [13, 26]. Let $\tilde{\mathbf{r}}$ be a non-negative real number. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in a normed linear space $\|.\|$, is said to be rough $\mathcal{I}$-convergent to x_* w.r.t. the roughness of degree $\tilde{\mathbf{r}}$ (or shortly: $\tilde{\mathbf{r}}$-$\mathcal{I}$-convergent to x_*) if for every $\varepsilon > 0$, the set

$$\{n \in \mathbb{N} : \|x_n - x_*\| \geq \tilde{\mathbf{r}} + \varepsilon\} \in \mathcal{I}$$

is satisfied and we denote this by $x_n \xrightarrow[\tilde{\mathbf{r}}]{\mathcal{I}} x_*$. If we take $\tilde{\mathbf{r}} = 0$, then we obtain the ordinary $\mathcal{I}$-convergence. The set $\mathcal{I} - LIM^{\tilde{\mathbf{r}}}x = \{x_* : x_n \xrightarrow[\tilde{\mathbf{r}}]{\mathcal{I}} x_*\}$ is called the rough $\mathcal{I}$-limit set w.r.t. the roughness of degree $\tilde{\mathbf{r}}$ (or shortly: $\tilde{\mathbf{r}}$-$\mathcal{I}$-limit set) of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be $\tilde{\mathbf{r}}$-$\mathcal{I}$-convergent if $\mathcal{I} - LIM^{\tilde{\mathbf{r}}}x \neq \emptyset$. One can also see [2, 7, 12, 29–31] for related works.

On further progress, we combine the approaches of $\mathcal{I}$-lacunary statistical convergence [10], rough $\mathcal{I}$-convergence [13, 26], statistical cluster point [3, 27] and weighted lacunary statistical convergence [21] and introduce new and more advance summability methods namely, rough weighted $\mathcal{I}$-lacunary statistical limit set and weighted $\mathcal{I}$-lacunary statistical cluster points set of a sequence in a metric space. Some new examples are constructed to ensure the deviation from basic results such as:

Theorem 1.7 [13, 26]. For a sequence $x = \{x_n\}_{n\in\mathbb{N}}$, we have the diameter of rough $\mathcal{I}$-limit set is $\leq 2\tilde{\mathbf{r}}$.

In general, it has no smaller bound.

Theorem 1.8 [13, 26]. For an arbitrary $c \in \mathcal{I}$-cluster points set of a sequence, then the distance between x_* and c is $\leq \tilde{\mathbf{r}}$ for all $x_* \in$ rough $\mathcal{I}$-limit set.

Theorem 1.9 [6, 24]. The set of all statistical cluster points for any sequence in a metric space is closed.

Theorem 1.10 [27]. If a sequence in a finite dimensional normed linear space is statistically bounded then the statistical cluster points set is non-empty.

Theorem 1.11 [4]. The set of statistical cluster points of a bounded sequence is a compact subset of $\mathbb{R}$.

So our main objective is to interpret the different behaviors of the new convergences and characterize both the sets with topological approach like closedness, boundedness, compactness etc.

2. Main Results

We begin with an example of a sequence which is $\mathcal{I}$-lacunary statistically convergent but neither lacunary statistically convergent nor statistically convergent.

Example 2.1. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be any lacunary sequence. Define a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ by

$$x_n = \begin{cases} 2, & \text{for } n \in I_r \text{ and } r = m^2 \text{ for all } m \in \mathbb{N}, \\ 1, & \text{otherwise.} \end{cases}$$

Let $0 < \varepsilon < 1$. Then,

$$\frac{1}{h_r}|\{k \in I_r : |x_k - 1| \geq \varepsilon\}| = \begin{cases} 1, & \text{if } r = m^2 \text{ for all } m \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

It is quite clear that the sequence $\{x_n\}_{n\in\mathbb{N}}$ is not lacunary statistically convergent. If we choose a lacunary sequence such that $1 < \liminf q_r \leq \limsup q_r < \infty$ then from Theorem 4 [17]: "For any lacunary sequence θ, $S = S_\theta$ iff $1 < \liminf q_r \leq \limsup q_r < \infty$. Also $S - \lim x = \ell$ implies and implied by $S_\theta - \lim x = \ell$." So we can say that the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is not statistically convergent to 1.

Next we assume that $\mathcal{I} = \mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$. Then for any $\delta > 0$, we get

$$\{r \in \mathbb{N} : \frac{1}{h_r}|\{k \in I_r : |x_k - 1| \geq \varepsilon\}| \geq \delta\} \subseteq \{1^2, 2^2, 3^2, 4^2, ...\} \in \mathcal{I}_d.$$

This shows that $\mathcal{I}$-lacunary statistical convergence is totally different from lacunary statistical convergence and statistical convergence.

In this paper we assume $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence of real numbers such that $t_n > \alpha$, $\forall\, n \in \mathbb{N}$ (where α is a positive real number) and $T_n = \sum_{k=1}^{n} t_k$ (where $n \in \mathbb{N}$ and $T_0 = 0$) and X denotes a metric space with metric ρ.

Now we introduce the definition of rough weighted $\mathcal{I}$-lacunary statistical convergence as follows:

Definition 2.2. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence, $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence and $\tilde{\mathbf{r}}$ be a non-negative real number. Then, the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in X is said to be rough weighted $\mathcal{I}$-lacunary statistically convergent to x_* if for every $\varepsilon, \delta > 0$,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, x_*) \geq \tilde{\mathbf{r}} + \varepsilon\}| \geq \delta\} \in \mathcal{I}.$$

We denote $x_n \xrightarrow[\tilde{\mathbf{r}}]{WS_\theta(\mathcal{I})} x_*$. The set $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x = \{x_* \in X : x_n \xrightarrow[\tilde{\mathbf{r}}]{WS_\theta(\mathcal{I})} x_*\}$ is called the rough weighted $\mathcal{I}$-lacunary statistical limit set of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ with degree of roughness $\tilde{\mathbf{r}}$. The sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is said to be $\tilde{\mathbf{r}}$-weighted $\mathcal{I}$-statistically convergent provided that $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x \neq \emptyset$.

Our next aim is to introduce the definition of weighted $\mathcal{I}$-lacunary statistically bounded sequence as follows:

Definition 2.3. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be a lacunary sequence and $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ in X is said to be weighted $\mathcal{I}$-lacunary statistically bounded if there exists an element ζ in X and a positive real number M such that for every $\delta > 0$,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, \zeta) \geq M\}| \geq \delta\} \in \mathcal{I}.$$

From the above Definition 2.3, a weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ of real numbers is said to be self weighted $\mathcal{I}$-lacunary statistically bounded if there exists a positive real number M such that for every $\delta > 0$,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k \geq M\}| \geq \delta\} \in \mathcal{I}.$$

Now we introduce the first theorem of rough weighted $\mathcal{I}$-lacunary statistical limit set as follows:

Theorem 2.4. The set $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ contains at most one element in X if the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ is not self weighted $\mathcal{I}$-lacunary statistically bounded.

Proof. Assume that there are two points $x_* \neq y_*$ such that $x_*, y_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$. Take $2\varepsilon = \rho(x_*, y_*)$.

Case 1: Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ be properly divergent to $+\infty$. Then,

$$\mathbb{N} \setminus \{\text{a finite subset of } \mathbb{N}\} = \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k \geq \frac{2\tilde{\mathbf{r}} + 2\varepsilon}{\rho(x_*, y_*)}\}| \geq 1\}$$

$$\subseteq \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, x_*) \geq \tilde{\mathbf{r}} + \varepsilon\}| \geq \frac{1}{2}\}+$$

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, y_*) \geq \tilde{\mathbf{r}} + \varepsilon\}| \geq \frac{1}{2}\} \in \mathcal{I},$$

which is a contradiction.

Case 2: Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ be unbounded but not properly divergent to $+\infty$. Then there exists two infinite subsets K and L (say) of $\mathbb{N}$ such that $K \cup L = \mathbb{N}$, $K \cap L = \emptyset$ and $\{t_n\}_{n\in K}$ is a unbounded subsequence and $\{t_n\}_{n\in L}$ is a bounded subsequence of $\{t_n\}_{n\in\mathbb{N}}$.

Subcase 2(i): Let $\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in K\}| \geq \delta\} \in \mathcal{I}$. Since $\{t_n\}_{n\in L}$ is a bounded subsequence of $\{t_n\}_{n\in\mathbb{N}}$, so there exists a positive real number G such that $t_n < G \ \forall\ n \in L$. Then,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k \geq G\}| \geq \delta\}$$

$$\subseteq \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in K\}| \geq \delta\} \in \mathcal{I},$$

which contradicts that $\{t_n\}_{n\in\mathbb{N}}$ is not self weighted $\mathcal{I}$-lacunary statistically bounded.

Subcase 2(ii): Let $\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in K\}| \geq \delta\} \notin \mathcal{I}$.

Then

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in K \setminus \{\text{a finite subset of } \mathbb{N}\}\}| \geq \delta\}$$

$$\subseteq \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k \geq \frac{2\tilde{\mathbf{r}} + 2\varepsilon}{\rho(x_*, y_*)}\}| \geq \delta\}$$

$$\subseteq \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, x_*) \geq \tilde{\mathbf{r}} + \varepsilon\}| \geq \frac{\delta}{2}\}\cup$$

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, y_*) \geq \tilde{\mathbf{r}} + \varepsilon\}| \geq \frac{\delta}{2}\} \in \mathcal{I},$$

which is a contradiction. Hence the proof is completed. □

If the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ is self weighted $\mathcal{I}$-lacunary statistically bounded then there exists a positive real number M such that

$$\{r\in\mathbb{N}: \frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]: t_k\geq M\}|\geq\delta\}\in\mathcal{I}$$

$$\Rightarrow \{r\in\mathbb{N}: 1-\frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]: k\in A\}|<\delta\}\in\mathcal{F}(\mathcal{I}),$$

where $A=\{k\in\mathbb{N}: t_k<M\}$. Then, the subsequence $\{t_n\}_{n\in A}$ of the sequence $\{t_n\}_{n\in\mathbb{N}}$ is bounded and so the limit inferior exists. The notation $\liminf_{n\in A} t_n$ denotes the limit inferior of the sequence $\{t_n\}_{n\in A}$ when the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ is self weighted $\mathcal{I}$-lacunary statistically bounded.

Theorem 2.5. For a sequence $x=\{x_n\}_{n\in\mathbb{N}}$, we have

$$0\leq \operatorname{diam}(WS_\theta(\mathcal{I})-LIM^{\tilde{\mathbf{r}}}x)\leq\begin{cases}\frac{2\tilde{\mathbf{r}}}{\liminf_{n\in A} t_n}, & \text{if } \{t_n\}_{n\in\mathbb{N}} \text{ is self weighted } \mathcal{I}\text{-lacunary statistically bounded,}\\ 0, & \text{otherwise.}\end{cases}$$

In general $\operatorname{diam}(WS_\theta(\mathcal{I})-LIM^{\tilde{\mathbf{r}}}x)$ has no smaller bound than $\frac{2\tilde{\mathbf{r}}}{\liminf_{n\in A} t_n}$ if the weighted sequence is self weighted $\mathcal{I}$-lacunary statistically bounded.

Proof. Case 1: Let $\{t_n\}_{n\in\mathbb{N}}$ be self weighted $\mathcal{I}$-lacunary statistically bounded sequence. By contradiction we assume that $\operatorname{diam}(WS_\theta(\mathcal{I})-LIM^{\tilde{\mathbf{r}}}x)>\frac{2\tilde{\mathbf{r}}}{\liminf_{n\in A} t_n}$. Then there exists a positive real number $\lambda\in(0,\liminf_{n\in A} t_n)$ such that $\operatorname{diam}(WS_\theta(\mathcal{I})-LIM^{\tilde{\mathbf{r}}}x)>\frac{2\tilde{\mathbf{r}}}{\lambda}>\frac{2\tilde{\mathbf{r}}}{\liminf_{n\in A} t_n}$.

So there exists $y,z\in WS_\theta(\mathcal{I})-LIM^{\tilde{\mathbf{r}}}x$ such that $\rho(y,z)>\frac{2\tilde{\mathbf{r}}}{\lambda}$. Since $\lambda<\liminf_{n\in A} t_n$ then, there exists a natural number l such that $\lambda<t_n$ for all $n>l$ and $n\in A$.

Let $\varepsilon\in(0,\frac{\lambda\rho(y,z)}{2}-\tilde{\mathbf{r}})$, $0<\delta<1$ and $A_l=\{1,2,3,...,l\}$. Then,

$$B=\{r\in\mathbb{N}: \frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]: t_k\rho(x_k,y)\geq\tilde{\mathbf{r}}+\varepsilon\}|<\frac{\delta}{3}\}\in\mathcal{F}(\mathcal{I}),$$

$$C=\{r\in\mathbb{N}: \frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]: t_k\rho(x_k,z)\geq\tilde{\mathbf{r}}+\varepsilon\}|<\frac{\delta}{3}\}\in\mathcal{F}(\mathcal{I}),$$

$$D=\{r\in\mathbb{N}: \frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]: t_k\geq M\}\cup A_l|<\frac{\delta}{3}\}\in\mathcal{F}(\mathcal{I}).$$

Since $B\cap C\cap D\in\mathcal{F}(\mathcal{I})$ and $\emptyset\notin\mathcal{F}(\mathcal{I})$, we can choose $r\in B\cap C\cap D$, such that

$$\frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]: t_k\rho(x_k,y)\geq\tilde{\mathbf{r}}+\varepsilon\}|<\frac{\delta}{3}<\frac{1}{3},$$

$$\frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]: t_k\rho(x_k,z)\geq\tilde{\mathbf{r}}+\varepsilon\}|<\frac{\delta}{3}<\frac{1}{3}$$

$$\text{and } \frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]: t_k\geq M\}\cup A_l|<\frac{\delta}{3}<\frac{1}{3}.$$

This implies

$$\frac{1}{(T_{k_r}-T_{k_{r-1}})}|\{k\in(T_{k_{r-1}},T_{k_r}]\setminus A_l: t_k\rho(x_k,y)\geq\tilde{\mathbf{r}}+\varepsilon\vee t_k\rho(x_k,z)\geq\tilde{\mathbf{r}}+\varepsilon\vee t_k\geq M\}|<1.$$

Since $(E \cup F \cup G)^c = E^c \cap F^c \cap G^c$ (where $'c'$ stands for the complement) so there exists a $k_0 \in (T_{k_{r-1}}, T_{k_r}] \setminus A_l$ such that $t_{k_0}\rho(x_{k_0}, y) < \tilde{\mathbf{r}} + \varepsilon$, $t_{k_0}\rho(x_{k_0}, z) < \tilde{\mathbf{r}} + \varepsilon$ and $t_{k_0} < M$.

$$\Rightarrow \lambda\rho(y,z) \leq \lambda\rho(x_{k_0}, y) + \lambda\rho(x_{k_0}, z) < t_{k_0}\rho(x_{k_0}, y) + t_{k_0}\rho(x_{k_0}, z) < 2(\tilde{\mathbf{r}} + \varepsilon) < \lambda\rho(y,z),$$

(since $\lambda < t_n$ for all $n \in A \cap \{l+1, l+2, l+3, ...\}$ and $\varepsilon < \frac{\lambda\rho(y,z)}{2} - \tilde{\mathbf{r}}$), which is a contradiction. Hence the proof of case 1 is completed.

Case 2: Let the weighted sequence is not self weighted $\mathcal{I}$-lacunary statistically bounded. Then from the Theorem 2.4, $\text{diam}(WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x) = 0$.

Let us now prove the second part of the theorem. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be any lacunary sequence, $t_n = \frac{1}{2} - \frac{1}{n+2}$ $\forall$ $n \in \mathbb{N}$ and $\mathcal{I} = \mathcal{I}_d$. We define a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ such that

$$x_n = \begin{cases} 2, \text{ if } n \in (T_{k_{r-1}}, T_{k_r}] \text{ and } r = m^2 \ \forall \ m \in \mathbb{N}, \\ 1, \text{ otherwise.} \end{cases}$$

For $0 < \varepsilon, \delta < \frac{1}{4}$ we get

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k|x_k - 1| \geq \varepsilon\}| \geq \delta\} \subseteq \{1^2, 2^2, 3^2, ...\} \in \mathcal{I}_d.$$

This shows that $x_n \xrightarrow[0]{WS_\theta(\mathcal{I}_d)} 1$. Let $\tilde{\mathbf{r}}$ be a positive real number and $\overline{B}_{2\tilde{\mathbf{r}}}(1) = [1 - 2\tilde{\mathbf{r}}, 1 + 2\tilde{\mathbf{r}}]$. For $y \in \overline{B}_{2\tilde{\mathbf{r}}}(1)$,

$$t_k|x_k - y| \leq t_k|x_k - 1| + t_k|1 - y| < \varepsilon + \frac{1}{2}2\tilde{\mathbf{r}} = \tilde{\mathbf{r}} + \varepsilon \ \forall \ k \in \mathbb{N} \setminus \{k \in \mathbb{N} : t_k|x_k - 1| \geq \varepsilon\}.$$

Therefore we get $y \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$. So $\overline{B}_{2\tilde{\mathbf{r}}}(1) \subseteq WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$. From the 1^{st} part of the Theorem 2.5, we get $\text{diam}(WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x) \leq \frac{2\tilde{\mathbf{r}}}{\liminf_{n\in\mathbb{N}} t_n} = 4\tilde{\mathbf{r}}$ (since $\{t_n\}_{n\in\mathbb{N}}$ is self weighted $\mathcal{I}$-lacunary statistically bounded and $\liminf_{n\in\mathbb{N}} t_n = \frac{1}{2}$) and $\text{diam}(\overline{B}_{2\tilde{\mathbf{r}}}(1)) = 4\tilde{\mathbf{r}}$. This shows that $\text{diam}(WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x) = 4\tilde{\mathbf{r}} = \frac{2\tilde{\mathbf{r}}}{\liminf_{n\in\mathbb{N}} t_n}$. So in general, the upper bound $\frac{2\tilde{\mathbf{r}}}{\liminf_{n\in\mathbb{N}} t_n}$ of the diameter of the set $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ can't decrease anymore. □

Remark 2.6. Aytar [1, Theorem 2.2], Pal et al. [26, Theorem 3.1], Dündar et al. [13, Theorem 2.3] and Dündar [12, Theorem 2.3] had shown that the diameter of a rough statistical limit set (or diameter of rough $\mathcal{I}$-limit set) is $\leq 2\tilde{\mathbf{r}}$. From the second part of above theorem we get, for the case of rough weighted $\mathcal{I}$-lacunary statistical convergence the diameter of rough weighted $\mathcal{I}$-lacunary statistical limit set may be strictly greater than $2\tilde{\mathbf{r}}$.

Theorem 2.7. The set $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is closed.

Proof. Case 1: Let $\{t_n\}_{n\in\mathbb{N}}$ be self weighted $\mathcal{I}$-lacunary statistically bounded and $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x \neq \emptyset$. Then there exists a sequence $p = \{p_n\}_{n\in\mathbb{N}}$ in $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ such that $p_n \to p_*$ as $n \to \infty$. We have to show that $p_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$.

Since $p_n \to p_*$ as $n \to \infty$, then for any $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $\rho(p_k, p_*) < \frac{\varepsilon}{2M}$ $\forall$ $k \geq k_0$.

Then, from the triangle inequality we get

$$\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, p_{k_0}) < r + \frac{\varepsilon}{2}\} \cap \{k \in (T_{k_{r-1}}, T_{k_r}] : t_k < M\} \subseteq \{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, p_*) < r + \varepsilon\}.$$

So, $p_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$. Hence, the set $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ is closed.

Case 2: Let $\{t_n\}_{n\in\mathbb{N}}$ be not self weighted $\mathcal{I}$-lacunary statistically bounded. Then by Theorem 2.4, the set $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ is closed. □

Remark 2.8. The set $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ is closed, bounded but not compact.

To prove this important fact of non-compactness, we consider the sequence $x = \{x_n\}_{n\in\mathbb{N}} = \{e_n\}_{n\in\mathbb{N}}$ (in l^2 space) where $e_n = \{\delta_n^k\}_{n\in\mathbb{N}}$ has the n^{th} term 1 and other terms are 0 and the weighted sequence is $t_n = \gamma + \frac{1}{n}$, for all $n \in \mathbb{N}$ and $\gamma > 0$. Let $\tilde{\mathbf{r}} = \gamma\sqrt{2}$ and $A = \{e_1, e_2, ...\}$. Then, $A \subseteq WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$. Since A is not compact, so the set $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ is not compact.

Theorem 2.9. A sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is weighted $\mathcal{I}$-lacunary statistically bounded iff there exists a non-negative real number r such that $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x \neq \emptyset$.

Proof. Proof of this theorem is similar to the proofs of Theorem 2.4 [1] and Theorem 3.2 [26]. So omitted. □

Next we proceed to introduce the definition of weighted $\mathcal{I}$-lacunary statistical cluster point of a sequence in a metric space as follows:

Definition 2.10. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be any lacunary sequence and $\{t_n\}_{n\in\mathbb{N}}$ be a weighted sequence. A point $c \in X$ is called a weighted $\mathcal{I}$-lacunary statistical cluster point of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ if for every $\varepsilon, \delta > 0$,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, c) < \varepsilon\}| \geq \delta\} \notin \mathcal{I}.$$

We denote the set of all weighted $\mathcal{I}$-lacunary statistical cluster points of the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ by $W_\theta\Gamma_x(\mathcal{I})$.

Remark 2.11. In [1, Lemma 2.9] Aytar had shown that for an arbitrary $c \in \Gamma_x$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ the distance between x_* and c is $\leq \tilde{\mathbf{r}}\ \forall\ x_* \in st - LIM^{\tilde{\mathbf{r}}}x$ (here $\mathcal{I} = \mathcal{I}_d$ so $\mathcal{I} - LIM^{\tilde{\mathbf{r}}}x = st - LIM^{\tilde{\mathbf{r}}}x$).

Similarly, Pal et al. [26, Proposition 3.1] and Dündar et al. [13, Lemma 2.9] had shown that for an arbitrary $c \in \Gamma_x(\mathcal{I})$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$ the distance between x_* and c is $\leq \tilde{\mathbf{r}}\ \forall\ x_* \in \mathcal{I} - LIM^{\tilde{\mathbf{r}}}x$.

For the case of weighted $\mathcal{I}$-lacunary statistical cluster points and rough weighted $\mathcal{I}$-lacunary statistical limit set, the distance between x_* and c may be strictly greater than $\tilde{\mathbf{r}}$ where $c \in W_\theta\Gamma_x(\mathcal{I})$ and $x_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$. Consider the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{t_n\}_{n\in\mathbb{N}}$ defined in Theorem 2.5 (second part). Then we get $1 \in W_\theta\Gamma_x(\mathcal{I})$ and $1 + 2\tilde{\mathbf{r}} \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$. Choose $x_* = 1 + 2\tilde{\mathbf{r}}$ and $c = 1$. It follows that the distance between x_* and c is $2\tilde{\mathbf{r}} > \tilde{\mathbf{r}}$. So in this case the result of Theorem 1.8 may not hold.

Now we give some important relations between the sets $W_\theta\Gamma_x(\mathcal{I})$ and $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$.

Theorem 2.12. For an arbitrary $c \in W_\theta\Gamma_x(\mathcal{I})$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$, we have

$$\rho(x_*, c) \leq \begin{cases} \frac{\tilde{\mathbf{r}}}{\liminf\limits_{n\in A} t_n}, \text{ if } \{t_n\}_{n\in\mathbb{N}} \text{ is self weighted } \mathcal{I}\text{-lacunary statistically bounded,} & \frac{\tilde{\mathbf{r}}}{\inf\limits_{n\in\mathbb{N}} t_n}, \text{ otherwise,} \end{cases}$$

for all $x_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$.

Proof. <u>Case 1:</u> Let $\{t_n\}_{n\in\mathbb{N}}$ be self weighted $\mathcal{I}$-lacunary statistically bounded sequence.

By contradiction we assume that there exist a point $c \in W_\theta\Gamma_x(\mathcal{I})$ and $x_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ such that $\rho(x_*, c) > \frac{\tilde{\mathbf{r}}}{\liminf\limits_{n\in A} t_n} > 0$.

This implies $\frac{(\liminf\limits_{n\in A} t_n)\rho(x_*, c) - \tilde{\mathbf{r}}}{3} > 0$. Then, there exists a positive real number $\lambda \in (0, \liminf\limits_{n\in A} t_n)$ such that

$$\frac{(\liminf\limits_{n\in A} t_n)\rho(x_*, c) - \tilde{\mathbf{r}}}{3} > \frac{\lambda\rho(x_*, c) - \tilde{\mathbf{r}}}{3} > 0.$$

Define $\varepsilon = \frac{\lambda\rho(x_*, c) - \tilde{\mathbf{r}}}{3} > 0$. Since $\lambda < \liminf\limits_{n\in A} t_n$ so there exists $k_0 \in \mathbb{N}$ such that $t_n > \lambda\ \forall\ n \geq k_0$ and $n \in A$ where $A = \{k \in \mathbb{N} : t_k < M\}$.

Let $A_0 = A \setminus \{1, 2, ..., k_0 - 1\}$ and $B = \{k \in \mathbb{N} : t_k\rho(x_k, c) < \varepsilon\}$. Then four subcases may arise.

Subcase 1(i): If $A_0 \cap B = \emptyset$ then $B \subseteq \mathbb{N} \setminus A_0$, which is a contradiction since $c \in W_\theta\Gamma_x(\mathcal{I})$. So this case can never happen.

Subcase 1(ii): If $B \subseteq A_0$, then $A_0 \cap B = B$.

Subcase 1(iii): If $A_0 \subseteq B$, then $A_0 \cap B = A_0$.

Subcase 1(iv): If $A_0 \cap B \neq \emptyset$, $A_0 \setminus B \neq \emptyset$ and $B \setminus A_0 \neq \emptyset$ then, $B \setminus (A_0 \cap B) \subseteq \mathbb{N} \setminus A_0$.

For any $\delta > 0$, the set

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in B \setminus (A_0 \cap B)\}| \geq \frac{\delta}{2}\} \subseteq$$

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in \mathbb{N} \setminus A_0\}| \geq \frac{\delta}{2}\} \in \mathcal{I},$$

$$\Rightarrow \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in B \setminus (A_0 \cap B)\}| \geq \frac{\delta}{2}\} \in \mathcal{I}. \ldots\ldots\ldots\ldots (I)$$

Since $c \in W_\theta\Gamma_x(\mathcal{I})$, then we get

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in B\}| \geq \delta\} \notin \mathcal{I}. \ldots\ldots\ldots\ldots\ldots\ldots (II)$$

Again $B = [B \setminus (A_0 \cap B)] \cup (A_0 \cap B)$. Then, we get

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in B\}| \geq \delta\}$$

$$\subseteq \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in B \setminus (A_0 \cap B)\}| \geq \frac{\delta}{2}\} \cup$$

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in A_0 \cap B\}| \geq \frac{\delta}{2}\}.$$

Since we know that if $K, L, M \subseteq \mathbb{N}$, $K \notin \mathcal{I}$, $L \in \mathcal{I}$ and $K \subseteq L \cup M$ then $M \notin \mathcal{I}$, this implies

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in A_0 \cap B\}| \geq \frac{\delta}{2}\} \notin \mathcal{I} \text{ (by equations } (I) \& (II)).$$

This shows that for all existing cases (i.e., 1(*ii*), 1(*iii*) and 1(*iv*)) we get

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : k \in A_0 \cap B\}| \geq \frac{\delta}{2}\} \notin \mathcal{I}.$$

So there exists a natural number $k \in A_0 \cap B$ such that

$$t_k\rho(x_k, x_*) \geq t_k\rho(x_*, c) - t_k\rho(x_k, c) > 3\varepsilon + \tilde{\mathbf{r}} - \varepsilon = \tilde{\mathbf{r}} + 2\varepsilon > \tilde{\mathbf{r}} + \varepsilon$$

(since $t_k > \lambda$ and $3\varepsilon = \lambda\rho(x_*, c) - \tilde{\mathbf{r}}$),

$$\Rightarrow A_0 \cap B \subseteq \{k \in \mathbb{N} : t_k\rho(x_k, x_*) \geq \tilde{\mathbf{r}} + \varepsilon\}.$$

Then,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, x_*) \geq \tilde{\mathbf{r}} + \varepsilon\}| \geq \frac{\delta}{2}\} \notin \mathcal{I}.$$

This contradicts the fact that $x_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$.

Case 2: Let the weighted sequence $\{t_n\}_{n\in\mathbb{N}}$ be not self weighted $\mathcal{I}$-lacunary statistically bounded.

By contradiction we assume that there exist a point $c \in W_\theta\Gamma_x(\mathcal{I})$ and $x_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ such that $\rho(x_*, c) > \frac{\tilde{\mathbf{r}}}{\inf\limits_{n\in\mathbb{N}} t_n}$. Setting $\varepsilon = \frac{\zeta\rho(x_*,c)-\tilde{\mathbf{r}}}{2}$, where $\zeta = \inf\limits_{n\in\mathbb{N}} t_n$. We know that

$$t_k\rho(x_*, x_k) \geq t_k\rho(x_*, c) - t_k\rho(x_k, c) \geq \zeta\rho(x_*, c) - t_k\rho(x_k, c) = \tilde{\mathbf{r}} + 2\varepsilon - t_k\rho(x_k, c)\ \forall\ k \in \mathbb{N},$$

$$\Rightarrow\ \{k \in \mathbb{N} : t_k\rho(x_*, c) < \varepsilon\} \subset \{k \in \mathbb{N} : t_k\rho(x_k, x_*) \geq \tilde{\mathbf{r}} + \varepsilon\}.$$

Then,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, c) < \varepsilon\}| \geq \delta\}$$

$$\subseteq \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})}|\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k\rho(x_k, x_*) \geq \tilde{\mathbf{r}} + \varepsilon\}| \geq \delta\}.$$

$x_* \in WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ contradicts the fact that $c \in W_\theta\Gamma_x(\mathcal{I})$. □

Note 2.13. If both the sets $W_\theta\Gamma_x(\mathcal{I})$ and $WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x$ are non-empty then from the Theorem 2.5 and Theorem 2.12 we get the set $W_\theta\Gamma_x(\mathcal{I})$ is bounded.

Theorem 2.14. **(a)** For an arbitrary $c \in W_\theta\Gamma_x(\mathcal{I})$ of a sequence $x = \{x_n\}_{n\in\mathbb{N}}$, we have

$$WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x \subseteq \begin{cases} \overline{B}_{\frac{r}{p}}(c), & \text{if } \{t_n\}_{n\in\mathbb{N}} \text{ is self weighted } \mathcal{I}\text{-lacunary statistically bounded}, \\ \overline{B}_{\frac{r}{q}}(c), & \text{otherwise} \end{cases}$$

where $p = \liminf\limits_{n\in A} t_n$, $q = \inf\limits_{n\in\mathbb{N}} t_n$ and $\overline{B}_\varepsilon(c) = \{y \in X : \rho(y, c) \leq \varepsilon\}$.
(b)

$$WS_\theta(\mathcal{I}) - LIM^{\tilde{\mathbf{r}}}x \subseteq \begin{cases} \bigcap\limits_{c\in W_\theta\Gamma_x(\mathcal{I})} \overline{B}_{\frac{r}{p}}(c) \subseteq \{x_* \in \mathbb{R} : W_\theta\Gamma_x(\mathcal{I}) \subseteq \overline{B}_{\frac{r}{p}}(x_*)\}, \\ \quad \text{if } \{t_n\}_{n\in\mathbb{N}} \text{ is self weighted } \mathcal{I}\text{-lacunary statistically bounded}, \\ \bigcap\limits_{c\in W_\theta\Gamma_x(\mathcal{I})} \overline{B}_{\frac{r}{q}}(c) \subseteq \{x_* \in \mathbb{R} : W_\theta\Gamma_x(\mathcal{I}) \subseteq \overline{B}_{\frac{r}{q}}(x_*)\}, \quad \text{otherwise.} \end{cases}$$

Proof. The results are obvious so omitted. □

Following Theorem 1.10, Pehlivan et al. [27, Corollary 1], had shown that if a sequence in a finite dimensional normed linear space is statistically bounded then the statistical cluster points set Γ_x is non-empty. For the case of weighted $\mathcal{I}$-lacunary statistical convergence, the weighted $\mathcal{I}$-lacunary statistical cluster points set $W_\theta\Gamma_x(\mathcal{I})$ may be empty even if the space is finite dimensional and the sequence is statistically bounded (or $\mathcal{I}$-lacunary statistically bounded).

To prove this important fact, we consider the sequence of real numbers $x_n = \frac{1}{n}$ and $t_n = n^2\ \forall\ n \in \mathbb{N}$ and $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be any lacunary sequence. Then the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is statistically bounded (or $\mathcal{I}$-lacunary statistically bounded) but $W_\theta\Gamma_x(\mathcal{I}) = \varnothing$.

In a finite dimensional normed linear space when we discuss the weighted $\mathcal{I}$-lacunary statistical convergence, the set $W_\theta\Gamma_x(\mathcal{I})$ may not be closed and nor even bounded however the sequence $x = \{x_n\}_{n\in\mathbb{N}}$ is bounded (or, statistically bounded) which differ the results in Theorems 1.9 & 1.11.

Example 2.15. Let $\theta = \{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ be any lacunary sequence, $\mathbb{N} = \bigcup\limits_{j=1}^{\infty}\Delta_j$ be a decomposition of $\mathbb{N}$ (i.e., $\Delta_m \cap \Delta_n = \phi$ for $m \neq n$). Assume that $\Delta_j = \{2^{j-1}(2s-1) : s \in \mathbb{N}\}$, for all $j \in \mathbb{N}$.
Setting

$$t_k = k\ \forall\ k \in \mathbb{N} \text{ and } x_k = \frac{1}{j} + \frac{1}{k^2}\ \forall\ k \in \Delta_j \text{ and } j = 1, 2, 3, \ldots.$$

Then, for fixed j, choose $\varepsilon > 0$ and $0 < \delta < \frac{1}{2^j}$. So we get,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})} |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k |x_k - \frac{1}{j}| < \varepsilon\}| \geq \delta\}$$

$$\supseteq \{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})} . \frac{(T_{k_r} - T_{k_{r-1}})}{2^j} \geq \delta\} = \mathbb{N} \setminus \{\text{a finite sub set of } \mathbb{N}\} \notin \mathcal{I}.$$

This shows that $\frac{1}{j} \in W_\theta \Gamma_x(\mathcal{I})$ for all $j \in \mathbb{N}$ and $\mathcal{I}$ be any ideal.

Next we assume $k \in \mathbb{N}$, then there exists an integer $j \in \mathbb{N}$ such that $k \in \Delta_j$ for some $j \in \mathbb{N}$. This implies k is of the form $k = 2^{j-1}(2s-1)$ where some $s \in \mathbb{N}$. Now for each $k \in \mathbb{N}$,

$$t_k |x_k| = 2^{j-1}(2s-1)\{\frac{1}{j} + \frac{1}{(2^{j-1}(2s-1))^2}\} = \frac{1}{2} . \frac{2^j}{j}(2s-1) + \frac{1}{2^{j-1}(2s-1)} > \frac{1}{2}.$$

Then,

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})} |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k |x_k - 0| < \varepsilon\}| \geq \delta\} \in \mathcal{I}.$$

This implies $0 \notin W_\theta \Gamma_x(\mathcal{I})$. So $W_\theta \Gamma_x(\mathcal{I})$ is not a closed set.

Next, we consider a sequence $y_k = j + \frac{1}{k^2}$ for all $k \in \Delta_j$ and $j = 1, 2, 3, ...$

Then, for each $j \in \mathbb{N}$ we get

$$\{r \in \mathbb{N} : \frac{1}{(T_{k_r} - T_{k_{r-1}})} |\{k \in (T_{k_{r-1}}, T_{k_r}] : t_k |y_k - j| < \varepsilon\}| \geq \delta\}$$

$$= \mathbb{N} \setminus \{\text{a finite sub set of } \mathbb{N}\} \notin \mathcal{I}.$$

This shows that $j \in W_\theta \Gamma_y(\mathcal{I})$ for all $j \in \mathbb{N}$. So $W_\theta \Gamma_y(\mathcal{I})$ is not a bounded set. This implies the set $W_\theta \Gamma_y(\mathcal{I})$ is not compact in $\mathbb{R}$.

Remark 2.16. The set $W_\theta \Gamma_x(\mathcal{I})$ is closed in X if the weighted sequence $\{t_n\}_{n \in \mathbb{N}}$ is self weighted $\mathcal{I}$-lacunary statistically bounded.

Proof. Proof of this remark is similar to the proof of Theorem 2.12. So we omit. □

In [9, 10, 20, 33] has been introduce the sequence $x = \{x_k\}_{k \in \mathbb{N}}$ in a normed linear space by

$$x_k = \begin{cases} ku, & \text{for } \; n - [\sqrt{\lambda_n}] + 1 \leq k \leq n, \; n \notin A, \ldots\ldots\ldots\ldots (I) \\ ku, & \text{for } \; n - \lambda_n + 1 \leq k \leq n, \; n \in A, \ldots\ldots\ldots\ldots\ldots (II) \\ \mathbf{0}, & \text{otherwise}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (III) \end{cases}$$

where $\lambda_n = 1$ for $n = 1$ to 10 and $\lambda_n = n - 10$ for all $n \geq 10$, $A = \{1^2, 2^2, 3^2, 4^2, ...\}$, u is a fixed element in a normed linear space with $||u|| = 1$, $\mathbf{0}$ is the null element of the normed linear space and $\mathcal{I} = \mathcal{I}_d$. Authors of these papers asserted that is $\mathcal{I}$-statistically convergent, but not statistically convergent. However, it is not so. Indeed,

$$x_k = \begin{cases} ku, & \text{for } \; 11 \leq k \leq 16 = 4^2 (\text{by } (II)), \\ ku, & \text{for } \; 11 \leq k \leq 25 = 5^2 (\text{by } (II)), \\ ku, & \text{for } \; 11 \leq k \leq 36 = 6^2 (\text{by } (II)), \\ ku, & \text{for } \; 11 \leq k \leq 49 = 7^2 (\text{by } (II)), \\ ku, & \text{for } \; 11 \leq k \leq 64 = 8^2 (\text{by } (II)), \\ \ldots\ldots . \end{cases}$$

Then $x_k = ku$ for all $k \geq 11$. This implies the $x = \{x_k\}_{k \in \mathbb{N}}$ is properly divergent sequence. So it is neither $\mathcal{I}$-statistically convergent nor statistically convergent.

3. Open Problem

It is not clear that $\mathcal{I}$-statistically convergent and statistically convergent are different or not. So it seems natural to ask is there exists a sequence which is $\mathcal{I}$-statistically convergent but is not statistically convergent?

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