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Well-Posedness and Exponential Stability for Coupled Lamé System with a Viscoelastic Damping

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Abstract. In this paper, we consider a coupled Lamé system with a viscoelastic damping in the first equation. We prove well-posedness by using Faedo-Galerkin method and establish an exponential decay result by introducing a suitable Lyaponov functional.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let us consider the following a coupled Lamé system :

$$\begin{cases} u_{tt}(x,t) + \alpha v - \Delta_e u(x,t) + \displaystyle\int_0^t g(s)\Delta u(t-s)ds - \mu_1 \Delta u_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ v_{tt}(x,t) + \alpha u - \Delta_e v(x,t) - \mu_2 \Delta v_t(x,t) = 0, & \text{in } \Omega \times (0,+\infty), \\ u(x,t) = v(x,t) = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ (u(x,0), v(x,0)) = (u_0(x), v_0(x)) & \text{in } \Omega, \\ (u_t(x,0), v_t(x,0)) = (u_1(x), v_1(x)) & \text{in } \Omega. \end{cases} \tag{1}$$

Where μ_1, μ_2 are positive constants and (u_0, u_1, v_0, v_1) are given history and initial data . Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu \Delta u + (\lambda + \mu)\nabla(\text{div u}), \quad u = (u_1, u_2, u_3)^T$$

and μ and λ are the Lamé constants which satisfy the conditions

$$\mu > 0, \quad \lambda + \mu \geq 0. \tag{2}$$

The problem of stabilization of coupled systems has also been studied by several authors see [1, 3, 6, 11, 17, 18]and the references therein.Under certain conditions imposed on the subset where the damping term is effective, Komornik [11] proves uniform stabilization of the solutions of a pair of hyperbolic systems

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coupled in velocities. Alabau and al.[1] studied the indirect internal stabilization of weakly coupled systems where the damping is effective in the whole domain. They prove that the behavior of the first equation is sufficient to stabilize the total system and to have polynomial decay for sufficiently smooth solutions.
For coupled systems in thermoelasticity, R.Racke [18] considered the following system:

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t-\tau) + b\theta_x(x,t) = 0, & \text{in } (0,L)\times(0,\infty), \\ \theta_t(x,t) - d\theta_{xx}(x,t) + bu_{tx}(x,t) = 0, & \text{in } (0,L)\times(0,\infty), \end{cases}$$

He proved that the internal time delay leads to ill-posedness of the system. However, the system without delay is exponentially stable.

In [14] M.I.Mustafa considered the following system:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + f_1(u,v) = 0, & \text{in } \Omega\times(0,+\infty), \\ v_{tt}(x,t) - \Delta v(x,t) + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + f_2(u,v) = 0, & \text{in } \Omega\times(0,+\infty), \\ u = v = 0 & \text{on } \partial\Omega\times(0,+\infty), \\ (u(.,0) = u_0, u_t(.,0) = u_1, v(.0) = v_0, v_t(.,0) = v_1 & \text{in } \Omega. \end{cases} \tag{3}$$

The author proved the well-posedness and, for a wider class of relaxation functions, establish a generalized stability result for this system.

Recently, Beniani and al. [3]considered the following Lamé system with time varying delay term:

$$\begin{cases} u''(x,t) - \Delta_e u(x,t) + \mu_1 g_1(u'(x,t)) + \mu_2 g_2(u'(x,t-\tau(t))) = 0 & \text{in } \Omega\times\mathbb{R}^+ \\ u(x,t) = 0 & \text{on } \partial\Omega\times\mathbb{R}^+ \end{cases} \tag{4}$$

and under suitable conditions, they proved general decay of energy.

The paper is organized as follows. The well-posedness of the problem is analyzed in Section 3 using the Faedo-Galerkin method. In Section 4, we prove the exponential decay of the energy when time goes to infinity.

2. Preliminaries and statement of main results

In this section, we present some materials that shall be used for proving our main results. For the relaxation function g, we have the folloing assumptions:

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$g \in L^1(0,\infty)\, g(0) > 0, \quad 0 < \beta(t) := \mu - \int_0^t g(s)\mathrm{d}s \quad \textit{and} \quad 0 < \beta_0 := \mu - \int_0^\infty g(s)\mathrm{d}s.$$

(A2) There exist a non-increasing differentiable function $\xi(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \le -\xi(t)g(t), \quad \forall t \ge 0 \quad \text{and} \quad \int_0^\infty \xi(t)\mathrm{d}t = +\infty.$$

These hypotheses imply that

$$\beta_0 \le \beta(t) \le \mu. \tag{5}$$

Let us introduce the following notations:

$$(g * h)(t) := \int_0^t g(t-s)h(s)ds,$$

$$(g \circ h)(t) := \int_0^t g(t-s)|h(t) - h(s)|^2 ds.$$

Lemma 2.1 ([10]). *For any $g, h \in C^1(\mathbb{R})$, the following equation holds*

$$2[g * h]h' = g' \circ h - g(t)|h|^2 - \frac{d}{dt}\Big\{g \circ h - \Big(\int_0^t g(s)ds\Big)|h|^2\Big\}.$$

The existence and uniqueness result is stated as follows:

Theorem 2.2. *Assume that $(A1)$ and $(A2)$ hold. Then given $(u_0, v_0) \in H^2(\Omega) \cap H_0^1(\Omega)$, $(u_1, v_1) \in L^2(\Omega)$, there exists a unique weak solution u, v of problem* (1) *such that*

$$(u, v) \in C([0, +\infty[, H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0, +\infty[, L^2(\Omega)).$$

For any regular solution of (1), we define the energy as

$$\begin{aligned} E(t) = &\frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{\beta(t)}{2}\int_\Omega |\nabla u|^2(x,t)dx + \frac{1}{2}\int_\Omega (g \circ \nabla u)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\text{div } u|^2 dx \\ &+ \frac{1}{2}\int_\Omega v_t^2(x,t)dx + \frac{\mu}{2}\int_\Omega |\nabla v|^2(x,t)dx + \frac{(\mu+\lambda)}{2}\int_\Omega |\text{div } v|^2 dx + 2\alpha\int_\Omega u(x,t)v(x,t)dx. \end{aligned} \tag{6}$$

Our decay result reads as follows:

Theorem 2.3. *Let (u, v) be the solution of* (1). *Assume that $(A1)$ and $(A2)$ hold. Then there exist two positive constants C and d, such that*

$$E(t) \le Ce^{-d\int_0^t \xi(s)ds}, \quad \forall t \ge 0. \tag{7}$$

3. Well-posedness of the problem

In this section, we will prove the existence and uniqueness of problem (1) by using Faedo-Galerkin method.

Proof. We divide the proof of Theorem2.2into two steps:the Faedo-Galerkin approximation and the energy estimates.

Step 1 :Faedo-Galerkin approximation.

We construct approximations of the solution (u, v) by the Faedo-Galerkin method as follows. For $n \ge 1$, let $W_n = \text{span}\{w_1, ..., w_i\}$ be a Hilbertian basis of the space H_0^1 and the projection of the initial data on the finite dimensional subspace W_n is given by

$$u_0^n = \sum_{i=1}^n a_i w_i, \quad v_0^n = \sum_{i=1}^n b_i w_i, \quad u_1^n = \sum_{i=1}^n c_i w_i, \quad v_1^n = \sum_{i=1}^n d_i w_i$$

where,$(u_0^n, v_0^n, u_1^n, v_1^n) \to (u_0, v_0, u_1, v_1)$ strongly in $H^2(\Omega) \cap H_0^1(\Omega)$ as $n \to \infty$. We search the approximate solutions

$$u^n(x,t) = \sum_{i=1}^n f_i^n(t) w_i(x), \quad v^n(x,t) = \sum_{i=1}^n h_i^n(t) w_i(x)$$

to the finite dimensional Cauchy problem:

$$\begin{cases} \int_\Omega u_{tt}^n w_i dx + \alpha \int_\Omega v^n w_i dx + \mu \int_\Omega \nabla u^n \nabla w_i dx + (\lambda+\mu) \int_\Omega \text{div } u^n.\text{div } w_i dx \\ - \int_\Omega (g(s) * \nabla u^n) \nabla w_i dx + \mu_1 \int_\Omega \nabla u_t^n \nabla w_i dx = 0, \\ \int_\Omega v_{tt}^n w_i dx + \alpha \int_\Omega u^n w_i dx + \mu \int_\Omega \nabla v^n \nabla w_i dx + (\lambda+\mu) \int_\Omega \text{div } v^n.\text{div } w_i dx + \mu_2 \int_\Omega \nabla v_t^n \nabla w_i dx = 0, \\ (u^n(0), v^n(0)) = (u_0^n, v_0^n) \quad (u_t^n(0), v_t^n(0)) = (u_1^n, v_1^n). \end{cases} \tag{8}$$

According to the standard theory of ordinary differential equations, the finite dimensional problem (8) has solution $f_i^n(t), h_i^n(t)$ defined on $[0,t)$.The a priori estimates that follow imply that in fact $t_n = T$.

Step 2: Energy estimates. Multiplying the first and the second equation of (8) by $\left(f_i^n(t)\right)'$ and $\left(h_i^n(t)\right)'$ respectively, we obtain:

$$\begin{aligned} &\int_\Omega u_{tt}^n u_t^n dx + \alpha \int_\Omega v^n u_t^n dx + \mu \int_\Omega \nabla u^n \nabla u_t^n dx + (\lambda+\mu) \int_\Omega \text{div } u^n.\text{div } u_t^n dx \\ &- \int_\Omega (g(s) * \nabla u^n) \nabla u_t^n dx + \mu_1 \int_\Omega |\nabla u_t^n|^2 dx = 0. \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\int_\Omega v_{tt}^n v_t^n dx + \alpha \int_\Omega u^n v_t^n dx + \mu \int_\Omega \nabla v^n \nabla v_t^n dx + (\lambda+\mu) \int_\Omega \text{div } v^n.\text{div } v_t^n dx \\ &+\mu_2 \int_\Omega |\nabla v_t^n|^2 dx = 0. \end{aligned} \tag{10}$$

Integrating (9) and (10) over $(0,t)$,and using Lemma (2.1), we obtain

$$\begin{aligned} &\mathcal{E}_n(t) + \mu_1 \int_0^t \int_\Omega |\nabla u_t^n|^2 dxds - \frac{1}{2} \int_\Omega (g' \circ \nabla u^n) dx + \frac{1}{2} \int_0^t \int_\Omega g(t) |\nabla u^n|^2 dxds + \mu_2 \int_0^t \int_\Omega |\nabla v_t^n|^2 dxds \\ &= \mathcal{E}_n(0) \end{aligned} \tag{11}$$

where

$$\begin{aligned} \mathcal{E}_n(t) = &\frac{1}{2} \int_\Omega (u_t^n)^2(x,t) dx + \frac{\beta(t)}{2} \int_\Omega |\nabla u^n|^2(x,t) dx + \frac{1}{2} \int_\Omega (g \circ \nabla u^n) dx + \frac{(\mu+\lambda)}{2} \int_\Omega |\text{div } u^n|^2 dx \\ &+ \frac{1}{2} \int_\Omega (v_t^n)^2(x,t) dx + \frac{\mu}{2} \int_\Omega |\nabla v^n|^2(x,t) dx + \frac{(\mu+\lambda)}{2} \int_\Omega |\text{div } v^n|^2 dx + 2\alpha \int_\Omega u^n(x,t) v^n(x,t) dx. \end{aligned} \tag{12}$$

Consequently, from11, we have the following estimate:

$$\mathcal{E}_n(t) - \frac{1}{2} \int_\Omega (g' \circ \nabla u^n) dx + \frac{1}{2} \int_0^t \int_\Omega g(t) |\nabla u^n|^2 dxds \le \mathcal{E}_n(0). \tag{13}$$

Now, since the sequences $\left(u_0^n\right)_{n\in\mathbb{N}'}$, $\left(u_1^n\right)_{n\in\mathbb{N}'}$, $\left(v_0^n\right)_{n\in\mathbb{N}'}$, $\left(v_1^n\right)_{n\in\mathbb{N}}$ converge and using (A2), in the both cases we can find a positive constant c independent of n such that

$$\mathcal{E}_n(t) \le c. \tag{14}$$

Therefore, using the fact that $\beta(t) \geq \beta(0)$, the estimate 14 together with 13 give us, for all $n \in \mathbb{N}, t_n = T$, we deduce

$$\begin{array}{lll} \left(u^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(v^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(u_t^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(\Omega)), \\ \left(v_t^n\right)_{n\in\mathbb{N}} & \text{is bounded in} & L^\infty(0,T;H_0^1(\Omega)). \end{array} \tag{15}$$

Consequently, we conclude that

$$\begin{array}{lll} u^n \rightharpoonup u & \text{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ v^n \rightharpoonup v & \text{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ u_t^n \rightharpoonup u_t & \text{weakly star in} & L^\infty(0,T;H_0^1(\Omega)), \\ v_t^n \rightharpoonup v_t & \text{weakly star in} & L^\infty(0,T;H_0^1(\Omega)). \end{array} \tag{16}$$

From 15, we have $\left(u^n\right)_{n\in\mathbb{N}}$ and $\left(v^n\right)_{n\in\mathbb{N}}$ are bounded in $L^\infty(0,T;H_0^1(\Omega))$.Then $\left(u^n\right)_{n\in\mathbb{N}}$ and $\left(v^n\right)_{n\in\mathbb{N}}$ are bounded in $L^2(0,T;H_0^1(\Omega))$. Consequently, $\left(u^n\right)_{n\in\mathbb{N}}$ and $\left(v^n\right)_{n\in\mathbb{N}}$ are bounded in $H^1(0,T;H^1(\Omega))$. Since the embedding

$$H^1(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;L^2(\Omega))$$

is compact,using AubinLion's theorem [12] ,we can extract subsequences $\left(u^k\right)_{k\in\mathbb{N}}$ of $\left(u^n\right)_{n\in\mathbb{N}}$ and $\left(v^k\right)_{k\in\mathbb{N}}$ of $\left(v^n\right)_{n\in\mathbb{N}}$ such that

$$u^k \to u \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega))$$

and

$$v^k \to v \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega))$$

Therefore,

$$u^k \to u \quad \text{strongly and a.e} \quad (0,T)\times(\Omega)$$

and

$$v^k \to v \quad \text{strongly and a.e} \quad (0,T)\times(\Omega)$$

The proof now can be completed arguing as in Theorem 3.1 of [12]

□

4. Exponential stability

In this section we study the asymptotic behavior of the system (1). For the proof of Theorem 2.3 we use the following lemmas.

Lemma 4.1. *Let* (u, v) *be the solution of* (1)*, Then we have the inequality*

$$\frac{dE(t)}{dt} \leq -\mu_1 \int_\Omega |\nabla u_t(x,t)|^2\,dx - \mu_2 \int_\Omega |\nabla v_t(x,t)|^2\,dx - \frac{1}{2}g(t)\int_\Omega |\nabla u(x,t)|^2\,dx + \frac{1}{2}\int_\Omega (g' \circ \nabla u)dx \tag{17}$$

Proof. From (6) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\Big(u_t^2+(\lambda+\mu)|\text{div }u|^2+v_t^2+\mu|\nabla v|^2+(\lambda+\mu)|\text{div }v|^2+2\alpha vu\Big)dx\\&=-\mu\int_\Omega\nabla u\nabla u_t dx-\mu_1\int_\Omega|\nabla u_t|^2dx-\mu_2\int_\Omega|\nabla v_t|^2dx+\int_\Omega\int_0^t g(s)\nabla u(s)\nabla u_t(t)dsdx\end{aligned}\tag{18}$$

From Lemma 2.1, the last term in the right-hand side of 18 can be rewritten as

$$\begin{aligned}&\int_0^t g(s)\int_\Omega\nabla u(s)\nabla u_t(t)dsdx+\frac{1}{2}g(t)\int_\Omega|\nabla u|^2(x,t)dx\\&=\frac{1}{2}\frac{d}{dt}\Big\{\int_0^t g(s)\int_\Omega|\nabla u|^2(x,t)dxds-\int_\Omega(g\circ\nabla u)(t)dx\Big\}+\frac{1}{2}\int_\Omega(g'\circ\nabla u)(t)dx\end{aligned}\tag{19}$$

So $\frac{dE}{dt}$ becomes:

$$\begin{aligned}\frac{dE}{dt}&=-\mu_1\int_\Omega|\nabla u_t|^2dx-\mu_2\int_\Omega|\nabla v_t|^2dx-\frac{1}{2}g(t)\int_\Omega|\nabla u|^2(x,t)dx\\&+\frac{1}{2}\int_\Omega(g'\circ\nabla u)(t)dx\\&\le 0.\end{aligned}\tag{20}$$

we show that (17) holds. The proof is complete. □

Now, we define the functional $\mathscr{D}(t)$ as follows

$$\mathscr{D}(t)=\int_\Omega uu_tdx+\int_\Omega vv_tdx+\frac{\mu_1}{2}\int_\Omega|\nabla u|^2dx+\frac{\mu_2}{2}\int_\Omega|\nabla v|^2dx.\tag{21}$$

Then, we have the following estimate.

Lemma 4.2. *The functional $\mathscr{D}(t)$ satisfies*

$$\begin{aligned}\mathscr{D}'(t)\le C\int_\Omega|\nabla u_t|^2dx+C\int_\Omega|\nabla v_t|^2dx+\Big(\delta+|\alpha|C-\beta(t)\Big)\int_\Omega|\nabla u|^2dx-(\lambda+\mu)\int_\Omega|div\ u|^2dx\\+\Big(|\alpha|C-\mu\Big)\int_\Omega|\nabla v|^2dx-(\lambda+\mu)\int_\Omega|div\ v|^2dx+\frac{\mu-\beta(t)}{4\delta}\int_\Omega(g\circ\nabla u)(t)dx\end{aligned}\tag{22}$$

Proof. Taking the derivative of $\mathscr{D}(t)$ with respect to t and using (1), we find that:

$$\begin{aligned}\mathscr{D}'(t)&=\int_\Omega u_t^2dx+\int_\Omega uu_{tt}dx+\int_\Omega v_t^2dx+\int_\Omega vv_{tt}dx+\mu_1\int_\Omega\nabla u_t\nabla udx+\mu_2\int_\Omega\nabla v_t\nabla vdx\\&=\int_\Omega u_t^2dx+\int_\Omega v_t^2dx-\beta(t)\int_\Omega|\nabla u|^2(x,t)dx+\int_\Omega\int_0^t g(s)(\nabla u(s)-\nabla u(t))\nabla u(t)dsdx\\&\quad-\mu\int_\Omega|\nabla v|^2dx-(\lambda+\mu)\int_\Omega|\text{div }u|^2dx-\int_\Omega(\lambda+\mu)|\text{div }v|^2dx-2\alpha\int_\Omega uv\,dx\end{aligned}\tag{23}$$

Using the fact that

$$\begin{aligned}\int_\Omega\int_0^t g(s)|\nabla u(s)-\nabla u(t)|\nabla u(t)dsdx&\le\delta\int_\Omega|\nabla u|^2(x,t)dx+\frac{1}{4\delta}\int_\Omega\Big(\int_0^t g(s)|\nabla u(s)-\nabla u(t)|ds\Big)^2dx\\&\le\delta\int_\Omega|\nabla u|^2(x,t)dx+\frac{\mu-\beta(t)}{4\delta}\int_\Omega(g\circ\nabla u)(t)dx.\end{aligned}\tag{24}$$

Inserting the estimate (24) into (23) and using Young's, Poincar's inequalities lead to the desired estimate. The proof is complete.

□

Proof. [Proof of Theorem 2.3] We define the Lyapunov functional

$$\mathscr{L}(t) = NE(t) + \epsilon\mathscr{D}(t), \tag{25}$$

where N and ϵ are positive constants that will be fixed later.

Taking the derivative of (25) with respect to t and making use of (17), (22), we obtain

$$\begin{aligned}\frac{d}{dt}\mathscr{L}(t) \leq & -\Big\{N\mu_1 - \epsilon C\Big\}\int_\Omega |\nabla u_t(x,t)|^2 dx - \Big\{N\mu_2 - \epsilon C\Big\}\int_\Omega |\nabla v_t(x,t)|^2 dx \\ & - \Big(\beta(t) - \delta - |\alpha|C\Big)\epsilon\int_\Omega |\nabla u|^2 dx - \Big(\mu - |\alpha|C\Big)\epsilon\int_\Omega |\nabla v|^2 dx \\ & - (\lambda+\mu)\epsilon\int_\Omega |\text{div } u|^2 dx \ - (\lambda+\mu)\epsilon\int_\Omega |\text{div } v|^2 dx \\ & + \frac{N}{2}\int_\Omega (g'\circ\nabla u)(t)dx + \frac{(\mu-\beta(t))\epsilon}{4\delta}\int_\Omega (g\circ\nabla u)(t)dx \\ & - \frac{N}{2}g(t)\int_\Omega |\nabla u|^2(x,t)dx.\end{aligned} \tag{26}$$

At this point, we choose our constants in (26), carefully, such that all the coefficients in (26) will be negative. It suffices to choose ϵ so small and N large enough such that

$$N\mu_1 - \epsilon C > 0,$$

and

$$N\mu_2 - \epsilon C > 0,$$

Further, we choose α small enough such that

$$\beta(t) - \delta - |\alpha|C > 0,$$

and

$$\mu - |\alpha|C > 0.$$

Consequently, from the above, we deduce that there exist there exists two positive constants η_1 and $\eta_2$such that (26) becomes

$$\frac{d\mathscr{L}(t)}{dt} \leq -\eta_1 E(t) + \eta_2\int_\Omega (g\circ\nabla u)dx \tag{27}$$

By multiplying (29) by $\xi(t)$, we arrive at

$$\xi(t)\mathscr{L}'(t) \leq -\eta_1\xi(t)E(t) + \eta_2\xi(t)\int_\Omega (g\circ\nabla u)dx \tag{28}$$

Recalling (A2) and using (17),we get

$$\begin{aligned}\xi(t)\mathscr{L}'(t) &\leq -\eta_1\xi(t)E(t) - \eta_2\int_\Omega (g'\circ\nabla u)dx \\ &\leq -\eta_1\xi(t)E(t) - 2\eta_2 E'(t)\end{aligned} \tag{29}$$

That is

$$\Big(\xi(t)\mathscr{L}(t) + 2\eta_2 E(t)\Big)' - \xi'(t)\mathscr{L} \leq -\eta_1 \xi(t) E(t)$$

Using the fact that $\xi'(t) \leq 0, \quad \forall t \geq 0$ and letting

$$\mathscr{F}(t) = \xi(t)\mathscr{L}(t) + 2\eta_2 E(t) \sim E(t) \tag{30}$$

we obtain

$$\mathscr{F}'(t) \leq -\eta_1 \xi(t) E(t) \leq -\eta_3 \xi(t) \mathscr{F}(t) \tag{31}$$

A simple integration of (31) over $(0, t)$ leads to

$$\mathscr{F}(t) \leq \mathscr{F}(0) e^{-\eta_3 \int_0^t \xi(s) ds}, \quad \forall t \geq 0 \tag{32}$$

A combination of (30) and (32) leads to (7). Then, the proof is complete. □

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