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Existence of Solutions for Infinite Systems of Differential Equations by Densifiability Techniques

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Abstract. A novel technique to state the existence of solutions for certain infinite systems of differential equations is proposed. Our main tool will be the so called degree of nondensifiability, which seems to work under more general conditions than the measures of noncompactness. In fact, in our main result, the required conditions proposed for the existence of solutions of such system are more general than others required in most of the results based on such measures.

1. Introduction

Many phenomena raised from the real word, such as branching processes, neural nets or dissociation of polymers can be modeled by infinite systems of ordinary differential equations (see [7, 9, 17, 18, 28, 31] and references within). For instance, assume that we have a system S which at every time t is in one of the countable states $n = 1, 2, \dots,$ and let $p_n(t)$ the probability that S is in the state i at the time t. Thus, under suitable conditions (see [9, Example 3, p. 2]) we can obtain formally the system

$$p_n'(t) + a_{nn}p_n(t) = \sum_{m\neq n} a_{nm}p_m(t), \quad \text{for each } t \geq 0 \text{ and } n \geq 1, \tag{1.1}$$

with the initial conditions

$$p_n(0) = c_n > 0 \text{ for each } n \geq 1 \text{ and } \sum_{n\geq 1} c_n = 1, \tag{1.2}$$

where the numbers a_{ij}, related with the conditioned probability of the states, and the probabilities at the initial time c_n are known. This is an example of a branching process.

Looking at the system (1.1)-(1.2), is clear that an infinite system of ordinary differential equations can be considered as an ordinary differential equation, ODE, posed in a suitable Banach space. That is, an ODE posed in some sequences Banach space (such as c_0 or ℓ_1) is, actually, an infinite system of ODEs. Thus, the study on the existence of solutions of an infinite systems of ordinary differential equations is equivalent to the study on the existence of solutions of ODEs in the mentioned Banach spaces.

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In this paper we consider the system

$$x_n'(t) = f_n(t, x_1(t), x_2(t), \ldots, x_n(t), \ldots), \text{ for } t \in I := [0, 1], \quad n = 1, 2, \ldots \tag{1.3}$$

with the initial condition

$$x_n(0) = x_n^0, \tag{1.4}$$

where, for every $n \geq 1$, $x_n^0 \in \mathbb{R}$ and $f_n : I \times \mathbb{R}^{\aleph_0} \longrightarrow \mathbb{R}$ are given, $\aleph_0$ being the first countable ordinal. As expected, some assumptions on the map f are necessary and will be given later.

There are several results, based in the so called *measures of noncompactness*, which we will expose briefly in in Section 2, to guarantee the existence of solutions for systems of type (1.3)-(1.4); see for instance [5, 6, 24–27] and references within.

On the other hand, our main tool will be the so called *degree of nondensifiability*, which is generated from a generalization of the space-filling curves, namely, the *α-dense curves*. We will detail these concepts in Section 2. It is worthy of remark that the degree of nondensifiability is not a measure of noncompactness but, as we point out in Section 3 (see also Example 3.1) works under more general conditions than these ones.

Further, as we will show in Section 4, the necessary conditions required for the existence of solution for the system (1.3)-(1.4) are more general than those required in most of the above cited works which use the Hausdorff measure of noncompactness; see also Remark 4.3. This fact will be illustrated in Examples 4.5 and 4.6.

2. Measures of noncompactness and the degree of nondensifiability

In order to make more comprehensive the manuscript, we recall the concepts of measure of noncompactness and degree of nondensifiability, as well as some relationships between them. Firstly, we need to fix the notation. In what follows, (E, d) will be a metric space, and $(X, \|\cdot\|)$ a Banach space. Also, we denote by $\mathfrak{B}_E$ (resp. $\mathfrak{B}_X$) the class of non-empty and bounded subsets of E (resp. of X), and given $B \subset E$ (or $B \subset X$), $\bar{B}$ denotes the closure of B.

Although the definition of measure of noncompactness may be slightly different according to the author (see, for instance, [2–4]), here we will adopt the following given in [14]:

Definition 2.1. *A map $\mu : \mathfrak{B}_E \longrightarrow \mathbb{R}_+ := [0, \infty)$ is said to be a measure of noncompactness, in short MNC, if satisfies the following properties:*

(i) Regularity: $\mu(B) = 0$ if, and only if, B is a precompact set.

(ii) Invariant under closure: $\mu(B) = \mu(\bar{B})$, for all $B \in \mathfrak{B}_E$.

(iii) Semi-additivity: $\mu(B_1 \cup B_2) = \max\{\mu(B_1), \mu(B_2)\}$, for all $B_1, B_2 \in \mathfrak{B}_E$.

To have a MNC in $(X, \|\cdot\|)$ it is needed to add the two following additional properties:

(I) Semi-homogeneity: $\mu(\lambda B) = |\lambda|\mu(B)$ for any number λ and $B \in \mathfrak{B}_X$.

(II) Invariant under translations: $\mu(x + B) = \mu(B)$ for any $x \in X$ and $B \in \mathfrak{B}_X$.

One of the most important examples of measure of noncompactness is the Hausdorff MNC defined as:

$$\chi(B) := \inf\left\{\varepsilon > 0 : B \text{ can be covered by finitely many balls with radii } \leq \varepsilon\right\},$$

for each $B \in \mathfrak{B}_E$. For instance, if B_X is the closed unit ball of X, then $\chi(B_X) = 1$ if X is of infinite dimension while $\chi(B_X) = 0$ otherwise (see, for instance, [3, Theorem 2.5, p. 23]).

On the other hand, in 1997 the concepts of α-dense curve and densifiable set were introduced by Mora and Cherruault [20] :

Definition 2.2. *Given $\alpha \geq 0$ and $B \in \mathfrak{B}_E$, a continuous map $\gamma : I := [0,1] \longrightarrow (E,d)$ is said to be an α-dense curve in B if the following conditions hold:*

(i) $\gamma(I) \subset B$.

(ii) For any $x \in B$, there is $y \in \gamma(I)$ such that $d(x,y) \leq \alpha$.

If for every $\alpha > 0$ there is an α-dense curve in B, then B is said to be densifiable.

Let us note that, given $B \in \mathfrak{B}_E$, always there is an α-dense curve in B for any $\alpha \geq \mathrm{Diam}(B)$ (the diameter of B). Indeed, fixed $x_0 \in B$, the map $\gamma(t) := x_0$, for all $t \in I$, is an α-dense curve in B whenever $\alpha \geq \mathrm{Diam}(B)$.

The positive parameter α of Definition 2.2 coincides with the Hausdorff distance from B to $\gamma(I)$ so, we can say that the α-dense curves are a generalization of the so called *space-filling curves* (see [29]). In fact, if $B := I^d$, a 0-dense curve in B is, precisely, a space-filling curve in B. Also, we can prove (see [23]) that the class of densifiable sets is strictly between the class of Peano Continua (i.e. those sets that are the continuous image of I) and the class of connected and precompact sets. For a detailed exposition of the α-dense curves and its applications, see [8, 11, 12, 19–22] and references therein.

From the α-dense curves the following definition, introduced in [22] and used in [14], can be stated:

Definition 2.3. *Let $\alpha \geq 0$ and $\Gamma_{\alpha,B}$ the class of all α-dense curves in $B \in \mathfrak{B}_E$. Then, we define the degree of nondensifiability, in short DND, $\phi_d : \mathfrak{B}_E \longrightarrow \mathbb{R}_+$ as*

$$\phi_d(B) := \inf\left\{\alpha \geq 0 : \Gamma_{\alpha,B} \neq \emptyset\right\},$$

for every $B \in \mathfrak{B}_E$.

For instance, in [22] it is show that in an infinite dimensional Banach space X, $\phi_d(B_X) = 1$ while $\phi_d(B_X) = 0$ otherwise, B_X being the closed unit ball of X.

Next, we show some properties of the DND ϕ_d proved in [14].

Proposition 2.4. *In a complete metric space (E,d), ϕ_d satisfies:*

(i) It is regular on the subclass $\mathfrak{B}_{a,E} \subset \mathfrak{B}_E$ of bounded and arc-connected sets bounded subsets of E, i.e. , $\phi_d(B) = 0$ if, and only if, B is precompact, with $B \in \mathfrak{B}_{a,E}$.

(ii) It is invariant under closure: $\phi_d(B) = \phi_d(\overline{B})$, for any $B \in \mathfrak{B}_E$.

(iii) It is semi-additive on sets B_1, $B_2 \neq \emptyset$ of $\mathfrak{B}_{a,E}$, provided that $B_1 \cap B_2 \neq \emptyset$, i.e. ,

$$\phi_d(B_1 \cup B_2) = \max\{\phi_d(B_1), \phi_d(B_2)\}.$$

Furthermore, if E is a Banach space, then ϕ_d also satisfies:

(I) $\phi_d\Big(\mathrm{Conv}(B)\Big) \leq \phi_d(B)$, $\forall B \in \mathfrak{B}_E$, *where* $\mathrm{Conv}(B)$ *stands for the convex hull of B.*

(II) It is semi-homogene, that is, $\phi_d(\lambda B) = |\lambda|\phi_d(B)$, for $\lambda \in \mathbb{R}$ and $B \in \mathfrak{B}_E$.

(III) It is invariant under translations, that is, $\phi_d(x + B) = \phi_d(B)$, for any $x \in E$ and $B \in \mathfrak{B}_E$.

Then, the DND ϕ_d shares some properties with the MNCs. However, we have to emphasize that the DND is not a MNC. The following example illustrates this fact.

Example 2.5. *Let $L^1(I)$ be the Banach space of the absolute value Lebesgue integrable functions defined on I, endowed its usual norm, and consider the set of the statistics density functions*

$$D := \Big\{ f \in L^1(I) : f \geq 0 \text{ and } \int_0^1 f(x)dx = 1 \Big\}.$$

Then, $\phi_d(D) = 2$ (see [14]) and therefore $1 = \phi_d(B_{L^1(I)}) = \phi_d(B_{L^1(I)} \cup D) < \max\{\phi_d(D), \phi_d(B_{L^1(I)})\} = 2$, where $B_{L^1(I)}$ is the closed unit ball of $L^1(I)$.

However, the DND ϕ_d and the Hausdorff MNC χ are related by the following inequalities (see [14, Theorem 2.5]):

Proposition 2.6. *For every $B \in \mathfrak{B}_X$ arc-connected, we have*

$$\chi(B) \leq \phi_d(B) \leq 2\chi(B),$$

and these inequalities are the best possible in infinite dimensional Banach spaces.

3. Auxiliary facts

In this section we provide a few facts having an auxiliary character which will be used further on. Let the following class of functions:

$$\Psi := \Big\{ \psi : \mathbb{R}_+ \to \mathbb{R}_+ : \psi \text{ is monotone increasing and } \lim_n \psi^n(r) = 0, \forall r \in \mathbb{R}_+ \Big\}.$$

Note that the continuity of the functions in the class Ψ is not required, and the exponent of ψ denotes the composition of ψ with itself.

Next, we recall the following generalization of the celebrated Darbo fixed point theorem (see, for instance, [3]) proved in [1, Theorem 2.2]:

Theorem 3.1. *Let $C \in \mathfrak{B}_X$ convex and closed, $T : C \longrightarrow C$ continuous and μ a MNC invariant under the convex hull. Assume that there is $\psi \in \Psi$ such that*

$$\mu(T(B)) \leq \psi(\mu(B)),$$

for each non-empty $B \subset C$. Then, T has some fixed point.

A useful fixed point result for our goals is the following, proved in [13, Theorem 3.2]:

Theorem 3.2. *Let $C \in \mathfrak{B}_X$ convex and closed and $T : C \longrightarrow C$ continuous. Assume that there is $\psi \in \Psi$ such that*

$$\phi_d(T(B)) \leq \psi(\phi_d(B)),$$

for each non-empty and convex $B \subset C$. Then, T has some fixed point.

Let us note that Theorems 3.1 and 3.2 are, in forms, very similar. However as it is shown in [13] by several examples (not exposed here for lack of space) both results are essentially different, as Theorem 3.2 works under more general conditions than Theorem 3.1.

An immediate consequence of Theorem 3.2 is the following *version* of the Darbo fixed point theorem for the DND ϕ_d:

Corollary 3.3. *Let $C \in \mathfrak{B}_X$ convex and closed and $T : C \longrightarrow C$ continuous. Assume that there is $0 < k < 1$ such that*

$$\phi_d(T(B)) \leq k\phi_d(B),$$

for each non-empty and convex $B \subset C$. Then, T has some fixed point.

As we have pointed out above, Theorem 3.2 can be applied under more general conditions than Theorem 3.1. Then, as expected, Corollary 3.3 works under more general conditions than Darbo fixed point theorem or its generalizations. This fact is evidenced in the following example.

Example 3.4. *Let $C(I)$ be the Banach space of the continuous functions defined on I, equipped the usual supremum norm $\|\cdot\|_\infty$. Define the convex and closed set $C := \{x \in C(I) : 0 = x(0) \leq x(t) \leq 1 = x(1), t \in I\}$ and the map $T : C \longrightarrow C$ as*

$$T(x)(t) := \begin{cases} \dfrac{1}{2}x(2t), & 0 \leq t \leq \dfrac{1}{2} \\ \dfrac{1}{2}x(2t-1) + \dfrac{1}{2}, & \dfrac{1}{2} < t \leq 1 \end{cases}$$

for each $x \in C$ and $t \in I$. Then, as it is shown in [3, Example 2, p. 169] we have

$$\chi(C) = \chi(T(C)) = \frac{1}{2}, \tag{3.1}$$

and consequently, neither Darbo fixed point theorem nor its generalizations (as that given in Theorem 3.1), for the Hausdorff MNC χ, can be applied here.

Now, let $B \subset C$ be non-empty and convex and $\gamma : I \longrightarrow (C(I), \|\cdot\|_\infty)$ an α-dense curve in B, for some $\alpha > \phi_d(B)$. So, given $x \in B$ there is $y \in \gamma(I)$ such that

$$\|x - y\|_\infty \leq \alpha. \tag{3.2}$$

Then, if $0 \leq t \leq 1/2$ from (3.1) we have

$$|T(x)(t) - T(y)(t)| = \frac{1}{2}|x(2t) - y(2t)| \leq \frac{1}{2}\|x - y\|_\infty \leq \frac{\alpha}{2},$$

and likewise, the same inequality holds for $1/2 < t \leq 1$. Therefore, for each $T(x) \in T(B)$ there is $T(y) \in T(\gamma(I))$ such that $\|T(x) - T(y)\|_\infty \leq \alpha/2$, that is, $T \circ \gamma$ is an $\alpha/2$-dense curve in $T(B)$. By the arbitrariness of $\alpha > \phi_d(B)$ we conclude that $\phi_d(T(B)) \leq \phi_d(B)/2$ and so, Corollary 3.3 states the existence of some fixed point of T.

At this point, we show a result which we will use later (see [13, Lemma 3.2]):

Lemma 3.5. *Let J be a bounded and closed interval and B a non-empty and bounded subset of the Banach space of the continuous maps $x : J \longrightarrow X$. Then, we have:*

$$\sup\left\{\phi_d\Big(\{x(t) : x \in B\}\Big) : t \in J\right\} \leq \phi_d(B).$$

4. Main result

Firstly, we need to recall the following concepts (see, for instance, [6, Definitions 1.11 and 1.12]).

Definition 4.1. *Let Y a linear metric space. Then, Y is called:*

(i) An FK-space if Y is a Fréchet space with continuous coordinates.

(ii) A BK-space if Y is a normed FK-space, i.e. a Banach space with continuous coordinates.

What follows, $(X, \|\cdot\|)$ will be a BK-space such that $X \subset \omega$, the space of all real sequences. Given $x \in X$ and $r > 0$ we denote by $\bar{B}(x, r)$ the closed ball centered at x and radius r. Also, given a bounded and closed interval J, $C(J, X)$ will be the space of continuous maps $x : J \longrightarrow (X, \|\cdot\|)$, endowed with the usual supremum norm $\|x\|_\infty := \sup\{\|x(t)\| : t \in J\}$. For $V \subset X$ non-empty, we put $C(J, V)$ the set of the maps $x \in C(J, X)$ with $x(J) \subset V$. Likewise, the integral of vector valued functions will mean in the Bochner sense (see, for instance, [30]) while the integral of scalar functions will mean in the Lebesgue sense.

Remark 4.2. *In most of the works cited in Section 1, which use the Hausdorff MNC χ, the required conditions to prove the existence of solutions for the system (1.3)-(1.4) depend strongly on the chosen Banach space of sequences (see also Example 4.6). This is so because the formula for the Hausdorff MNC is, of course, different from one space to another. However, our conditions do not depend of the chosen Banach space due, mainly, to the inequalities of Proposition 2.6.*

Let the following conditions:

(C1) The map $f := (f_1, \ldots, f_n, \ldots) : I \times X \longrightarrow X$ is continuous and the initial condition $x_0 := (x_n^0)_{n\geq 1} \in X$.

(C2) There is $h : I \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\|f(t, x_1(t), \ldots, x_n(t), \ldots)\| \leq h(t, R)$ for almost everywhere $t \in I$ and $x \in C(I, X)$, whenever $\|x\|_\infty \leq R$. Also, there are $R_0 > 0$ and $K > 0$ such that

$$\frac{h(t, R_0)}{R_0} \leq K, \quad \text{for almost everywhere } t \in I. \tag{4.1}$$

(C3) Given $C \subset C(I, X)$ non-empty, closed and convex, there is $\beta : I \longrightarrow \mathbb{R}_+$ such that for each non-empty and convex $B \subset C$

$$\phi_d(f(t, B)) \leq \beta(t)\phi_d(\{x(t) : x \in B\}), \tag{4.2}$$

for almost everywhere $t \in I$, where $f(t, B) := \{f(t, x_1(t), \ldots, x_n(t), \ldots) : (x_n)_{n\geq 1} \in B\}$ for each $t \in J$. Moreover, there is $0 < b \leq 1$ that:

$$0 < \int_0^b \beta(s)ds < 1. \tag{4.3}$$

Remark 4.3. *Many works (see, for instance, [10, 15, 16]) need the Lipschitz condition $\|f(t, x) - f(t, y)\| \leq L\|x - y\|$, for some $L > 0$ and each $t \in I$, $x, y \in X$, to state and prove the existence of solutions for ordinary differential equations posed in a (possibly of infinite dimension) Banach space. Looking at Example 3.2, is clear that condition (C3) is less restrictive than the Lipschitz one. This fact is also manifested in results based on MNCs.*

In [5, Theorem 3], for $X := c_0$ the Banach space of null sequences, the following conditions are required to prove the existence of solutions for the system (1.3)-(1.4):

(D1) $x_0 := (x_n^0)_{n\geq 1} \in c_0$

(D2) The map $f := (f_1, \ldots, f_n, \ldots)$ acts from the set $I \times c_0$ into c_0 and it is continuous.

(D3) There exists an increasing sequence $(k_n)_{n\geq 1}$ of positive integers such that for any $t \in I$, $x = (x_n)_{n\geq 1} \in c_0$ and $n \geq 1$ the following inequality holds:

$$|f_n(t, x_1, \ldots, x_n, \ldots)| \leq p_n(t) + q_n(t) \sup\{|x_i| : i \geq k_n\},$$

where $(p_n(t))_{n\geq 1}$ and $(q_n(t))_{n\geq 1}$ are real functions defined and continuous on I such that the sequence $(p_n(t))_{n\geq 1}$ converges uniformly on I to the function vanishing identically and the sequence $(q_n(t))_{n\geq 1}$ is equibounded on I.

Let us note that condition (C1), for $X := c_0$, is equal to conditions (D1)-(D2), and putting $q(t) := \sup\{q_n(t) : n \geq 1\}$, for each $t \in I$, the numbers $P := \sup\{p(t) : t \in I\}$ and $Q := \sup\{q(t) : n \geq 1\}$ are well defined from condition (D3). Moreover, from condition (D3) we can deduce easily (following the proof of [5, Theorem 3]) that fixed a closed interval $J \subset I$ and $C \subset \mathcal{C}(J, X)$ non-empty

$$\|f(t, x(t))\| \leq P + Q\|x\|_\infty, \tag{4.4}$$

for each $t \in I$, $x \in C$ and

$$\chi(f(t, B)) \leq q(t)\chi(B), \tag{4.5}$$

for each non-empty and convex $B \subset C$. Consequently, we find that inequality (4.1) follows from (4.4) taking $h(t, R) := P + QR$, $R_0 := 1$, $K := P + Q$ and condition (C4) follows from the inequality (4.5) and Proposition 2.6 taking $\beta(t) := 2q(t)$ and $0 < b < 1/2Q$. However, as we will show in Example 4.5, conditions (C1)-(C3) may be satisfied but not the (D1)-(D3) ones.

Likewise, conditions (C1)-(C3) can be deduced from the conditions required in [5, Theorem 5] for $X := \ell_1$ the Banach space of absolute value summable sequences. The same can be said for [25, Theorem 3.8], where $X := \ell_p$ with $1 \leq p < \infty$.

Summarizing, conditions (C1)-(C3) are more general than those required in others results which are based in the Hausdorff MNC χ.

Now, we are ready to state and show our main result:

Theorem 4.4. *Let the conditions (C1)-(C3), and $0 < \rho < \min\{1, 1/K, b\}$ with K and b defined in (4.1) and (4.3), respectively. Then, the system (1.3)-(1.4) has some solution $x \in \mathcal{C}([0, \rho], \bar{B}(x_0, R_0))$.*

Proof. Define the map $F : \mathcal{C}(J, X) \longrightarrow \mathcal{C}(J, X)$ as

$$F(x)(t) := x_0 + \int_0^t f(s, x(s))ds,$$

for each $x := (x_n)_{n\geq 1} \in \mathcal{C}(J, X)$ and $t \in [0, \rho]$. So, we only need to prove the existence of some fixed point of F. For this, we will apply Corollary 3.3. Clearly, F is well defined and is continuous from condition (C1).

Define $C := \mathcal{C}([0, \rho], \bar{B}(x_0, R_0))$, where $R_0 > 0$ is given in condition (C2), that clearly is non-empty, bounded and convex. First, we will prove that $F(C) \subset C$. Indeed, given any $t \in [0, \rho]$, from inequality (4.1) of condition (C2)

$$\frac{\|F(x)(t) - x_0\|}{R_0} \leq \int_0^t \frac{\|f(s, x(s))\|}{R_0} ds \leq \int_0^t \frac{h(s, R_0)}{R_0} ds \leq \int_0^\rho \frac{h(s, R_0)}{R_0} ds \leq \rho K < 1,$$

and therefore $\|F(x)(t) - x_0\| \leq R_0$ for every $t \in [0, \rho]$. So, $F(C) \subset C$ as claimed.

Now, let $B \subset C$ be non-empty and convex, and for each $s \in [0, \rho]$ let $\alpha_s := \phi_d(\{x(s) : x \in B\})$. From inequality (4.2) of condition (C3), for almost everywhere $s \in [0, \rho]$, we have

$$\phi_d(f(s, B)) \leq \beta(s)\alpha_s,$$

and therefore, noticing Definition 2.3, given any $\varepsilon > 0$ there is a continuous $\gamma_s : I \longrightarrow \mathcal{C}(J, X)$, put $\tau \in I \longmapsto \gamma_s(\tau)$ with $\gamma_s(I) \subset f(s, B)$ and such that for every $x \in B$ there is $\tau \in I$ satisfying

$$\|f(s, x(s)) - \gamma_s(\tau)\| \leq \beta(s)\alpha_s + \varepsilon. \tag{4.6}$$

Define $\Gamma : I \longrightarrow \mathcal{C}(J, X)$ as

$$\tau \in I \longmapsto \Gamma(\tau, t) := x_0 + \int_0^t \gamma_s(\tau)ds \quad \text{for all } t \in [0, \rho],$$

which is well defined, is continuous (due to the continuity of γ_s) and $\Gamma(I) \subset C(J, X)$.

Then, from (4.6), for each $t \in [0, \rho]$ there is $\tau \in I$ such that

$$\|F(x)(t) - \Gamma(\tau, t)\| \leq \int_0^t \|f(s, x(s)) - \gamma_s(\tau)\| ds \leq \int_0^t (\beta(s)\alpha_s + \varepsilon) ds, \tag{4.7}$$

and by Lemma 3.5

$$\int_0^t (\beta(s)\alpha_s + \varepsilon) ds \leq \phi_d(B) \int_0^t \beta(s) ds + \rho\varepsilon \leq \phi_d(B) \int_0^\rho \beta(s) ds + \rho\varepsilon \leq \phi_d(B) \int_0^b \beta(s) ds + \rho\varepsilon. \tag{4.8}$$

So, joining (4.7) and (4.8) and letting $\varepsilon \to 0$, we find that

$$\|F(x)(t) - \Gamma(\tau, t)\| \leq \tilde{\beta}\phi_d(B),$$

where $\tilde{\beta} := \int_0^b \beta(s) ds$.

Finally, by the arbitrariness of t, the inequality $\|F(x) - Y\|_\infty \leq \tilde{\beta}\phi_d(B)$ holds for $Y(t) := \Gamma(\tau, t) \in \Gamma(I)$ and therefore Γ is a $\tilde{\beta}\phi_d(B)$-dense curve in $F(B)$. Then, as $0 < \tilde{\beta} < 1$, the conditions of Corollary 3.3 are fulfilled and the proof is now complete.

□

From the above considerations the examples exposed, for instance, in [5, 25] can be solved by Theorem 4.4. However, to close the paper, we will show a pair of examples which can not be solved by the results proved in the cited papers.

Example 4.5. *Let $X := c_0$, endowed the usual supremum norm $\|\cdot\|$, and consider the system*

$$x_n'(t) = x_n^2(t), \quad \text{for } t \in I, n = 1, 2, \ldots$$

with the initial condition $x_n(0) = 0$ for each $n \geq 1$. First, we will show that the above condition (D3) is not satisfied. Otherwise, from the inequality given in condition (D3), we have $\|f(t, x_1, \ldots, x_n, \ldots)\| \leq P + Q\|x\|$ for each $t \in I$ and $x := (x_1, \ldots, x_n, \ldots) \in c_0$ where the numbers P and Q have been defined above. So,

$$\frac{\|f(t, x)\|}{\|x\|} \leq \frac{P}{\|x\|} + Q,$$

for each $x \in c_0$, not null. Thus, taking $x := (R, 0, \ldots, 0, \ldots) \in c_0$, from the above inequality must be $Q \geq R - P/R$ which is contradictory if we let $R \to \infty$.

On the other hand, condition (C1) holds trivially, and taking $h(t, R) := R^2$ the inequality (4.2) of condition (C2) is satisfied, for instance, taking $R_0 = K = 1$. Next, given $C \subset C(I, c_0)$ non-empty, closed and convex, we can check easily that

$$\phi_d(f(t, B)) \leq 2\phi_d(\{x(t) : x \in B\}),$$

for each non-empty and convex $B \subset C$. So, condition (C3) is fulfilled and then for each $0 < \rho < 1/2$ Theorem 4.4 guarantees the existence of some solution $x \in C([0, \rho], c_0)$ for this system.

Example 4.6. *Let the system (1.1)-(1.2)*

$$\begin{cases} x_n'(t) + a_{nn}x_n(t) = \displaystyle\sum_{m \neq n} a_{nm}x_m(t), & \text{for } t \in I, n = 1, 2, \ldots \\ x_n(0) = c_n > 0 \quad \text{such that} & \displaystyle\sum_{n \geq 1} c_n = 1. \end{cases}$$

where $a_{nm} \in I$ *and* $M := \sup\{\sum_{m\neq n} a_{nm} : m \geq 1\} < \infty$. *We take* $X := \ell_\infty$, *the Banach space of bounded sequences endowed the usual supremum norm* $\|\cdot\|$. *The results proved in [5, 25, 26] can not be applied here, because these results do not contemplate the case of this particular Banach space.*

Condition (C1) is clearly satisfied, so we will show that condition (C2) holds. Given $t \in I$ *and* $x := (x_n)_{n\geq 1} \in \mathcal{C}(I, \ell_\infty)$, *we have*

$$\|f(t, x(t))\| = \sup\{|f_n(t, x_n(t)| : n \geq 1\} \leq \sup\{|x_n(t)| + \sum_{m\neq n} a_{nm}|x_m(t)| : n \geq 1\},$$

and therefore, if $\|x\|_\infty \leq R$ *for some* $R > 0$, *the above inequality yields the following:*

$$\|f(t, x(t))\| \leq (1 + M)R.$$

So, condition (C2) holds for $h(t, R) := (1 + M)R$, $K := 1 + M$ *and any* $R_0 > 0$. *Now, let* $C \subset \mathcal{C}(I, \ell_\infty)$ *non-empty, closed and convex, and* $B \subset C$ *non-empty and convex. We can check easily that for each* $t \in I$

$$\phi_d(f(t, B)) \leq (1 + M)\phi_d(B).$$

Then, condition (C3) holds for $\beta(t) := 1 + M$ *and therefore, by Theorem 4.4, fixed* $R_0 > 0$ *this system has some solution* $x \in \mathcal{C}([0, \rho], \bar{B}(x_0, R_0))$, *for every* $0 < \rho < 1/(1 + M)$.

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