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Incomplete q-Chebyshev Polynomials

Elif Ercan[a], Mirac Cetin[b], Naim Tuglu[a]

[a]*Gazi University, Faculty of Science, Departments of Mathematics Ankara/TURKEY*
[b]*Başkent University, Faculty of Education, Departments of Mathematics Education Ankara/TURKEY*

Abstract. In this paper, we get the generating functions of the q-Chebyshev polynomials using η_z operator, which is $\eta_z(f(z)) = f(qz)$ for any given function $f(z)$. Also considering explicit formulas of the q-Chebyshev polynomials, we give new generalizations of the q-Chebyshev polynomials called the incomplete q-Chebyshev polynomials of the first and second kind. We obtain recurrence relations and several properties of these polynomials. We show that there are connections between the incomplete q-Chebyshev polynomials and the some well-known polynomials.

1. Introduction

The Chebyshev polynomials are of great importance in many area of mathematics, particularly approximation theory. The Chebyshev polynomials of the second kind can be expressed by the formula

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x) \quad n \geq 2$$

with initial conditions $U_0 = 1,\ U_1(x) = 2x$ and the Chebyshev polynomials of the first kind can be defined as

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \quad n \geq 2$$

with initial conditions $T_0(x) = 1,\ T_1(x) = x$ in [13].

The well-known Fibonacci and Lucas sequences are defined by the recurrence relations

$$F_{n+1} = F_n + F_{n-1} \quad n \geq 1$$

$$L_{n+1} = L_n + L_{n-1} \quad n \geq 1$$

with initial conditions $F_0 = 0,\ F_1 = 1$ and $L_0 = 2,\ L_1 = 1$, respectively. In [10], Filipponi introduced a generalization of the Fibonacci numbers. Accordingly, the incomplete Fibonacci and Lucas numbers are determined by:

$$F_n(k) = \sum_{j=0}^{k} \binom{n-1-j}{j}, \qquad 0 \leq k \leq \left\lfloor \tfrac{n-1}{2} \right\rfloor \tag{1}$$

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Email addresses: elifercan06@gmail.com (Elif Ercan), mcetin@baskent.edu.tr (Mirac Cetin), naimtuglu@gazi.edu.tr (Naim Tuglu)

and

$$L_n(k) = \sum_{j=0}^{k} \frac{n}{n-j}\binom{n-j}{j}, \qquad 0 \le k \le \left\lfloor \tfrac{n}{2} \right\rfloor, \tag{2}$$

where $n \in \mathbb{N}$. Note that $F_n(\lfloor \frac{n-1}{2} \rfloor) = F_n$ and $L_n(\lfloor \frac{n}{2} \rfloor) = L_n$. In [16], the generating functions of incomplete Fibonacci and Lucas polynomials were given by Pintér and Srivastava. For more results on the incomplete Fibonacci numbers, the readers may refer to [6–9, 17, 20, 21].

We need q-integer and q-binomial coefficient. There are several equivalent definition and notation for the q-binomial coefficients [2, 11, 12, 15, 19]. Let $q \in \mathbb{C}$ with $0 < |q| < 1$ as an indeterminate and nonnegative integer n. The q-integer of the number n is defined by

$$[n]_q := \frac{1-q^n}{1-q}$$

with $[0]_q = 0$. The Gaussian or $q-$binomial coefficients are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k}, \qquad 0 \le k \le n$$

with $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$ for $n < k$, where $(x;q)_n$ is the $q-$shifted factorial, that is, $(x;q)_0 = 1$,

$$(x;q)_n = \prod_{i=0}^{n-1} \left(1 - q^i x\right).$$

The q-binomial coefficient satisfies the recurrence relations and properties:

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \tag{3}$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \tag{4}$$

$$\frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q + \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q \tag{5}$$

$$q^k \frac{[n]_q}{[n-k]_q} \begin{bmatrix} n-k \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-k \\ k \end{bmatrix}_q + q^n \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}_q. \tag{6}$$

The q-analogues of the Fibonacci polynomials are studied by Carlitz in [3]. Also, a new q-analogue of the Fibonacci polynomials is defined by Cigler and obtain some of its properties in [5]. In [14], Pan study some arithmetic properties of the q-Fibonacci numbers and the q-Pell numbers. Cigler defined the q-analogues of the Chebyshev polynomials and study properties of these polynomials in [4].

In this paper, we derive generating functions of the q-Chebyshev polynomials of the first and second kind. More generally, we define the incomplete q-Chebyshev polynomials of the first and second kind. We get recurrence relations and several properties of these polynomials. We show that there are the relationships between q-Chebyshev polynomials and the incomplete q-Chebyshev polynomials.

2. q-Chebyshev Polynomials

Definition 2.1. *The q-Chebyshev polynomials of the second kind are defined by*

$$\mathcal{U}_n(x,s,q) = (1+q^n)x\,\mathcal{U}_{n-1}(x,s,q) + q^{n-1}s\,\mathcal{U}_{n-2}(x,s,q) \qquad n \ge 2 \tag{7}$$

with initial conditions $\mathcal{U}_0(x,s,q) = 1$ and $\mathcal{U}_1(x,s,q) = (1+q)x$ in [4].

Definition 2.2. *The q-Chebyshev polynomials of the first kind are defined by*

$$\mathcal{T}_n(x,s,q) = (1+q^{n-1})x\mathcal{T}_{n-1}(x,s,q) + q^{n-1}s\mathcal{T}_{n-2}(x,s,q) \qquad n \geq 2 \tag{8}$$

with initial conditions $\mathcal{T}_0(x,s,q) = 1$ *and* $\mathcal{T}_1(x,s,q) = x$ *in [4].*

It is clear that $\mathcal{U}_n(x,-1,1) = U_n(x)$ and $\mathcal{T}_n(x,-1,1) = T_n(x)$. The q-Chebyshev polynomials of the second kind is determined as the combinatorial sum

$$\mathcal{U}_n(x,s,q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j}}{(-q;q)_j} s^j x^{n-2j}, \qquad n \geq 0 \tag{9}$$

and the q-Chebyshev polynomials of the first kind is determined as

$$\mathcal{T}_n(x,s,q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} q^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j-1}}{(-q;q)_j} s^j x^{n-2j}, \qquad n > 0 \tag{10}$$

with $\mathcal{T}_0(x,s,q) = 1$ in [4].

2.1. Generating Functions of q-Chebyshev Polynomials

Andrews [1] obtain the generating function for Schur's polynomials, which is defined by $S_n(q) = S_{n-1}(q) - q^{n-2}S_{n-2}(q)$ for $n > 1$ with intial conditions $S_0(q) = 0$ and $S_1(q) = 1$. The generating funtions of $S_n(q)$ is

$$\sum_{n=0}^{\infty} S_n(q)x^n = \frac{x}{1 - x - x^2\,\eta_z} \tag{11}$$

where is η_z is an operator on functions of z defined by $\eta_z\,(f(z)) = f(qz)$ in [1]. We give the following theorems for generating functions of q-Chebyshev polynomials of the second and first kind with an operator η_z.

Theorem 2.3. *The generating function of the q-Chebyshev polynomials of the second kind is*

$$G(z) = \frac{1}{1 - zx - (xqz + sqz^2)\,\eta_z}. \tag{12}$$

Proof. Let $G(z) = \sum_{n=0}^{\infty} \mathcal{U}_n z^n$. Thus we write

$$\begin{aligned}\left(1 - xz - \left(xqz + sqz^2\right)\eta_z\right)G(z) &= \sum_{n=0}^{\infty} \mathcal{U}_n z^n - x\sum_{n=0}^{\infty} \mathcal{U}_n z^{n+1} - x\sum_{n=0}^{\infty} \mathcal{U}_n q^{n+1} z^{n+1} - s\sum_{n=0}^{\infty} \mathcal{U}_n q^{n+1} z^{n+2} \\ &= \mathcal{U}_0 + \mathcal{U}_1 z - x\,(1+q)\,\mathcal{U}_0 z + \sum_{n=2}^{\infty} \left(\mathcal{U}_n - x\,(1+q^n)\,\mathcal{U}_{n-1} - s\,\mathcal{U}_{n-2}q^{n-1}\right)z^n.\end{aligned}$$

Therefore we have from Eq. (7) and $\mathcal{U}_0 = 1$, $\mathcal{U}_1 = (1+q)x$, we get

$$\left(1 - xz - \left(xqz + sqz^2\right)\eta_z\right)G(z) = 1.$$

□

Theorem 2.4. *The generating function of the q-Chebyshev polynomials of the first kind is*

$$S(z) = \frac{1 - xz}{1 - xz - (xz - sqz^2)\,\eta_z}. \tag{13}$$

Proof. Let $S(z) = \sum_{n=0}^{\infty} \mathcal{T}_n z^n$. Then

$$\left(1 - xz - \left(xz - sqz^2\right)\eta_z\right) S(z) = \sum_{n=0}^{\infty} \mathcal{T}_n z^n - x\sum_{n=1}^{\infty} \mathcal{T}_{n-1} z^n - x\sum_{n=1}^{\infty} \mathcal{T}_{n-1}\, q^{n-1} z^n - s\sum_{n=2}^{\infty} \mathcal{T}_{n-2}\, q^{n-1} z^n$$

$$= \mathcal{T}_0 + \mathcal{T}_1 z - 2x\mathcal{T}_0\, z + \sum_{n=2}^{\infty}\left(\mathcal{T}_n - x\left(1+q^{n-1}\right)\mathcal{T}_{n-1} - sq^{n-1}\,\mathcal{T}_{n-2}\right) z^n,$$

using Eq. (8) and $\mathcal{T}_0 = 1$ ve $\mathcal{T}_1 = x$, we conclude that

$$S(z) - xzS(z) - xz\,\eta_z S(z) - sqz^2\,\eta_z S(z) = 1 - xz,$$

finally we obtain

$$S(z) = \frac{1 - xz}{1 - xz - (xz - sqz^2)\,\eta_z}. \tag{14}$$

□

3. Incomplete q-Chebyshev Polynomials

In this section, we define the incomplete q-Chebyshev polynomials of the first and second kind. We give several properties for these polynomials.

Definition 3.1. *For n is a nonnegative integer, the incomplete q-Chebyshev polynomials of the second kind are defined as*

$$\mathcal{U}_n^k(x,s,q) = \sum_{j=0}^{k} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j}}{(-q;q)_j}\, s^j x^{n-2j} \quad 0 \le k \le \left\lfloor \tfrac{n}{2} \right\rfloor. \tag{15}$$

When $k = \left\lfloor \frac{n}{2} \right\rfloor$ in (15), $\mathcal{U}_n^k(x,s,q) = \mathcal{U}_n(x,s,q)$, we get the q-Chebyshev polynomials of the second kind in [4].

Definition 3.2. *For n is a nonnegative integer, the incomplete q-Chebyshev polynomials of the first kind are defined by*

$$\mathcal{T}_n^k(x,s,q) = \sum_{j=0}^{k} q^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j-1}}{(-q;q)_j}\, s^j x^{n-2j} \quad 0 \le k \le \left\lfloor \tfrac{n}{2} \right\rfloor. \tag{16}$$

Theorem 3.3. *The incomplete q-Chebyshev Polynomials of the second kind satisfy*

$$\mathcal{U}_{n+2}^{k+1} = (1+q^{n+2})x\,\mathcal{U}_{n+1}^{k+1} + q^{n+1}s\,\mathcal{U}_n^k \tag{17}$$

for $0 \le k \le \frac{n-1}{2}$.

Proof. From Eq. (15), we can write

$$(1+q^{n+2})x\,\mathcal{U}_{n+1}^{k+1} + q^{n+1}s\,\mathcal{U}_n^k = (1+q^{n+2})x \sum_{j=0}^{k+1} q^{j^2} \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j+1}}{(-q;q)_j} s^j x^{n+1-2j}$$

$$+ q^{n+1}s \sum_{j=0}^{k} q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j}}{(-q;q)_j} s^j x^{n-2j}$$

$$= \sum_{j=0}^{k+1} q^{j^2} \left\{ \left(\begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + q^{n-2j+2} \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix}_q \right) \right.$$

$$\left. + q^{n-j+2}\left(q^j \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q + \begin{bmatrix} n-j+1 \\ j-1 \end{bmatrix} j - 1_q \right) \right\} \frac{(-q;q)_{n-j+1}}{(-q;q)_j} s^j x^{n-2j+2}.$$

Thus using Eq. (3) and Eq. (4), we get

$$\begin{aligned}(1+q^{n+2})x\,\mathcal{U}_{n+1}^{k+1}+q^{n+1}s\,\mathcal{U}_{n}^{k} &= \sum_{j=0}^{k+1} q^{j^2}\begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j+2}}{(-q;q)_j} s^j x^{n-2j+2} \\ &= \mathcal{U}_{n+2}^{k+1}.\end{aligned}$$

□

Corollary 3.4. *The incomplete q-Chebyshev Polynomials of the second kind satisfy the non-homogeneous recurrence relation*

$$\mathcal{U}_{n+2}^{k} = (1+q^{n+2})x\,\mathcal{U}_{n+1}^{k}+q^{n+1}s\,\mathcal{U}_{n}^{k} - q^{n+1+k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{(-q;q)_{n-k}}{(-q;q)_k} s^{k+1}x^{n-2k}. \tag{18}$$

Theorem 3.5. *For $0 \le k \le \frac{n+1}{2}$, the following equality give a relationships between the incomplete q-Chebyshev polynomials of the first and second kind*

$$\mathcal{T}_{n+2}^{k} = x\,\mathcal{U}_{n+1}^{k}+q^{n+1}s\,\mathcal{U}_{n}^{k-1}. \tag{19}$$

Proof. Using Eq. (15), Eq. (4) and Eq. (6) we obtain

$$\begin{aligned}\mathcal{U}_{n+1}^{k}+q^{n+1}s\,\mathcal{U}_{n}^{k-1} &= x\sum_{j=0}^{k} q^{j^2}\begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j+1}}{(-q;q)_j} s^j x^{n+1-2j} + q^{n+1}s\sum_{j=0}^{k-1} q^{j^2}\begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j}}{(-q;q)_j} s^j x^{n-2j} \\ &= \sum_{j=0}^{k} q^{j^2}\frac{[n+2]_q}{[n-j+2]_q}\begin{bmatrix} n-j+2 \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j+1}}{(-q;q)_j} s^j x^{n-2j+2} \\ &= \mathcal{T}_{n+2}^{k}.\end{aligned}$$

□

Theorem 3.6. *The incomplete q-Chebyshev polynomials of the first kind satisfy*

$$\mathcal{T}_{n+2}^{k+1} = (1+q^{n+1})x\mathcal{T}_{n+1}^{k+1}+q^{n+1}s\mathcal{T}_{n}^{k} \tag{20}$$

for $0 \le k \le \frac{n-1}{2}$.

Proof. By using Eq. (17) and Eq. (19), we get

$$\begin{aligned}\mathcal{T}_{n+2}^{k+1} &= x\,\mathcal{U}_{n+1}^{k+1}+q^{n+1}s\,\mathcal{U}_{n}^{k} \\ &= (1+q^{n+1})x^2\,\mathcal{U}_{n}^{k+1}+q^{n}sx\,\mathcal{U}_{n-1}^{k}+q^{n+1}s(1+q^n)x\,\mathcal{U}_{n-1}^{k}+q^{2n}s^2\,\mathcal{U}_{n-2}^{k-1} \\ &=(1+q^{n+1})x\mathcal{T}_{n+1}^{k+1}+q^{n+1}s\mathcal{T}_{n}^{k}.\end{aligned}$$

□

Corollary 3.7. *The incomplete q-Chebyshev polynomials of the first kind satisfy the non-homogeneous recurrence relation*

$$\mathcal{T}_{n+2}^{k} = (1+q^{n+1})\mathcal{T}_{n+1}^{k}+q^{n+1}s\mathcal{T}_{n}^{k} - q^{n+1+k^2}\frac{[n]_q}{[n-k]_q}\begin{bmatrix} n-k \\ k \end{bmatrix}_q \frac{(-q;q)_{n-k-1}}{(-q;q)_k} s^{k+1}x^{n-2k}. \tag{21}$$

Theorem 3.8. *For $0 \le k \le \frac{n+1}{2}$, then*

$$\mathcal{T}_{n+2}^{k} = x\,\mathcal{U}_{n+1}^{k}(x,q^2s,q)+qs\,\mathcal{U}_{n}^{k-1}(x,q^2s,q) \tag{22}$$

holds.

Proof. We obtain from Eq. (15) and (3), we have

$$x\,\mathcal{U}_{n+1}^{k}(x,q^2s,q)+qs\,\mathcal{U}_{n}^{k-1}(x,q^2s,q)=\sum_{j=0}^{k}q^{j^2}\left\{q^{2j}\begin{bmatrix}n-j+1\\ j\end{bmatrix}_q+(1+q^j)\begin{bmatrix}n-j+1\\ j-1\end{bmatrix}_q\right\}\frac{(-q;q)_{n+1-j}}{(-q;q)_j}s^jx^{n+2-2j}$$
$$=\sum_{j=0}^{k}q^{j^2}\frac{[n+2]_q}{[n+2-j]_q}\begin{bmatrix}n-j+2\\ j\end{bmatrix}_q\frac{(-q;q)_{n+1-j}}{(-q;q)_j}s^jx^{n+2-2j}$$
$$=\mathcal{T}_{n+2}^{k}.$$

□

Theorem 3.9. *We have*

$$(1+q^{n+2})\mathcal{T}_{n+2}^{k}=\mathcal{U}_{n+2}^{k}+q^{2n+3}s\,\mathcal{U}_{n}^{k-1},\qquad 0\le k\le\left\lfloor\tfrac{n}{2}\right\rfloor. \tag{23}$$

Proof. From Eq. (17) and Eq. (15), we get

$$\mathcal{U}_{n+2}^{k}+q^{2n+3}s\,\mathcal{U}_{n}^{k-1}=\sum_{j=0}^{k}q^{j^2}\left\{\begin{bmatrix}n-j+1\\ j\end{bmatrix}_q+q^{n+1-2j+1}(1+q^j)\begin{bmatrix}n-j+1\\ j-1\end{bmatrix}_q\right\}\frac{(-q;q)_{n+1-j}}{(-q;q)_j}s^jx^{n+2-2j}$$
$$+q^{n+2}\sum_{j=0}^{k}q^{j^2}\left\{\begin{bmatrix}n-j+1\\ j\end{bmatrix}_q+q^{n+1-2j+1}(1+q^j)\begin{bmatrix}n-j+1\\ j-1\end{bmatrix}_q\right\}\frac{(-q;q)_{n+1-j}}{(-q;q)_j}s^jx^{n+2-2j}$$

We get the following result from Eq. (4) and Eq. (6)

$$\mathcal{U}_{n+2}^{k}+q^{2n+3}s\,\mathcal{U}_{n}^{k-1}=\mathcal{T}_{n+2}^{k}+q^{n+2}\mathcal{T}_{n+2}^{k}.$$

□

Lemma 3.10. *We have*

$$\frac{d\,\mathcal{U}_n}{dx}=nx^{-1}\,\mathcal{U}_n-2x^{-1}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}jq^{j^2}\begin{bmatrix}n-j\\ j\end{bmatrix}_q\frac{(-q;q)_{n-j}}{(-q;q)_j}s^jx^{n-2j} \tag{24}$$

and

$$\frac{d\,\mathcal{T}_n}{dx}=nx^{-1}\mathcal{T}_n-2x^{-1}\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}jq^{j^2}\frac{[n]_q}{[n-j]_q}\begin{bmatrix}n-j\\ j\end{bmatrix}_q\frac{(-q;q)_{n-j-1}}{(-q;q)_j}s^jx^{n-2j}. \tag{25}$$

Proof. By using Eq. (9), we have

$$\frac{d\,\mathcal{U}_n}{dx}=\frac{d}{dx}\left\{\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}q^{j^2}\begin{bmatrix}n-j\\ j\end{bmatrix}_q\frac{(-q;q)_{n-j}}{(-q;q)_j}s^jx^{n-2j}\right\}$$
$$=nx^{-1}\,\mathcal{U}_n-2\sum_{j=0}^{\lfloor\frac{n}{2}\rfloor}jq^{j^2}\begin{bmatrix}n-j\\ j\end{bmatrix}_q\frac{(-q;q)_{n-j}}{(-q;q)_j}s^jx^{n-2j-1}.$$

Similarly, from Eq. (10), we get Eq. (25). □

Using Lemma 3.10 , we can prove the following theorem.

Theorem 3.11. *We have*

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_n^k = \left(\left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{U}_n + \frac{x}{2} \frac{d\,\mathcal{U}_n}{dx}. \tag{26}$$

Proof. From Eq. (15), we have

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_n^k &= \left(q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q;q)_n}{(-q;q)_0} x^n \right) + \left(q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q;q)_n}{(-q;q)_0} x^n + q \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q;q)_{n-1}}{(-q;q)_1} s x^{n-2} \right) + \cdots \\ &\quad + \left(q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q;q)_n}{(-q;q)_0} x^n + q \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q;q)_{n-1}}{(-q;q)_1} s x^{n-2} + \cdots + q^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n - \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_q \frac{(-q;q)_{n-\lfloor \frac{n}{2} \rfloor}}{(-q;q)_{\lfloor \frac{n}{2} \rfloor}} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \right) \\ &= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \left(q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q;q)_n}{(-q;q)_0} x^n \right) + \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 - 1 \right) \left(q \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q;q)_{n-1}}{(-q;q)_1} s x^{n-2} \right) + \cdots \\ &\quad + \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 - \left\lfloor \frac{n}{2} \right\rfloor \right) \left(q^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n - \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_q \frac{(-q;q)_{n-\lfloor \frac{n}{2} \rfloor}}{(-q;q)_{\lfloor \frac{n}{2} \rfloor}} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \right) \\ &= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \mathcal{U}_n - \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j}}{(-q;q)_j} s^j x^{n-2j}. \end{aligned}$$

Then by using Lemma 3.10, we get

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{U}_n^k = \left(\left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{U}_n + \frac{x}{2} \frac{d\,\mathcal{U}_n}{dx}.$$

□

Theorem 3.12. *We have*

$$\sum_{k=0}^{[\frac{n}{2}]} \mathcal{T}_n^k = \left(\left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{T}_n + \frac{x}{2} \frac{d\,\mathcal{T}_n}{dx}. \tag{27}$$

Proof. We have from Eq. (16) and Lemma 3.10

$$\begin{aligned} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \mathcal{T}_n^k &= (q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q;q)_{n-1}}{(-q;q)_0} x^n) + (q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q;q)_{n-1}}{(-q;q)_0} x^n + q \frac{[n]_q}{[n-1]_q} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q;q)_{n-2}}{(-q;q)_1} s x^{n-2}) + \cdots \\ &\quad + \Bigg(q^0 \begin{bmatrix} n \\ 0 \end{bmatrix}_q \frac{(-q;q)_{n-1}}{(-q;q)_0} x^n + q \frac{[n]_q}{[n-1]_q} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{(-q;q)_{n-2}}{(-q;q)_1} s x^{n-2} + \cdots \\ &\quad + q^{\lfloor \frac{n}{2} \rfloor^2} \frac{[n]_q}{\left[n - \lfloor \frac{n}{2} \rfloor \right]_q} \begin{bmatrix} n - \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n}{2} \rfloor \end{bmatrix}_q \frac{(-q;q)_{n-\lfloor \frac{n}{2} \rfloor - 1}}{(-q;q)_{\lfloor \frac{n}{2} \rfloor}} s^{\lfloor \frac{n}{2} \rfloor} x^{n-2\lfloor \frac{n}{2} \rfloor} \Bigg) \\ &= \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \mathcal{T}_n - \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} j q^{j^2} \frac{[n]_q}{[n-j]_q} \begin{bmatrix} n-j \\ j \end{bmatrix}_q \frac{(-q;q)_{n-j-1}}{(-q;q)_j} s^j x^{n-2j} \\ &= \left(\left\lfloor \frac{n}{2} \right\rfloor - \frac{n}{2} + 1 \right) \mathcal{T}_n + \frac{x}{2} \frac{d\,\mathcal{T}_n}{dx}. \end{aligned}$$

□

4. Graphs of The Incomplete q-Chebyshev polynomials

In this section, we display the graphs of the q-Chebyshev polynomials and the incomplete q-Chebyshev polynomials.

In Figures 1, 2 the graphs of the q-Chebyshev polynomials of first and second kind for $s = -1$, $q = -0.5,\ 0.5,\ 0.9999$, $n = 0, 1, 2, 3, 4, 5$ and $-1 \le x \le 1$ are shown.

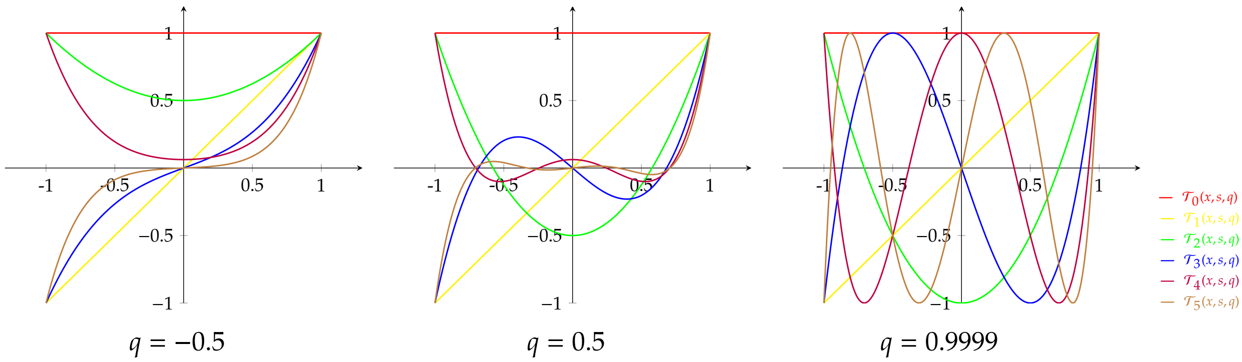

Figure 1: Graphs of $\mathcal{T}_n(x,s,q)$ for $s = -1, q = -0.5, 0.5, 0.9999,\ n = 0, 1, 2, 3, 4, 5$

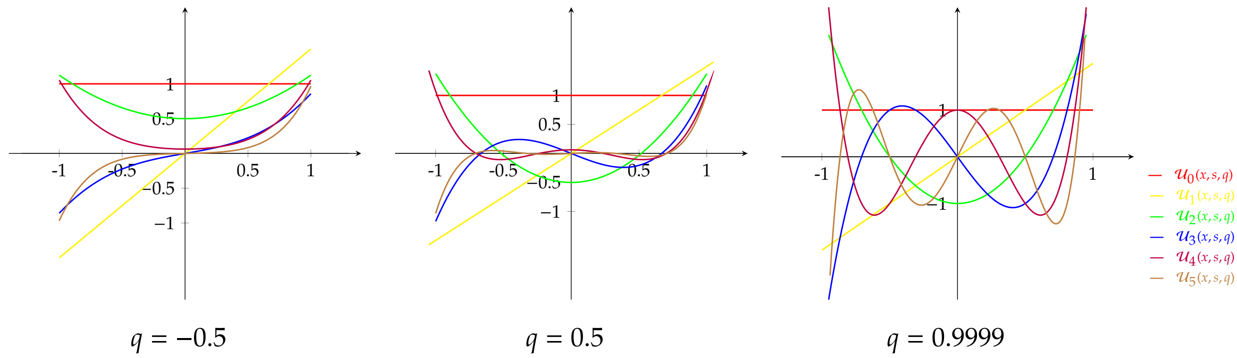

Figure 2: Graphs of $\mathcal{U}_n(x,s,q)$ for $s = -1, q = -0.5, 0.5, 0.9999,\ n = 0, 1, 2, 3, 4, 5$

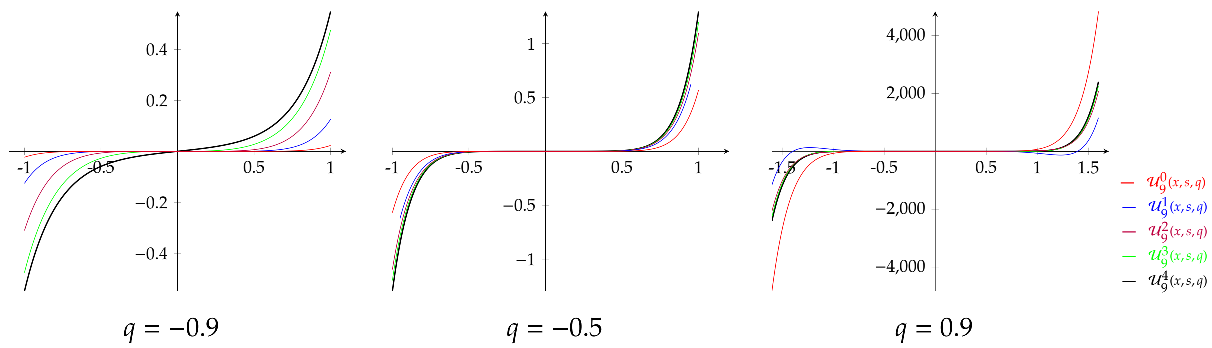

Figure 3: Graphs of $\mathcal{U}_9^k(x,s,q)$ for $s = -1,\ q = -0.9,\ -0.5,\ 0.9, k = 0, 1, 2, 3, 4$

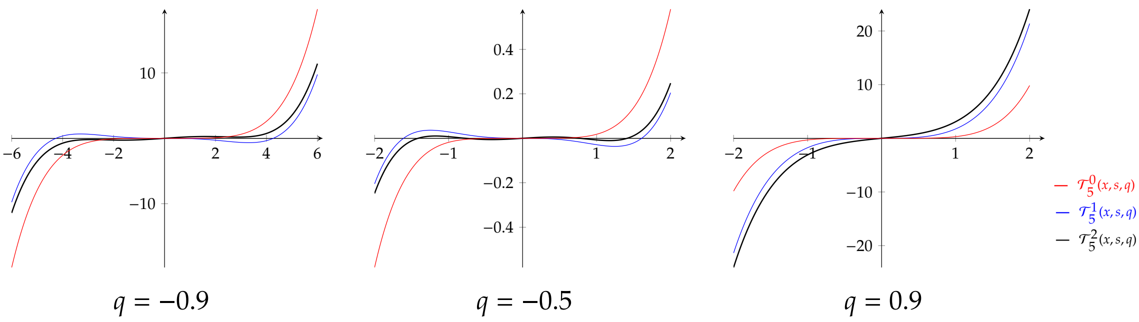

Figure 4: Graphs of $\mathcal{T}_5^k(\frac{x}{2},s,q)$ for $s=1, q=-0.9,\ -0.5,\ 0.9,\ k=0,1,2$

In Figure 3 the graphs of the incomplete q-Chebyshev polynomials of second kind $\mathcal{U}_9^k(x,s,q)$ for $s=-1$, $q=-0.9,-0.5,0.9,\ k=0,1,2,3,4$ are shown.

In Figure 4 the graphs of the incomplete Lucas polynomials $\mathcal{T}_5^k(\frac{x}{2},s,q)$ for $s=1$, $q=-0.9,\ -0.5,\ 0.9$, $k=0,1,2$ are shown.

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