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Legendre Curves on 3-Dimensional Kenmotsu Manifolds Admitting Semisymmetric Metric Connection

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Abstract. The object of the present paper is to study biharmonic Legendre curves, locally ϕ-symmetric Legendre curves and slant curves in 3-dimensional Kenmotsu manifolds admitting semisymmetric metric connection. Finally, we construct an example of a Legendre curve in a 3-dimensional Kenmotsu manifold.

1. Introduction

In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D.E. Blair [1].

The study of Legendre curves on S^3 with pseudo-Hermitian connection has been initiated by J. T. Cho [4]. Again C. Özgür and M.M. Tripathi [12] studied Legendre curves on α-Sasakian manifolds. Moreover Legendre curves and slant curves have been studied by several authors such as Calin et al. [3] and Welyczko ([21], [22]) and many others. The notion of Kenmotsu manifolds was introduced by K. Kenmotsu [10]. Kenmotsu manifolds have been studied by several authors such as G. Pitis [18], U.C. De and G. Pathak [5], J-B. Jun, U.C. De and G. Pathak [9], V.F. Kirichenko [11] and many others.

In the present paper, we are interested to study Legendre curves in 3-dimensional Kenmotsu manifolds. The present paper is organized as follows: After preliminaries in section 3, we study biharmonic Legendre curves with respect to semisymmetric metric connection in a 3-dimensional Kenmotsu manifold. In the next section we consider locally ϕ-symmetric Legendre curves with respect to semisymmetric metric connection in a 3-dimensional Kenmotsu manifold. Section 5 is devoted to study slant curves in 3-dimensional Kenmotsu manifolds admitting semisymmetric metric connection. Finally, we construct an example of a Legendre curve in a 3-dimensional Kenmotsu manifold.

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2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1, 1)$ tensor field, ξ is a vector field, η is an 1-form and g is a compatible Riemannian metric such that [2]

$$\phi^2(X) = -X + \eta(X)\xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2}$$

$$g(X, \xi) = \eta(X), g(\phi X, Y) = -g(X, \phi Y), \tag{3}$$

for all $X, Y \in \chi(M)$.

An almost contact manifold is called a Kenmotsu manifold if

$$(\nabla_X \phi)Y = -g(X, \phi Y) - \eta(Y)\phi X \tag{4}$$

and

$$\nabla_X \xi = X - \eta(X)\xi. \tag{5}$$

In [5], De and Pathak studied 3-dimensional Kenmotsu manifolds and obtain the expression of the curvature as follows:

$$\begin{aligned} R(X, Y)Z &= (\frac{r}{2} + 2)[g(Y, Z)X - g(X, Z)Y] - g(Y, Z)(\frac{r}{2} + 3)\eta(X)\xi + g(X, Z)(\frac{r}{2} + 3)\eta(Y)\xi \\ &\quad + (\frac{r}{2} + 3)\eta(Y)\eta(Z)X - (\frac{r}{2} + 3)\eta(X)\eta(Z)Y. \end{aligned} \tag{6}$$

A linear connection $\tilde{\nabla}$ on a Riemannian manifold is called symmetric if the torsion tensor of the connection is zero on the manifold. The connection is called semisymmetric [20] if the torsion $\tilde{\tau}^t$ of $\tilde{\nabla}$ satisfies

$$\tilde{\tau}^t(X, Y) = \eta(Y)X - \eta(X)Y, \tag{7}$$

where η is a nonzero 1-form.

The connection $\tilde{\nabla}$ is said to be a semisymmetric metric connection if

$$\tilde{\nabla} g = 0.$$

The semisymmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ on an almost contact metric manifold are related by [17]

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \tag{8}$$

for all vector fields X, Y on M. Semisymmetric metric connections have been studied by several authors such as U. C. De ([6]) and Özgür et al. ([13], [14], [15]) and many others.

A curve γ on M is called a Frenet curve with respect to the semisymmetric metric connection if

$$\tilde{\nabla}_T T = \tilde{\kappa} N, \tag{9}$$

$$\tilde{\nabla}_T N = -\tilde{\kappa} T + \tilde{\tau} B, \tag{10}$$

$$\tilde{\nabla}_T B = \tilde{\tau} N, \tag{11}$$

where $\tilde{\kappa} = |\tilde{\nabla}_T T|$ and $\tilde{\tau}$ are the curvature and torsion with respect to the semisymmetric metric connection, $T = \dot{\gamma}, N, B$ is the Frenet frame along γ.

For the semisymmetric metric connection, we can also define the helix, circle and geodesic by using the same method as in [4]. A helix is a curve where both its curvature and torsion with respect to the semisymmetric metric connection are constants. In particular, curves with constant nonzero curvature and zero torsion are called circles with respect to the semisymmetric metric connection. Note that the geodesics are regarded as helices where both their curvature and torsion are zero.

A Frenet curve γ in an almost contact metric manifold is said to be a Legendre curve if it is an integral curve of the contact distribution $D = ker\eta$. Formally, it is also said that $g(\dot{\gamma}, \dot{\gamma}) = 1, \eta(\dot{\gamma}) = 0$.

3. Biharmonic Legendre Curves with Respect to Semisymmetric Metric Connection

Definition 3.1. *[8] A Legendre curve on a 3-dimensional Kenmotsu manifold will be called biharmonic with respect to semisymmetric metric connection $\tilde{\nabla}$ if it satisfies*

$$\tilde{\nabla}_T^3 T + \tilde{\nabla}_T \tilde{\tau}^t(\tilde{\nabla}_T T, T) + \tilde{R}(\tilde{\nabla}_T T, T)T = 0, \tag{12}$$

where $\tilde{\tau}^t$ is torsion of semisymmectric connection and T is tangent vector field of the curve.

Let us consider a Legendre curve γ. Let T be the tangent vector field of γ. We take $\{T, \phi T, \xi\}$ be an orthonormal right handed system when $\phi T = -N, \phi N = T$.

For a semisymmetric metric connection

$$\tilde{\nabla}_T \tilde{\tau}^t(\tilde{\nabla}_T T, T) = 0. \tag{13}$$

Hence (12) reduces to

$$\tilde{\nabla}_T^3 T + \tilde{\kappa}\tilde{R}(N, T)T = 0. \tag{14}$$

Let $\tilde{R}$ and R be the Riemannian curvature tensor with respect to the semisymmetric metric connection and Levi-Civita connection respectively. Then the relation between $\tilde{R}$ and R is given by [20]

$$\tilde{R}(X, Y)Z = R(X, Y)Z - L(Y, Z)X + L(X, Z)Y - g(Y, Z)FX + g(X, Z)FY, \tag{15}$$

where L is a tensor field of type $(0, 2)$ given by

$$L(Y, Z) = (\nabla_Y \eta)Z - \eta(Y)\eta(Z) + \frac{1}{2}\eta(\xi)g(Y, Z) \tag{16}$$

F is a tensor field of type $(1, 1)$ given by $g(FY, Z) = L(Y, Z)$ for any vector field Y, Z on M.

Using (5) in (16), we have

$$L(Y, Z) = \frac{3}{2}g(Y, Z) - 2\eta(Y)\eta(Z). \tag{17}$$

Considering $g(FY, Z) = L(Y, Z)$ in the above equation we obtain

$$FY = \frac{3}{2}Y - 2\eta(Y)\xi. \tag{18}$$

Substituting (17) and (18) into (15) we get for a 3-dimensional Kenmotsu manifold [17]

$$\begin{aligned}\tilde{R}(X, Y)Z &= R(X, Y)Z - 3[g(Y, Z)X - g(X, Z)Y] + 2[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi] \\ &\quad +2\eta(Z)[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{19}$$

For a Legendre curve $\eta(T) = 0$, $\eta(N) = 0$, because we have considered Frenet Frame as $\{T, \phi T, \xi\}$ where $\phi T = -N$. Using this facts in (19) and considering (6), we get

$$\tilde{R}(N, T)T = (\frac{r}{2} - 1)N. \tag{20}$$

By Frenet formula (9),(10) and (11) we get

$$\tilde{\nabla}_T^3 T = -3\tilde{\kappa}\tilde{\kappa}'T + (\tilde{\kappa}'' - \tilde{\kappa}^3 - \tilde{\kappa}\tilde{\tau}^3)N + (2\tilde{\tau}\tilde{\kappa}' + \tilde{\kappa}\tilde{\tau}')B, \tag{21}$$

where $N = -\phi T$, $B = \xi$.

In view of (20) and (21) we obtain

$$\tilde{\nabla}_T^3 T + \tilde{\kappa}\tilde{R}(N, T)T = -3\tilde{\kappa}\tilde{\kappa}'T + [(\tilde{\kappa}''' - \tilde{\kappa}^3 - \tilde{\kappa}\tilde{\tau}^2) + \tilde{\kappa}(\frac{r}{2} - 1)N] + (2\tilde{\tau}\tilde{\kappa}' + \tilde{\kappa}\tilde{\tau}')B. \tag{22}$$

By virtue of (14) and observing the component from right hand side of (22), we get $\tilde{k}$ and $\tilde{\tau}$ e are constant such that $\tilde{k}^2 + \tilde{\tau}^2 = 1 - \frac{1}{2}r$. Hence we can state the following theorem:

Theorem 3.2. *Let γ be a nongeodesic biharmonic Legendre curve with respect to the semisymmetric metric connection on a 3-dimensional Kenmotsu manifold. Then it is a helix of the semisymmetric metric connection such that $\tilde{k}^2 + \tilde{\tau}^2 = 1 - \frac{1}{2}r$. The converse statement is true when the torsion tensor is constant along γ.*

4. Locally ϕ-Symmetric Legendre Curves

The notion of Locally ϕ-symmetric Legendre curves was introduced in the paper [19].

Definition 4.1. *With respect to the semisymmetric metric connection a Legendre curve on a 3-dimensional Kenmotsu manifold is called locally ϕ-symmetric if it satisfies*

$$\tilde{\phi}^2(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T)T = 0. \tag{23}$$

Let us consider a Legendre curve γ. Now proceeding in the same way as in the previous section, we get

$$\tilde{R}(\tilde{\nabla}_T T, T)T = \tilde{\kappa}(\frac{r}{2} - 1)N. \tag{24}$$

By definition of covariant differentiation of $\tilde{R}$ we get

$$(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T) = \tilde{\nabla}_T\tilde{R}(\tilde{\nabla}_T T, T)T - \tilde{R}(\tilde{\nabla}_T^2 T, T)T - \tilde{R}(\tilde{\nabla}_T T, \tilde{\nabla}_T T)T - \tilde{R}(\tilde{\nabla}_T T, T)\tilde{\nabla}_T T. \tag{25}$$

Using Frenet formula, we get from the above relation

$$(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T) = \tilde{\nabla}_T\tilde{R}(\tilde{\nabla}_T T, T)T - \tilde{\kappa}'\tilde{R}(N, T)T + \tilde{\kappa}^2\tilde{R}(T, T)T - \tilde{\kappa}\tilde{\tau}\tilde{R}(B, T)T - \tilde{\kappa}^2\tilde{R}(N, N)T - \tilde{\kappa}^2\tilde{R}(N, T)N. \tag{26}$$

After some straight forward calculations we have

$$\begin{aligned}(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T) &= \tilde{\kappa}'(\frac{r}{2} - 1)N + \tilde{\kappa}(\frac{r}{2} - 1)'N + \tilde{\kappa}(\frac{r}{2} - 1)(-\tilde{\kappa}T + \tilde{\tau}B) \\ &\quad -\tilde{\kappa}'(\frac{r}{2} - 1)N - \tilde{\kappa}\tilde{\tau}(\frac{r}{2} - 1)B - \tilde{\kappa}^2(1 - \frac{r}{2})T,\end{aligned} \tag{27}$$

where $'$ denotes differentiation. From (27) we have

$$(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T) = \tilde{\kappa}(\frac{r}{2} - 1)'N. \tag{28}$$

Applying ϕ^2 in both sides of the above equation

$$\phi^2(\tilde{\nabla}_T\tilde{R})(\tilde{\nabla}_T T, T) = -\tilde{\kappa}(\frac{r}{2} - 1)'N. \tag{29}$$

By virtue of (4.1) in Definition 2 and the above relation we observe that: A necessary and sufficient condition for a Legendre curve on a 3-dimensional Kenmotsu manifold to be locally ϕ-symmetric with respect to the semisymmetric metric connection $\tilde{\nabla}$ is either $\tilde{\kappa} = 0$, or r=constant.

If r =constant, the original manifold is locally ϕ-symmetric and hence the curve is trivially locally ϕ-symmetric. If r is a non-zero constant, then the curve will be locally ϕ-symmetric if and only if $\tilde{\kappa} = 0$. So, we can state the following:

Theorem 4.2. *There exists no non-geodesic locally ϕ-symmetric Lengendre curves on a three dimensional Kenmotsu manifolds with respect to the semisymmetric metric connection.*

5. Slant curves in 3-dimensional Kenmotsu manifolds admitting semisymmetric metric connection

Definition 5.1. *[4] A Frenet curve is called a slant curve if it makes a constant angle with the Reeb vector field ξ.*

Hence a curve γ on an almost contact metric manifold is a slant curve if $\eta(\dot{\gamma}) = \cos\theta$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$, where θ is a constant, called the slant angle. In particular, if the angle is $\pi/2$, the curve becomes a Legendre curve. Slant curves in trans Sasakian manifolds have been studied by S. Güvenç and C. Özgür [7]. They have given examples of Slant curves.

In this section, we are interested to study slant curves on 3-dimensional Kenmotsu manifolds with respect to semisymmetric metric connection. Let us consider a slant curve γ on a Kenmotsu manifold with respect to the semisymmetric metric connection. Here $\gamma'(s) = T(s)$ is given by

$$\begin{aligned} \cos\theta(s) &= \eta(\gamma'(s)) \\ &= g(T(s), \xi), \end{aligned} \tag{30}$$

where θ is a constant slant angle. Since $\tilde{\nabla}$ is semi-symmetric metric connection, $\tilde{\nabla} g = 0$. So,

$$\tilde{\nabla}_T g(T, \xi) - g(\tilde{\nabla}_T T, \xi) - g(T, \tilde{\nabla}_T \xi) = 0. \tag{31}$$

Now $g(T, \xi) = \cos\theta(s)$. Hence $\tilde{\nabla}_T g(T, \xi) = -\sin\theta(s)\theta'(s)$.

Again from the relation between $\tilde{\nabla}$ and ∇, we get

$$\tilde{\nabla}_T \xi = \nabla_T \xi - T - \cos\theta\xi.$$

Using the above results and (2.9) in (5.2), we get

$$-\sin\theta(s)\theta'(s) = \tilde{\kappa}\eta(N) - 2\cos^2\theta.$$

For θ=constant, the above equation reduces to

$$\tilde{\kappa}\eta(N) = 2\cos^2\theta. \tag{32}$$

Consider $\theta \neq \pi/2$, from (5.3), we get $\tilde{\kappa} \neq 0$.

Thus we can state the following:

Theorem 5.2. *A non-Legendre slant curve on a three-dimensional Kenmotsu manifold with respect to the semisymmetric metric connection is never a geodesic.*

6. Example of a Legendre Curve in a 3-Dimensional Kenmotsu Manifold

Let us consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : (x, y, z) \neq (0, 0, 0)\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = e^{-z}\frac{\partial}{\partial x},\ e_2 = e^{-z}\frac{\partial}{\partial y},\ e_3 = \frac{\partial}{\partial z},$$

are linear independent at each point of M. Let g be the metric defined by

$$\begin{aligned} g(e_i, e_j) &= 1, \quad for\ i = j, \\ &= 0, \quad for\ i \neq j. \end{aligned}$$

Here i and j runs from 1 to 3.

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any vector field Z tangent to M. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi e_1 = -e_2,\ \phi e_2 = e_1,\ \phi e_3 = 0.$$

Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2.$$

From Koszul's formula, the Riemannian connection we have

$$\nabla_{e_1} e_1 = -e_3, \quad \nabla_{e_1} e_3 = e_1,$$

$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3, \quad \nabla_{e_2} e_3 = e_2,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

$\nabla_X \xi = X - \eta(X)\xi$ is satisfied for $\xi = e_3$. Hence the manifold is a Kenmotsu manifold.

Consider the curve $\gamma : I \longrightarrow M = \mathbb{R}^3$ by $\gamma(s) = (\frac{\sqrt{2}}{3}s, \sqrt{\frac{1}{3}}s, 1)$.

Thus

$$\begin{aligned} g(\dot{\gamma}, e_3) &= \eta(\dot{\gamma}) \\ &= g(\frac{\sqrt{2}}{3}e_1 + \sqrt{\frac{1}{3}}e_2, e_3) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} g(\dot{\gamma}, \dot{\gamma}) &= g(\frac{\sqrt{2}}{3}e_1 + \sqrt{\frac{1}{3}}e_2, \frac{\sqrt{2}}{3}e_1 + \sqrt{\frac{1}{3}}e_2) \\ &= 1. \end{aligned}$$

Therefore the curve γ is a Legendre curve.

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