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Fixed Point Theorems and Tiling Problems

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Abstract. Let (X, d) be a complete metric space and let $f : X \to X$ satisfy $\inf\{\alpha(x, y)d(f^m(x), f^m(y)) : m \in \mathbb{J}\} \leq Kd(x, y)$ for all $x, y \in X$ and some $K \in (0, 1)$ and $\alpha : X \times X \to [0, \infty)$, where $\mathbb{J}$ is a set of positive integers. In this paper, we prove fixed point theorems for this mapping f. We also discuss the connection with tiling problems and give a titling proof of a fixed point theorem.

1. Introduction and preliminaries

The well-known Banach contraction principle states that every contraction from a complete metric space into itself has a unique fixed point. It has played a fundamental role in various areas of pure and applied sciences. During the last 50 years, it has been generalized and extended in many ways by a number of authors. In [4], following interesting conjecture, connected with Banach's fixed point theorem, was considered.

Conjecture I. Let (X, d) be a complete metric space and let $f : X \to X$ satisfy the following condition:

$$\inf\{d(f^n(x), f^n(y)) : n \in \mathbb{J}\} \leq Kd(x, y) \tag{1}$$

for all $x, y \in X$ and some $K \in (0, 1)$, where $\mathbb{J}$ is a set of positive integers. Then f has a fixed point.

Remark

1. The condition (1) does not imply the continuity of f.
2. If $\mathbb{J} = \{1\}$ in the condition (1), then T is a contraction on X.
3. If f^k is a contraction, then the condition (1) holds.

We also note that the case $\mathbb{J} = \{1\}$ corresponds to the Banach contraction principle and the case $\mathbb{J} = \{k\}$, where $k \in \mathbb{N}$, to a result in [3]. It was shown in [4, 8] that Conjecture I is not true when $\mathbb{J} = \mathbb{N}$, the set of natural numbers.

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Example 1.1. [8] Let $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define a mapping $f : X \to X$ by

$$f(x) = \sqrt{x^2 + 1}$$

for all $x \in X$. Then $f^n(x) = \sqrt{x^2 + n}$ for all $x \in X$ and, for all $x, y \in X$ with $x < y$, we can find $K \in (0, 1)$ such that

$$\inf\{|f^n(x) - f^n(y)| : n \in \mathbb{N}\} \leq K|x - y|.$$

However, it is clear that f has no fixed points.

Let $f : X \to X$ be a mapping and $\alpha : X \times X \to [0, \infty)$. We say that f is (1) α-admissible [7] if $x, y \in X$ and $\alpha(x, y) \geq 1$ implies $\alpha(f(x), f(y)) \geq 1$; (2) triangular α-admissible [5] if (a) $\alpha(x, y) \geq 1$ implies $\alpha(f(x), f(y)) \geq 1$, $x, y \in X$; (b) $\begin{cases} \alpha(x, y) \geq 1 \\ \alpha(y, z) \geq 1 \end{cases}$ implies $\alpha(x, z) \geq 1$, $x, y, z \in X$. Fixed point results for α-admissible mappings can be found in [1, 2, 5–7].

In this paper, we consider the following generalization:

Conjecture II. Let (X, d) be a complete metric space and let $f : X \to X$ satisfy the following condition:

$$\inf\{\alpha(x, y)d(f^m(x), f^m(y)) : m \in \mathbb{J}\} \leq Kd(x, y) \tag{2}$$

for all $x, y \in X$ and some $K \in (0, 1)$ and $\alpha : X \times X \to [0, \infty)$, where $\mathbb{J}$ is a set of positive integers. Then f has a fixed point.

Example 1.2. Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ for all $x, y \in X$. Define a mapping $f : X \to X$ by

$$f(x) = \begin{cases} \sqrt{x^2 + 1} & \text{if } x \geq 0 \\ 2x & \text{otherwise,} \end{cases}$$

and $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, \infty) \\ 0 & \text{otherwise.} \end{cases}$$

Then $f^n(x) = \sqrt{x^2 + n}$ for $x \in [0, \infty)$ and $f^n(x) = 2^n x$ otherwise. Note that for all $x, y \in X$, we can find $K \in (0, 1)$ such that

$$\inf\{\alpha(x, y)|f^n(x) - f^n(y)| : n \in \mathbb{N}\} \leq K|x - y|.$$

So f satisfies (2). However, f does not satisfy (1). To see this, let $x = -1$ and $y = 0$, then

$$\inf\{|f^n(-1) - f^n(0)| : n \in \mathbb{N}\} = \inf\{2^n + \sqrt{n} : n \in \mathbb{N}\} > K| - 1 - 0|$$

for all $K \in [0, 1)$. It is clear that f has no fixed points.

Taking $\alpha(x, y) = 1$ for all $x, y \in X$, it follows that if Conjecture 1 holds then Conjecture 2 holds as well. We note that Conjecture II is not true for infinite $\mathbb{J}$. One is led to conjecture whether Conjectures I and II are true if $\mathbb{J}$ is finite. In [8], Stein established that Conjecture I holds for the class of strongly continuous mappings and $\mathbb{J} = \{1, 2, . . , n\}$. In [4], the authors showed that Conjecture I is true if $\mathbb{J} = \{1, 2\}$ without any additional assumption on f. In this paper we show that Conjecture II is true for $\mathbb{J} = \{1, 2\}$. We also give a titling proof of our result.

2. Main results

2.1. Fixed point theorems

Theorem 2.1. *Let (X, d) be a metric space and let $f : X \to X$ satisfy the following condition:*

$$\inf\{\alpha(x, y)d(f^m(x), f^m(y)) : m \in \mathbb{N}\} \leq Kd(x, y)$$

for all $x, y \in X$, some $K \in (0, 1)$ and $\alpha : X \times X \to [0, \infty)$. Suppose that
(i) f is $\alpha-$admissible;
(ii) there exist $x_0 \in X$ with $\alpha(x_0, f(x_0)) \geq 1$ and $m \in \mathbb{N}$ such that $f^m(x_0) = x_0$.
Then x_0 is a fixed point of f.

Proof. Let $x_0 \in X$ with $\alpha(x_0, f(x_0)) \geq 1$ and $f^m(x_0) = x_0$. Define the sequence $\{x_i\}$ in X by $x_{i+1} = f(x_i)$ for $i \in \mathbb{N} \cup \{0\}$. Then it follows from $\alpha-$admissibility of f that $\alpha(x_0, x_1) \geq 1 \Rightarrow \alpha(x_1, x_2) = \alpha(f(x_0), f(x_1)) \geq 1$ and thus, by induction, $\alpha(x_i, x_{i+1}) \geq 1$ for all i. Choose L such that $K < L < 1$. Now for each $i \in \{0, 1, \ldots, m-1\}$, there is $m_i \in \mathbb{N}$ such that

$$\alpha(x_i, x_{i+1})d(f^{m_i}(x_i), f^{m_i}(x_{i+1})) \leq Ld(x_i, x_{i+1})$$

and so

$$d(f^{m_i}(x_i), f^{m_i}(x_{i+1})) \leq \alpha(x_i, x_{i+1})d(f^{m_i}(x_i), f^{m_i}(x_{i+1})) \leq Ld(x_i, x_{i+1}).$$

Since $f^m(x_0) = x_0$, following arguments as in Lemma 1 of [4], we can find a sequence $\{k_i\}$ in $\{0, 1, \ldots, m-1\}$ such that

$$d(f^{k_i}(x_0), f^{k_i+1}(x_0)) \leq Ld(x_{k_{i-1}}, x_{k_{i-1}+1}).$$

Since $k_i \in \{0, 1, \ldots, m-1\}$, we can find i and j in $\mathbb{N}$ such that $k_{i+j} = k_i$. Thus

$$\begin{aligned} d(f^{k_i}(x_0), f^{k_i+1}(x_0)) &= d(f^{k_{i+j}}(x_0), f^{k_{i+j}+1}(x_0)) \\ &\leq L^j d(x_{k_i}, x_{k_{i+1}}) \\ &= L^j d(f^{k_i}(x_0), f^{k_i+1}(x_0)). \end{aligned}$$

Since $L < 1$, we have $d(f^{k_i}(x_0), f^{k_i+1}(x_0)) = 0$ and so $f^{k_i}(x_0) = f^{k_i+1}(x_0) = f(f^{k_i}(x_0))$. That is, $f^{k_i}(x_0)$ is a fixed point of f. But $m - k_i > 0$ and $f^{m-k_i}(f^{k_i}(x_0)) = f^{k_i}(x_0)$, that is, $f^{k_i}(x_0)$ is a fixed point of f^{m-k_i}. This implies that $f^m(x_0) = f^{k_i}(x_0)$. But $f^m(x_0) = x_0$. Therefore $f^m(x_0) = f^{k_i}(x_0) = x_0$. Hence x_0 is a fixed point of f. □

Theorem 2.2. *Let (X, d) be a metric space and let $f : X \to X$ satisfy the following condition:*

$$\min\{\alpha(x, y)d(f^m(x), f^m(y)) : m \in \mathbb{J}\} \leq Kd(x, y)$$

for all $x, y \in X$ and some $K \in (0, 1)$ and $\alpha : X \times X \to [0, \infty)$, where $\mathbb{J}$ is a finite set of positive integers. Suppose that
(i) f is triangular $\alpha-$admissible;
(ii) there exists $x, z \in X$ such that $\alpha(x, f(x)) \geq 1$, $\alpha(z, f(z)) \geq 1$, $\alpha(z, x) \geq 1$ and for any $\epsilon > 0$, there is an integer $N = N(\epsilon)$ such that $d(z, f^{i+N}(x)) < \epsilon$ for any $i \in \{0\} \cup \mathbb{J}$.
Then f has a fixed point.

Proof. Let $\epsilon > 0$ and let $\delta = \frac{\epsilon}{1+K}$. Choose $N = N(\delta)$ as mentioned in the hypothesis such that $d(z, f^{i+N}(x)) < \delta$ for any $i \in \{0\} \cup \mathbb{J}$. By (i) and (ii), $\alpha(x, f^N(x)) \geq 1$ and $\alpha(z, x) \geq 1$ and so $\alpha(z, f^N(x)) \geq 1$. Also there exists $m \in \mathbb{J}$ such that

$$\alpha(z, f^N(x))d(f^m(z), f^m(f^N(x))) \leq Kd(z, f^N(x)).$$

and so

$$\begin{aligned} d(f^m(z), f^m(f^N(x))) &\leq \alpha(z, f^N(x))d(f^m(z), f^m(f^N(x))) \\ &\leq Kd(z, f^N(x)) < K\delta. \end{aligned}$$

As a result, we have

$$\begin{aligned} d(z, f^m(z)) &\leq d(z, f^{m+N}(x)) + d(f^{m+N}(x), f^m(z)) \\ &\leq \delta + K\delta = \epsilon. \end{aligned}$$

Since $\mathbb{J}$ is finite, there exists $m \in \mathbb{J}$ such that $f^m(z) = z$. By Theorem 2.1, z is a fixed point of f. □

Theorem 2.3. *Let (X, d) be a complete metric space and let $f : X \to X$ be a continuous mapping satisfying*

$$\inf\{\alpha(x, y)d(f^m(x), f^m(y)) : m \in \mathbb{N}\} \leq Kd(x, y)$$

for all $x, y \in X$ and some $K \in (0, 1)$ and $\alpha : X \times X \to [0, \infty)$. Suppose that
(i) f is $\alpha-$admissible;
(ii) there exist $x_0 \in X$ with $\alpha(x_0, f(x_0)) \geq 1$ and an increasing sequence $\{k_i\}$ of integers such that
(a) for all $i \in \mathbb{N}$, $d(f^{k_i}(x_0), f^{k_{i-1}}(x_0)) \leq CL^{k_{i-1}}$ for some $0 < L < 1$ and $C > 0$;
(b) there is a positive integer m such that $k_i - k_{i-1} = m$ for infinitely many i.
Then f has a fixed point.

Proof. It follows from (a) that the sequence $\{f^{k_i}(x_0)\}$ is Cauchy and so converges to x (say) by the completeness of X. The continuity of f further implies that the limit $\lim_{i\to\infty} f^m(f^{k_i}(x_0))$ exists. By virtue of (b), there is a cofinal subsequence $\{i_n\}$ such that $f^m(f^{k_{i_n}}(x_0)) = f^{k_{i_n+1}}(x_0)$. Thus $\{f^m(f^{k_i}(x_0))\}$ and $\{f^{k_i}(x_0))\}$ have a common cofinal subsequence and so have the same limits. As a result, we have

$$f^m(x) = f^m(\lim_{i\to\infty} f^{k_i}(x_0)) = \lim_{i\to\infty} f^m(f^{k_i}(x_0)) = \lim_{i\to\infty} f^{k_i}(x_0) = x.$$

Hence f has a periodic point and the result now follows from Theorem 2.1. □

2.2. On a tiling problem and tiling proof of a fixed point theorem

Let (X, d) be a complete metric space, let $x_0 \in X$, and let $f : X \to X$ be triangular α-admissible with $\alpha(x_0, f(x_0)) \geq 1$ and satisfy the following condition:

$$\min\{\alpha(x, y)d(f^n(x), f^n(y)) : n \in \mathbb{J}\} \leq Kd(x, y)$$

for all $x, y \in X$ and some $K \in (0, 1)$, where $\mathbb{J} = \{1, 2, ..., N\}$.

Let $T(q, q + k)$ denote a tile of length k that starts from q. Our aim is to have a usable bound for the term $d(f^q(x_0), f^{q+k}(x_0))$, which implies the sequence of iterates is Cauchy and its limit is the fixed point. The idea here is to able to tile that segment of the line that goes from q to $q + k$ with $T(q, q + l)$, a tile that starts from q and is of length l. In order to obtain a collection of titles whose metric analog is a Cauchy sequence, following [4] we have the following notion. A set of tiles $\mathcal{E}$ is called a good collection of titles if there exist $C > 0$ and $0 < L < 1$ such that for all titles $T(q, q + k)$ in $\mathcal{E}$,

$$d(f^q(x_0), f^{q+k}(x_0)) \leq CL^q.$$

Our titling problem affects fixed point theorems and consists of an initial good collection of titles, set of rules which enable us to enlarge the collection, and a goal showing that the good collection can be enlarged according to rules such that it includes a pre-determined sub-collection of titles. For instance, our objective is to enlarge the original good collection such that it contains all but finitely many adjacent titles of the same length. If a tile of length of 4 starts from 5 and covers $5-6$, $6-7$, $7-8$, $8-9$, then the next adjacent titles of length 4 starts at 9 and covers $9-10$, $10-11$, $11-12$, $12-13$. Suppose that the good collection of titles consists of adjacent titles of length 4, starting at 5, which cover all but a finite portion of the real line corresponds to showing that the sequence $\{f^5(x_0), f^9(x_0), f^{13}(x_0), ...\}$ Cauchy and thus converges. If f satisfies assumption of results of previous section, then f has a periodic point and thus has a fixed point.

We present some rules defined in [4] which lead to tiling proof of our fixed point theorem.

Rule 1. Suppose that we have a good collection $\mathcal{E}$ of tiles. Then there is a good collection $\mathcal{E}'$ with $\mathcal{E} \subset \mathcal{E}'$ having the following property: If $T(q, q+k)$ lies in $\mathcal{E}'$, then at least one of the tiles $T(q+1, q+k+1)$, $T(q+2, q+k+2)$, ..., $T(q+N, q+k+N)$ lies in $\mathcal{E}'$.

If $T(q, q+k)$ lies in $\mathcal{E}$, then $d(f^q(x_0), f^{q+k}(x_0)) \leq CL^q$. Since $\alpha(x_0, f(x_0)) \geq 1$, f is triangular α-admissible, $\alpha(f(x_0), f^2(x_0)) \geq 1$, and so $\alpha(x_0, f^2(x_0)) \geq 1$. By induction, we have $\alpha(x_0, f^k(x_0)) \geq 1$ for all $k \in \mathbb{N}$. Since f is α-admissible, we have, by induction, $\alpha(f^q(x_0), f^{q+k}(x_0)) \geq 1$. By the assumption on f, there exists $j_1 \in \{1, ..., N\}$ such that

$$\alpha(f^q(x_0), f^{q+k}(x_0))d(f^{q+j_1}(x_0), f^{q+k+j_1}(x_0)) \leq Kd(f^q(x_0), f^{q+k}(x_0))$$

and so

$$\begin{aligned} d(f^{q+j_1}(x_0), f^{q+k+j_1}(x_0)) &\leq \alpha(f^q(x_0), f^{q+k}(x_0))d(f^{q+j_1}(x_0), f^{q+k+j_1}(x_0)) \\ &\leq Kd(f^q(x_0), f^{q+k}(x_0)) \leq KCL^q. \end{aligned}$$

Continuing in this way, we can find a sequence $\{j_n : n = 1, 2, 3, ...\}$ such that $d(f^{q+j_1}(x_0), f^{q+k+j_1}(x_0)) \leq K^n CL^q$ and $1 \leq j_{n+1} - j_n \leq N$. Thus $n \leq j_n \leq nN$ which implies $n \geq \frac{j_n}{N}$ and $K^n \leq K^{\frac{j_n}{N}}$. So $d(f^{q+j_n}(x_0), f^{q+k+j_n}(x_0)) \leq CR^{q+j_n}$, where $R = \max\{K^{\frac{1}{N}}, L\}$. Consequently, the collection $\mathcal{E}'$ is obtained from $\mathcal{E}$ by adjoining all tiles of the from $T(q+i_n, q+i_n+k)$ with constants of the collections $\mathcal{E}'$, $C > 0$ and $0 < R < 1$.

Rule 2. Suppose that we have a good collection $\mathcal{E}$ of tiles. Then there is a good collection $\mathcal{E}'$ with $\mathcal{E} \subset \mathcal{E}'$ having the following property: If T_1 and T_2 are adjacent tiles in $\mathcal{E}$ with T_1 preceding T_2, then $\mathcal{E}'$ contains the tile that begins at the start of T_1 and ends at the end of T_2.

If $T(q, q+k)$ lies in $\mathcal{E}$, then $d(f^q(x_0), f^{q+k}(x_0)) \leq CL^q$. Suppose that $i < n < p$ and that $T_1 = T(i, i+n)$ and $T_2 = T(i+n, i+n+p)$. Then

$$\begin{aligned} d(f^i(x_0), f^{i+n+p}(x_0)) &\leq d(f^i(x_0), f^{i+n}(x_0)) + d(f^{i+n}(x_0), f^{i+n+p}(x_0)) \\ &\leq CL^i + CL^{i+n} = C(1+L^n)L^i \leq 2CL^i. \end{aligned}$$

The collection $\mathcal{E}'$ is obtained from $\mathcal{E}$ by adjoining all sum of two adjacent tiles in $\mathcal{E}$ with constants of the collections $\mathcal{E}'$, $2C > 0$ and $0 < L < 1$.

Rule 3. Suppose that we have a good collection $\mathcal{E}$ of tiles and that $q \in \mathbb{N}$ is fixed. Then there is a good collection $\mathcal{E}'$ with $\mathcal{E} \subset \mathcal{E}'$ having the following property: If $\mathcal{E}$ contains two tiles which either begin or end at the same point, then the longer tile is of length less than or equal to q, then $\mathcal{E}'$ contains the difference of the shorter and longer tiles.

Suppose $i < n < p$. We consider two cases. Case 1: If the titles $T(i, i+p)$ and $T(i, i+n)$ belong to $\mathcal{E}$, then

$$\begin{aligned} d(f^{i+n}(x_0), f^{i+p}(x_0)) &\leq d(f^{i+n}(x_0), f^i(x_0)) + d(f^i(x_0), f^{i+p}(x_0)) \\ &\leq CL^i + CL^i = 2CL^i = \frac{2C}{L^{n-i}}L^n \leq \frac{2C}{L^q}L^n. \end{aligned}$$

Here we assume that the longer tile is of length less than or equal to q.
Case 2: If the tiles $T(i, i+p)$ and $T(i+n, i+p)$ belong to $\mathcal{E}$, then

$$\begin{aligned} d(f^i(x_0), f^{i+n}(x_0)) &\leq d(f^i(x_0), f^{i+p}(x_0)) + d(f^{i+p}(x_0), f^{i+n}(x_0)) \\ &\leq CL^i + CL^{i+n} = C(1+L^n)L^i \leq \frac{2C^i}{L} \leq \frac{2C}{L^q}L^i. \end{aligned}$$

The collection $\mathcal{E}'$ is obtained from $\mathcal{E}$ by adjoining all differences of tiles of length less than or equal to q in $\mathcal{E}$ which begin or end at the same point with constants of the collections $\mathcal{E}'$, $\frac{2C}{L^q} > 0$ and $0 < L < 1$.

Applying the above rules, we are able to prove the following fixed point theorem. Note that any finite collection of tiles is a good collection for any constant $L < 1$, by choosing the constant C sufficiently large.

Theorem 2.4. *Let (X, d) be a complete metric space and let $f : X \to X$ satisfy the following condition:*

$$\min\{\alpha(x, y)d(f^m(x), f^m(y)) : m = 1, 2\} \leq Kd(x, y)$$

for all $x, y \in X$ and some $K \in (0, 1)$ and $\alpha : X \times X \to [0, \infty)$. Suppose that
(i) f is continuous and triangular $\alpha-$admissible;
(ii) there exist $x_0 \in X$ with $\alpha(x_0, f(x_0)) \geq 1$.
Then the sequence $\{f^q(x_0)\}$ is Cauchy and f has a fixed point (which is the limit of the sequence).

Proof. We follow [4]. Let $\mathcal{E}_0$ be the good collection consisting of tiles $T(0, 1)$ and $T(0, 2)$. Applying Rule 1 to $\mathcal{E}_0$ to get a good collection $\mathcal{E}_1$. Observe that if $T(q, q + 1)$, with $q \geq 1$, does not lie in $\mathcal{E}_1$, then both $T(q - 1, q)$ and $T(q + 1, q + 2)$ lie in $\mathcal{E}_1$. Similarly, if $T(q, q + 2)$, with $q \geq 1$, does not lie in $\mathcal{E}_1$, then both $T(q - 1, q + 1)$ and $T(q + 1, q + 3)$ lie in $\mathcal{E}_1$. Since all tiles in $\mathcal{E}_1$ are of length less than or equal to 2, we obtain a good collection $\mathcal{E}_2$ by applying Rule 3 to $\mathcal{E}_1$. We claim that $\mathcal{E}_2$ includes the tile $T(q, q + 1)$ for $q \geq 2$. If $T(q, q + 1)$ lies in $\mathcal{E}_1$, then we are done since $\mathcal{E}_1 \subset \mathcal{E}_2$. If $T(q, q + 1)$ does not lie in $\mathcal{E}_1$, then both $T(q - 1, q)$ and $T(q + 1, q + 2)$ lie in $\mathcal{E}_1$. If $T(q - 1, q + 1)$ lies in $\mathcal{E}_1$, then applying Rule 3 to tiles $T(q - 1, q + 1)$ and $T(q - 1, q)$ both belonging to $\mathcal{E}_1$ to get the tile $T(q, q + 1)$ lies in $\mathcal{E}_2$. If $T(q - 1, q + 1)$ does not lie in $\mathcal{E}_1$, then applying Rule 3 to tiles $T(q, q + 2)$ and $T(q + 1, q + 2)$ both belonging to $\mathcal{E}_1$ to get the tile $T(q, q + 2)$ lies in $\mathcal{E}_1$ and $T(q, q + 1)$ lies in $\mathcal{E}_2$. This implies that $d(f^q(x_0), f^{q+1}(x_0)) \leq CL^q$ for $q \geq 2$. Suppose $p = q + k$ for $k \geq 1$. Then

$$\begin{aligned}
d(f^q(x_0), f^p(x_0)) &\leq d(f^q(x_0), f^{q+1}(x_0)) + d(f^{q+1}(x_0), f^{q+2}(x_0)) + \\
&\quad ... + d(f^{n+k-1}(x_0), f^{n+k}(x_0)) \\
&\leq CL^q + CL^{q+1} + ... + CL^{q+k} \\
&\leq \sum_{n=q}^{\infty} CL^n.
\end{aligned}$$

Since $0 < L < 1$, the sequence $\{f^q(x_0)\}$ is Cauchy and so converges to $x \in X$. Since f is continuous, $\{f^{q+1}(x_0)\}$ converges to $f(x)$. Since $d(x, f(x)) = \lim_{n\to\infty} d(f^q(x_0), f^{q+1}(x_0)) = 0$, this implies that x is a fixed point of f. □

Taking $\alpha(x, y) = 1$ for all $x, y \in X$, we get the following corollary. Note that in this case we do not require the continuity of f instead we apply Theorem 2.2. Indeed, $\{f^q(x_0)\}$ is a Cauchy sequence as above and so converges to $x \in X$. Thus for any $\epsilon > 0$, there is an integer $N = N(\epsilon)$ such that $d(f^{i+N}(x_0), z) < \epsilon$ for all $i \in \{0\} \cup \mathbb{J}$. So, by Theorem 2.2, f has a fixed point. For uniqueness, choose L such that $K < L < 1$. If $x = f(x)$ and $y = f(y)$ with $x \neq y$. Then there exist $m \in \{1, 2\}$ such that $d(f^m(x), f^n(y)) \leq Ld(x, y)$. This implies $d(x, y) \leq Ld(x, y)$. This is a contradiction since $L < 1$.

Corollary 2.5. *[4] Let (X, d) be a complete metric space and let $f : X \to X$ satisfy the following condition:*

$$\min\{d(f^m(x), f^m(y)) : m = 1, 2\} \leq Kd(x, y)$$

for all $x, y \in X$ and some $K \in (0, 1)$. Then f has a unique fixed point.

We end the paper with the following problem.

Problem Let (X, d) be a complete metric space, let $x_0 \in X$, and let $f : X \to X$ be an α-admissible with $\alpha(x_0, f(x_0)) \geq 1$ and satisfy the following condition:

$$\min\{\alpha(x,y)d(f^n(x), f^n(y)) : n \in \mathbb{J}\} \leq Kd(x,y)$$

for all $x, y \in X$ and some $K \in (0,1)$, where $\mathbb{J} = \{1, 2, ..., N\}$. Does f have a fixed point?

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