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Meir-Keeler Type Contractions on Modular Metric Spaces

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Abstract. In this paper we introduce contraction mappings of Meir-Keeler types on modular metric spaces and investigate the existence and uniqueness of their fixed points. We give an example which demonstrates our theoretical results.

1. Introduction

One of the interesting generalizations of metric space was proposed by Chistyakov [3, 5] under the name of metric modular and modular metric spaces (or metric modular spaces). Indeed, this new notion, metric modular, generates a metric space by providing a weaker convergence called the modular convergence having a non-metrizable topology. Accordingly, the notion of modular metric space not only generalizes the concept of metric space but also extends the notions of metric linear space, and classical modular linear spaces founded by Nakano as extensions of Lebesgue, Riesz, and Orlicz spaces of integrable functions [11, 12]. For more details of the structure of metric modular and modular metric space , see e.g. [5].

Besides the technical observations in the topological structure of metric modular and modular metric space, Chistyakov [3] successfully established a fixed point theorem for contractive maps in modular metric spaces. Following this initial results, a number of authors have reported several fixed point results for certain mappings in modular metric spaces, see e.g. [1, 7, 10] and related references therein.

In this paper, our main goal is to investigate the existence and uniqueness of fixed point of Meir-Keeler types mappings in the context on modular metric spaces. To illustrate the presented results we shall consider an example.

2. A brief review on modular metric spaces

In this section, we recollect some basic definitions and fundamental results on modular metric spaces. For further details on the subject, see [3, 5].

Let X be a nonempty set, $w : (0,\infty)\times X\times X \longrightarrow [0,\infty]$. We write $w_\lambda(x, y) := w(\lambda, x, y)$ for all $\lambda > 0$, $x, y \in X$ so that $w = \{w_\lambda\}_{\lambda>0}$ for which $w_\lambda : X \times X \longrightarrow [0,\infty]$.

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Definition 2.1. *A function* $w : (0, \infty) \times X \times X \longrightarrow [0, \infty]$ *is said to be a* **(metric) modular** *on* X *if it satisfies the following three conditions for all* $\lambda, \mu > 0$*:*

a) $w_\lambda(x, y) = 0$ *if and only if* $x = y$*;*

b) $w_\lambda(x, y) = w_\lambda(y, x)$*;*

c) $w_{\lambda+\mu}(x, z) \leq w_\lambda(x, y) + w_\mu(y, z)$ *.*

for all $x, y, z \in X$*.*

A mapping w is said to be a **pseudomodular** on X, if, instead of a), the function w satisfies only

$$w_\lambda(x, x) = 0 \text{ for all } \lambda > 0. \tag{1}$$

A mapping w is called a **strict modular** on X if it satisfies (1) and for given $x, y \in X$, if there exists a number $\lambda > 0$, which may depend on x and y so that $w_\lambda(x, y) = 0$ implies $x = y$.

If for all $\lambda, \mu > 0$, a modular (pseudomodular, strict modular) w on X satisfies

$$w_{\lambda+\mu}(x, z) \leq \frac{\lambda}{\lambda + \mu} w_\lambda(x, y) + \frac{\mu}{\lambda + \mu} w_\mu(y, z). \tag{2}$$

instead of c), it is called as **convex**.

A convex modular satisfies

$$w_\lambda(x, y) \leq \frac{\mu}{\lambda} w_\mu(x, y) \leq w_\mu(x, y) \tag{3}$$

for all $x, y \in X$ and $0 < \mu \leq \lambda$, [3]. The condition (c) of Definition 2.1 implies that, for all $x, y \in X$,

$$w_{\lambda_2}(x, y) \leq w_{\lambda_1}(x, y) \tag{4}$$

holds for $\lambda_1 < \lambda_2$ for a modular w.

Definition 2.2. *[3] Let* w *be a pseudomodular on* X *and* $x_0 \in X$*. Then the sets*

$$X_w = X_w(x_0) = \{x \in X : w_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

$$X_w^* = X_w^*(x_0) = \{x \in X : \exists \lambda = \lambda(x) > 0, \text{ such that } w_\lambda(x, x_0) < \infty\}$$

are said to be modular metric spaces (around x_0*).*

It can be seen that, $X_w \subset X_w^*$ holds. If w is a metric modular on X, then the modular space X_w can be equipped with a (nontrivial) metric generated by w given by

$$d_w(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq \lambda\}$$

for any $x, y \in X_w$. If w is a convex modular on X, then $X_w = X_w^*$ holds and they are endowed with the metric

$$d_w^*(x, y) = \inf\{\lambda > 0 : w_\lambda(x, y) \leq 1\}.$$

Definition 2.3. *[3, 4] Let* X_w *and* X_w^* *be modular metric spaces.*

- *A sequence* $\{x_n\}$ *in* X_w *(or* X_w^**) is said to be* w*-convergent to* $x \in X$ *if and only if* $w_\lambda(x_n, x) \to 0$ *as* $n \to \infty$ *for some* $\lambda > 0$*. Then* x *is called the modular limit of* $\{x_n\}$*.*
- *A sequence* $\{x_n\}$ *in* X_w *is said to be* w*-Cauchy if* $w_\lambda(x_n, x_m) \to 0$ *as* $m, n \to \infty$ *for some* $\lambda > 0$*.*
- *A subset* M *of* X_w *or* X_w^* *is said to be* w*-complete if any* w*-Cauchy sequence in* M *is an* w*-convergent sequence and its* w*-limit is in* M*.*

In [4], it is shown that, if w is a pseudomodular on X, the modular metric spaces X_w and X_w^* are closed with respect to w-convergence. In addition, if w is strict, then the modular limit is unique, if it exists. Moreover, it can be easily seen that, $\lim_{n\to\infty} w_\lambda(x_n, x) = 0$ for some $\lambda > 0$ implies $\lim_{n\to\infty} w_\mu(x_n, x) = 0$ for all $\mu > \lambda > 0$.

3. Fixed point results for Meir-Keeler type contractions on modular metric spaces

In this section, we discuss contraction mappings of Meir-Keeler type and restate their definitions in the frame of modular metric spaces. After that, we give various fixed point results for such contractions and study their generalizations.

In a metric space, Meir and Keeler defined a contraction mapping and proved a fixed point theorem, which we call as Meir-Keeler fixed point theorem, generalizing Banach contraction principle, see [9].

Definition 3.1. *A mapping T on a metric space (X, d) is said to be a Meir-Keeler contraction if, given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$,*

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Longrightarrow d(Tx, Ty) < \varepsilon.$$

Theorem 3.2. *Let (X, d) be a complete metric space and let T be a Meir-Keeler contraction on X. Then T has a unique fixed point x_0, and for all $x \in X$, $T^n x \to x_0$.*

Later, it was pointed out by Ćirić [6] and Matkowski [8] that making a slight change on the contraction condition of Meir-Keeler extends the class of mappings and they introduced a class of mappings containing the class of Meir-Keeler type mappings. Then, a fixed point theorem generalizing the Meir-Keeler fixed point theorem was given, see [6, 8].

Definition 3.3. *A mapping T on a metric space (X,d) is said to be a Ćirić-Matkowski contraction if $d(Tx, Ty) < d(x, y)$ for every $x, y \in X$ with $x \neq y$ and for given $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$,*

$$\varepsilon < d(x, y) < \varepsilon + \delta \Longrightarrow d(Tx, Ty) \leq \varepsilon.$$

Theorem 3.4. *Let (X, d) be a complete metric space and let T be a Ćirić-Matkowski contraction on X. Then T has a unique fixed point x_0, and for all $x \in X$, $T^n x \to x_0$.*

Observe that the difference between the two types of contraction mappings is that equality is allowed in the contraction inequality of Ćirić-Matkowski mappings while in Meir-Keeler type mappings it is not.

Inspired from the above definitions of Meir-Keeler and Ćirić-Matkowski contractions, we define the following modular space versions of such type of mappings.

Definition 3.5. *Let X be a nonempty set, w a metric modular on X and X_w^* be a modular metric space induced by w.*

1. *A map $T : X_w^* \to X_w^*$ is called a* **Meir-Keeler type contraction** *on X_w^* if it satisfies the following condition. Given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X_w^*$*

$$\varepsilon \leq w_\lambda(x, y) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx, Ty) < \varepsilon, \tag{5}$$

whenever w is convex and

$$\varepsilon \leq w_\lambda(x, y) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx, Ty) < k\varepsilon, \tag{6}$$

whenever w is nonconvex, for some number $0 < k < 1$ and all $0 < \lambda \leq \lambda_0$ where $\lambda_0 > 0$.

2. *A map $T : X_w^* \to X_w^*$ is called a* **generalized Meir-Keeler type contraction** *on X_w^* if it satisfies the following condition.*
Given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X_w^*$

$$\varepsilon \leq M_\lambda(x, y) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx, Ty) < \varepsilon, \tag{7}$$

if w is convex and

$$\varepsilon \leq M_\lambda(x, y) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx, Ty) < k\varepsilon, \tag{8}$$

if w is nonconvex, for some number $0 < k < 1$ and all $0 < \lambda \leq \lambda_0$ where $\lambda_0 > 0$, and

$$M_\lambda(x, y) = \max\{w_\lambda(x, y), w_\lambda(x, Tx), w_\lambda(y, Ty)\}. \tag{9}$$

Definition 3.6. *Let X be a nonempty set, w be a metric modular on X and X^*_w be a modular metric space induced by w.*

1. *A map $T : X^*_w \to X^*_w$ is called a* **Ćirić-Matkowski type contraction** *on X^*_w if it satisfies the following condition.*
   Given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X^*_w$
   $$\varepsilon < w_\lambda(x, y) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx, Ty) \leq \varepsilon, \tag{10}$$
   whenever w is convex and
   $$\varepsilon < w_\lambda(x, y) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx, Ty) \leq k\varepsilon, \tag{11}$$
   whenever w is nonconvex, for some number $0 < k < 1$ and all $0 < \lambda \leq \lambda_0$ where $\lambda_0 > 0$.
2. *A map $T : X^*_w \to X^*_w$ is called a* **generalized Ćirić-Matkowski type contraction** *on X^*_w if it satisfies the following condition.*
   Given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X^*_w$
   $$\varepsilon < M_\lambda(x, y) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx, Ty) \leq \varepsilon, \tag{12}$$
   if w is convex and
   $$\varepsilon < M_\lambda(x, y) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx, Ty) \leq k\varepsilon, \tag{13}$$
   if w is nonconvex, for some number $0 < k < 1$ and all $0 < \lambda \leq \lambda_0$ where $\lambda_0 > 0$, and
   $$M_\lambda(x, y) = \max\{w_\lambda(x, y), w_\lambda(x, Tx), w_\lambda(y, Ty)\}. \tag{14}$$

Before starting to state and prove our fixed point results for the contractions defined above, we first prove some auxiliary results to be used in our further discussion on modular metric spaces.

Lemma 3.7. *Let w be a convex metric modular on X and X^*_w be a modular metric space induced by w. Let $\{x_n\}$ be a sequence in X^*_w such that*

$$w_\lambda(x_0, x_1) < \infty, \tag{15}$$

for all $\lambda > 0$. Suppose also that there exists $\lambda_0 > 0$ and $0 < k < 1$ such that

$$w_{k\lambda}(x_{n+1}, x_{n+2}) \leq w_\lambda(x_n, x_{n+1}), \tag{16}$$

for all $n \in \mathbb{N}_0$ where $0 < \lambda \leq \lambda_0$. Then, the sequence $\{x_n\} \in X^*_w$ is w-Cauchy.

Proof. We shall show that the sequence $\{x_n\} \in X^*_w$, satisfying (15) and (16), is a w-Cauchy by a constructive proof. We first, observe some estimations.

Owing to the fact that $k^n\lambda < \lambda < \lambda_0$, the inequality (16) yields

$$w_{k^{n+1}\lambda}(x_{n+1}, x_{n+2}) = w_{k(k^n\lambda)}(x_{n+1}, x_{n+2}) \leq w_{k^n\lambda}(x_n, x_{n+1}).$$

Recursively, we obtain that

$$w_{k^{n+1}\lambda}(x_{n+1}, x_{n+2}) \leq w_\lambda(x_0, x_1), \tag{17}$$

for all $n \in \mathbb{N}_0$ and $0 < \lambda < \lambda_0$. Now, by letting $\lambda_1 = (1 - k)\lambda_0 < \lambda_0$, we conclude from (15) that

$$w_{k^n\lambda_1}(x_n, x_{n+1}) \leq w_{\lambda_1}(x_0, x_1) < \infty, \tag{18}$$

for all $n \in \mathbb{N}_0$.

Regarding the above observations, we set $\lambda_l = k^l\lambda_1$ for $l = 2, 3 \ldots$.

In what follows, we shall prove that the sequence $\{x_n\} \in X_w^*$ is w-Cauchy. Since w is convex, then for any positive integers m, n with $n > m$ we have

$$\begin{aligned} w_{\lambda_c}(x_m, x_n) &\leq \frac{\lambda_m}{\lambda_c} w_{\lambda_m}(x_m, x_{m+1}) + \frac{\lambda_{m+1}}{\lambda_c} w_{\lambda_{m+1}}(x_{m+1}, x_{m+2}) \\ &+ \cdots + \frac{\lambda_{n-1}}{\lambda_c} w_{\lambda_{n-1}}(x_{n-1}, x_n), \\ &= \frac{k^m \lambda_1}{\lambda_c} w_{k^m \lambda_1}(x_m, x_{m+1}) + \frac{k^{m+1}\lambda_1}{\lambda_c} w_{k^{m+1}\lambda_1}(x_{m+1}, x_{m+2}) \\ &+ \cdots + \frac{k^{n-1}\lambda_1}{\lambda_c} w_{k^{n-1}\lambda_1}(x_{n-1}, x_n) \\ &= \frac{\lambda_1}{\lambda_c} \sum_{l=m}^{n-1} k^l w_{k^l \lambda_1}(x_l, x_{l+1}) \\ &\leq \frac{\lambda_1}{\lambda_c} w_{\lambda_1}(x_0, x_1) \sum_{l=m}^{n-1} k^l \\ &= \frac{\lambda_1}{\lambda_c} w_{\lambda_1}(x_0, x_1) k^m \frac{1-k^{n-m}}{1-k}, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \lambda_c &= \lambda_m + \lambda_{m+1} + \cdots + \lambda_{n-1} \\ &= k^m\lambda_1 + k^{m+1}\lambda_1 + \cdots + k^{n-1}\lambda_1 \\ &= \sum_{l=m}^{n-1} k^l \lambda_1 = k^m \lambda_1 \left(\frac{1-k^{n-m}}{1-k} \right). \end{aligned}$$

Since $\lambda_1 = (1-k)\lambda_0$, then $\lambda_0 = \dfrac{\lambda_1}{1-k}$ and we have $\lambda_c = k^m(1-k^{n-m})\lambda_0 < \lambda_0$. Hence, the inequalities (3) and (19) imply

$$0 \leq w_{\lambda_0}(x_m, x_n) \leq \frac{\lambda_c}{\lambda_0} w_{\lambda_c}(x_m, x_n) \leq \frac{\lambda_1}{\lambda_0} w_{\lambda_1}(x_0, x_1) k^m \frac{1-k^{n-m}}{1-k} \leq k^m w_{\lambda_1}(x_0, x_1).$$

Accordingly, we find that

$$\lim_{m\to\infty} w_{\lambda_0}(x_m, x_n) = 0,$$

which completes the proof. Thus, the sequence $\{x_n\}$ is w-Cauchy in X_w^*. □

The following Cauchy criteria is considered for a metric modular without convexity assumption.

Lemma 3.8. *Let w be a metric modular on X and X_w^* be a modular metric space induced by w. Let $\{x_n\}$ be a sequence in X_w^* such that*

$$w_\lambda(x_0, x_1) < \infty, \tag{20}$$

for all $\lambda > 0$. Suppose also that there exists $\lambda_0 > 0$ and $0 < k < 1$ such that

$$w_{k\lambda}(x_{n+1}, x_{n+2}) \leq k w_\lambda(x_n, x_{n+1}), \tag{21}$$

for all $n \in \mathbb{N}_0$ where $0 < \lambda \leq \lambda_0$. Then the sequence $\{x_n\} \in X_w^*$ is a w-Cauchy sequence.

Proof. Let the sequence $\{x_n\} \in X_w^*$ satisfy (20) and (21). Define a function v_λ as

$$v_\lambda(x, y) = \frac{w_\lambda(x, y)}{\lambda},$$

for all $\lambda > 0$ and $x, y \in X_w^*$. Then $v_\lambda(x, y)$ is a convex metric modular on X. The inequality (21) can be written in terms of v_λ as

$$k\lambda v_{k\lambda}(x_{n+1}, x_{n+2}) \le k\lambda v_\lambda(x_n, x_{n+1}), \tag{22}$$

that yields (16). Then, the conditions of the Lemma 3.7 hold and hence, the sequence $\{x_n\} \in X_w^*$ is w-Cauchy. □

One of the main drawbacks of the metric modular is that it is not necessarily finite valued. Therefore, one need to impose some finiteness conditions to guarantee the existence and uniqueness of fixed points of contraction mappings on modular metric spaces. In our theorems we use the conditions stated below.

(C1) $w_\lambda(x, Tx) < \infty$ for all $\lambda > 0$ and $x \in X_w^*$.

(C2) $w_\lambda(x, y) < \infty$ for all $\lambda > 0$ and $x, y \in X_w^*$.

In the following, we present various existence and uniqueness theorems based on the fixed points of the mappings given in Definition 3.5. Our first existence-uniqueness theorem is related with fixed points of generalized Meir-Keeler type contractive mappings on modular metric spaces induced by convex metric modular.

Theorem 3.9. *Let X be a nonempty set and let w be a strict convex metric modular on X. Suppose that X_w^* is a complete modular metric space induced by w and $T : X_w^* \to X_w^*$ is a generalized Meir-Keeler type contraction, that is, T satisfies* (7)*. If the condition (C1) is satisfied, then the mapping T has a fixed point in X_w^*. If in addition, the condition (C2) is satisfied, then the fixed point of T is unique.*

Proof. Let x_0 be any element in X_w^*. Define the sequence $\{x_n\} \in X_w^*$ as $x_n = T^n x_0$ for all $n \in \mathbb{N}$ and assume that $w_\lambda(x_n, x_{n+1}) > 0$ for all $\lambda > 0$ and $n \in \mathbb{N}_0$. Indeed, if $w_{\lambda_f}(x_{n_0}, x_{n_0+1}) = 0$ for some $\lambda_f > 0$ and $n_0 \in \mathbb{N}_0$, then x_{n_0} would be a fixed point of T.

In what follows, we shall show that $\{x_n\}$ is a Cauchy sequence. Since T satisfies the condition (7), then for any $\varepsilon > 0$ there exists $\delta > 0$ and $\lambda_0 > 0$ such that

$$\varepsilon \le M_\lambda(x_n, x_{n+1}) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx_n, Tx_{n+1}) < \varepsilon. \tag{23}$$

for all $0 < \lambda \le \lambda_0$, with $0 < k < 1$, where

$$\begin{aligned} M_\lambda(x_n, x_{n+1}) &= \max\{w_\lambda(x_n, x_{n+1}), w_\lambda(x_n, Tx_n), w_\lambda(x_{n+1}, Tx_{n+1})\} \\ &= \max\{w_\lambda(x_n, x_{n+1}), w_\lambda(x_n, x_{n+1}), w_\lambda(x_{n+1}, x_{n+2})\} \\ &= \max\{w_\lambda(x_n, x_{n+1}), w_\lambda(x_{n+1}, x_{n+2})\}. \end{aligned} \tag{24}$$

Since $w_\lambda(x_n, x_{n+1}) > 0$ for all $\lambda > 0$ and $n \in \mathbb{N}_0$, then, $\varepsilon \le M_\lambda(x_n, x_{n+1})$ for some $\varepsilon > 0$. On the other hand, the condition $w_\lambda(x, Tx) < \infty$ for all $\lambda > 0$ and $x \in X_w^*$ implies that $M_\lambda(x_n, x_{n+1}) < \infty$. Thus we have

$$\varepsilon \le M_\lambda(x_n, x_{n+1}) < \varepsilon + \delta.$$

If for some $n \in \mathbb{N}_0$ we have $M_\lambda(x_n, x_{n+1}) = w_\lambda(x_{n+1}, x_{n+2})$, then the condition (23) implies

$$\varepsilon \le w_\lambda(x_{n+1}, x_{n+2}) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(x_{n+1}, x_{n+2}) < \varepsilon,$$

and hence,

$$w_{k\lambda}(x_{n+1}, x_{n+2}) < \varepsilon \le w_\lambda(x_{n+1}, x_{n+2}),$$

which contradicts (4) since $k\lambda < \lambda$. Therefore, for all $n \in \mathbb{N}_0$ we have $M_\lambda(x_n, x_{n+1}) = w_\lambda(x_n, x_{n+1})$ and the condition (23) implies

$$\varepsilon \leq w_\lambda(x_n, x_{n+1}) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(x_{n+1}, x_{n+2}) < \varepsilon,$$

and hence,

$$w_{k\lambda}(x_{n+1}, x_{n+2}) < w_\lambda(x_n, x_{n+1}).$$

Then, by Lemma 3.7, the sequence $\{x_n\} \in X_w^*$ is a w-Cauchy sequence and since X_w^* is w-complete, it converges to a limit $x \in X_w^*$.

We will show next that this limit x is a fixed point of T. Since the sequence $\{x_n\} \in X_w^*$ converges to $x \in X_w^*$, then for any $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}_0$ such that

$$w_{\lambda_0}(x_n, x) < \frac{\varepsilon}{2}, \quad \text{for all } n \geq N_0, \tag{25}$$

and suppose that $\frac{\varepsilon}{2} \leq w_{\lambda_0}(x_{N_0-1}, x)$, that is $N_0 \in \mathbb{N}$ is the smallest number for which (25) holds. From the third condition of metric modular we have

$$w_{(k+1)\lambda_0}(x, Tx) \leq w_{\lambda_0}(x, x_{N_0}) + w_{k\lambda_0}(x_{N_0}, Tx) < \frac{\varepsilon}{2} + w_{k\lambda_0}(Tx_{N_0-1}, Tx). \tag{26}$$

On the other hand, the contractive condition (7) of the mapping implies that for this ε there exists $\delta > 0$ such that

$$\frac{\varepsilon}{2} \leq M_{\lambda_0}(x_{N_0-1}, x) < \frac{\varepsilon}{2} + \delta \Longrightarrow w_{k\lambda_0}(Tx_{N_0}, Tx) < \frac{\varepsilon}{2}. \tag{27}$$

where $0 < k < 1$, $\lambda_0 > 0$ and

$$M_{\lambda_0}(x_{N_0-1}, x) = \max\{w_{\lambda_0}(x_{N_0-1}, x), w_{\lambda_0}(x_{N_0-1}, Tx_{N_0-1}), w_{\lambda_0}(x, Tx)\}. \tag{28}$$

Clearly,

$$\frac{\varepsilon}{2} \leq w_{\lambda_0}(x_{N_0-1}, x) \leq M_{\lambda_0}(x_{N_0-1}, x) < \frac{\varepsilon}{2} + \delta,$$

which implies

$$w_{k\lambda_0}(Tx_{N_0}, Tx) < \frac{\varepsilon}{2}.$$

Then, we conclude that for each $\varepsilon > 0$,

$$w_{(k+1)\lambda_0}(x, Tx) \leq w_{\lambda_0}(x, x_{N_0}) + w_{k\lambda_0}(x_{N_0}, Tx) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \tag{29}$$

and hence, $w_{(k+1)\lambda_0}(x, Tx) = 0$. Therefore, x is a fixed point of T.

Finally, we will prove the uniqueness of the fixed point. Assume that $x \neq y$ are fixed points of T. Then, we have $w_{\lambda_0}(x, Tx) = w_{\lambda_0}(y, Ty) = 0$ and hence

$$M_{\lambda_0}(x, y) = \max\{w_{\lambda_0}(x, y), w_{\lambda_0}(x, Tx), w_{\lambda_0}(y, Ty)\} = w_{\lambda_0}(x, y) > 0.$$

If for all $\varepsilon > 0$ we have $w_{\lambda_0}(x, y) < \varepsilon$, then we would conclude $w_{\lambda_0}(x, y) = 0$ which completes the proof of uniqueness. Otherwise, since by the assumption $w_{\lambda_0}(x, y) < \infty$, then there exists $\delta > 0$ such that $\varepsilon \geq M_{\lambda_0}(x, y) = w_{\lambda_0}(x, y) < \varepsilon + \delta$ which implies $w_{k\lambda_0}(Tx, Ty) = w_{k\lambda_0}(x, y) < \varepsilon$. However, $k\lambda_0 < \lambda_0$ and hence we get $\varepsilon \leq w_{\lambda_0}(x, y) \leq w_{k\lambda_0}(x, y) < \varepsilon$ which is a contradiction. We conclude that

$$w_{k\lambda_0}(x, y) = 0,$$

that is, the fixed point of T is unique. $\square$

Our next result is an existence-uniqueness theorem for fixed points of generalized Meir-Keeler type contractive mappings on modular metric spaces induced by a nonconvex metric modular.

Theorem 3.10. *Let X be a nonempty set and let w be a strict metric modular on X. Assume that X_w^* is a complete modular metric space induced by w and $T : X_w^* \to X_w^*$ is a generalized Meir-Keeler contraction mapping, that is* (8) *holds. If the condition (C1) is satisfied, then the mapping T has a fixed point in X_w^*. If in addition, the condition (C2) is satisfied, then the fixed point of T is unique.*

Proof. The proof mimics the proof of the previous theorem. First, we take an arbitrary x_0 in X_w^* and define the sequence $\{x_n\} \in X_w^*$ as $x_n = T^n x_0$ for all $n \in \mathbb{N}_0$. Suppose that $w_\lambda(x_n, x_{n+1}) > 0$ for all $\lambda > 0$ and $n \in \mathbb{N}_0$, otherwise the existence proof would be done. We will show that $\{x_n\}$ is a Cauchy sequence. From the condition (8), for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\varepsilon \le M_\lambda(x_n, x_{n+1}) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx_n, Tx_{n+1}) < k\varepsilon. \tag{30}$$

for all $0 < \lambda \le \lambda_0$, with some $\lambda_0 > 0$ and $0 < k < 1$. Here

$$\begin{aligned} M_\lambda(x_n, x_{n+1}) &= \max\{w_\lambda(x_n, x_{n+1}), w_\lambda(x_n, Tx_n), w_\lambda(x_{n+1}, Tx_{n+1})\} \\ &= \max\{w_\lambda(x_n, x_{n+1}), w_\lambda(x_n, x_{n+1}), w_\lambda(x_{n+1}, x_{n+2})\} \\ &= \max\{w_\lambda(x_n, x_{n+1}), w_\lambda(x_{n+1}, x_{n+2})\} \end{aligned} \tag{31}$$

The condition $w_\lambda(x, Tx) < \infty$ for all $\lambda > 0$ and $x \in X_w^*$ implies that $M_\lambda(x_n, x_{n+1})$ is finite, that is,

$$\varepsilon \le M_\lambda(x_n, x_{n+1}) < \varepsilon + \delta.$$

On the other hand we cannot have $M_\lambda(x_n, x_{n+1}) = w_\lambda(x_{n+1}, x_{n+2})$ for any $n \in \mathbb{N}_0$ because this would yield

$$w_{k\lambda}(x_{n+1}, x_{n+2}) < k\varepsilon < \varepsilon \le w_\lambda(x_{n+1}, x_{n+2}),$$

which is not possible. Therefore, for all $n \in \mathbb{N}_0$ we have $M_\lambda(x_n, x_{n+1}) = w_\lambda(x_n, x_{n+1})$ and the condition (30) implies

$$\varepsilon \le w_\lambda(x_n, x_{n+1}) < \varepsilon + \delta \Longrightarrow w_{k\lambda}(x_{n+1}, x_{n+2}) < k\varepsilon,$$

and hence,

$$w_{k\lambda}(x_{n+1}, x_{n+2}) < k w_\lambda(x_n, x_{n+1}).$$

By the Lemma 3.8, the sequence $\{x_n\} \in X_w^*$ is a w-Cauchy sequence and since X_w^* is w-complete, it converges to a limit $x \in X_w^*$. The proof of the fact that the limit x of the sequence $\{x_n\}$ is a fixed point of the map T, and the uniqueness of the fixed point is similar to the proof given in Theorem 3.9, hence, we omit it. □

Further, we state an existence and uniqueness theorem for generalized Ćirić-Matkowski type contractions. The proof is quite similar to the proofs of Theorems 3.9 and 3.10, therefore, we give only the statement.

Theorem 3.11. *Let X be a nonempty set and let w be a strict metric modular on X. Let X_w^* be a complete modular metric space induced by w and $T : X_w^* \to X_w^*$ be a generalized Ćirić-Matkowski type contraction, that is, T satisfies* (12) *if w is convex and* (13) *if w is nonconvex. If the condition (C1) is satisfied, then the mapping T has a fixed point in X_w^*. If in addition, the condition (C2) is satisfied, then the fixed point of T is unique.*

These results have various consequences some of which we give below as corollaries. First, due to the fact that

$$w_\lambda(x, y) \le M_\lambda(x, y) = \max\{w_\lambda(x, y), w_\lambda(x, Tx), w_\lambda(y, Ty)\},$$

we can easily conclude the following fixed point results for Meir-Keeler or a Ćirić-Matkowski contractions on modular metric spaces.

Corollary 3.12. *Let X be a nonempty set and let w be a strict metric modular on X. Assume that X^*_w is a complete modular metric space induced by w and that $T : X^*_w \to X^*_w$ is a Meir-Keeler or a Ćirić-Matkowski type contraction mapping, that is, T satisfies either* (5) *or* (10) *if w is convex and satisfies either* (6) *or* (11) *if w is nonconvex. If the condition (C1) is satisfied, then the mapping T has a fixed point in X^*_w. If in addition, the condition (C2) is satisfied, then the fixed point of T is unique.*

Our last consequences can be observed directly from the inequality

$$\frac{w_\lambda(x,y) + w_\lambda(x,Tx) + w_\lambda(y,Ty)}{3} \le \max\{w_\lambda(x,y), w_\lambda(x,Tx), w_\lambda(y,Ty)\}.$$

Corollary 3.13. *Let X be a nonempty set and let w be a strict convex metric modular on X. Let X^*_w be a complete modular metric space induced by w and $T : X^*_w \to X^*_w$ satisfy either of the conditions (A) or (B).*
Given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X^*_w$

(A)

$$\varepsilon \le \frac{w_\lambda(x,y) + w_\lambda(x,Tx) + w_\lambda(y,yT)}{3} < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx,Ty) < \varepsilon. \tag{32}$$

(B)

$$\varepsilon < \frac{w_\lambda(x,y) + w_\lambda(x,Tx) + w_\lambda(y,yT)}{3} < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx,Ty) \le \varepsilon. \tag{33}$$

for some number $0 < k < 1$ and all $0 < \lambda \le \lambda_0$ with $\lambda_0 > 0$. If the condition (C1) is satisfied, then the mapping T has a fixed point in X^*_w. If in addition, the condition (C2) is satisfied, then the fixed point of T is unique.

Corollary 3.14. *Let X be a nonempty set and let w be a strict metric modular on X. Let X^*_w be a complete modular metric space induced by w and $T : X^*_w \to X^*_w$ satisfy either of the conditions (I) or (II).*
Given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X^*_w$

(I)

$$\varepsilon \le \frac{w_\lambda(x,y) + w_\lambda(x,Tx) + w_\lambda(y,yT)}{3} < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx,Ty) < k\varepsilon. \tag{34}$$

(II)

$$\varepsilon < \frac{w_\lambda(x,y) + w_\lambda(x,Tx) + w_\lambda(y,yT)}{3} < \varepsilon + \delta \Longrightarrow w_{k\lambda}(Tx,Ty) \le k\varepsilon. \tag{35}$$

for some number $0 < k < 1$ and all $0 < \lambda \le \lambda_0$ where $\lambda_0 > 0$. If the condition (C1) is satisfied, then the mapping T has a fixed point in X^*_w. If in addition, the condition (C2) is satisfied, then the fixed point of T is unique.

Motivated by the example of modular metric discussed in [13] we give next an example of a generalized Meir-Keeler type mapping on a modular metric space to illustrate our relevant fixed point results.

Example 3.15. *Let $X = A \cup B \subset \mathbb{R}^2$, where $A = \{(a,0) \mid 0 \le a \le 1\}$ and $B = \{(0,b) \mid 0 \le b \le 1\}$. Consider the mapping T as*

$$Tx = \begin{cases} \left(0, \dfrac{a}{2}\right) & \text{if} \quad x = (a,0) \in A \\ \left(\dfrac{b}{2}, 0\right) & \text{if} \quad x = (0,b) \in B. \end{cases}$$

Notice that if we take the metric on $\mathbb{R}^2$ as

$$d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|,$$

and $x = (1,0), y = (0,1)$, *we compute*

$$\begin{aligned} M((1,0),(0,1)) &= \max\{d((1,0),(0,1)), d((1,0),T(1,0)), d((0,1),T(0,1))\} \\ &= \max\{2, \frac{3}{2}, \frac{3}{2}\} = 2, \end{aligned}$$

and

$$d(T(1,0),T(0,1)) = d((0,\frac{1}{2}),(\frac{1}{2},0)) = 1.$$

Then, for $\varepsilon = \frac{1}{2}$ *whenever* $\frac{1}{2} \leq M((1,0),(0,1)) = 2 < \frac{1}{2} + \delta$,

$$d(T(1,0),T(0,1)) = 1 > \frac{1}{2}.$$

Thus on the metric space (X,d), T *is not a generalized Meir-Keeler contraction.*

Now, we define the modular $w_\lambda : (0,\infty] \times X \times X \to [0,\infty]$ *as*

$$w_\lambda(x,y) = \begin{cases} \dfrac{|a_1 - a_2|}{\lambda} & \text{if} \quad x = (a_1,0), y = (a_2,0) \in A \\ \dfrac{|b_1 - b_2|}{\lambda} & \text{if} \quad x = (0,b_1), y = (0,b_2) \in B \\ \dfrac{a+b}{\lambda} & \text{if} \quad x = (a,0) \in A, y = (0,b) \in B. \end{cases}$$

It can be seen that w_λ *is a strict nonconvex metric modular on* X. *In addition, the set*

$$X_w^* = \{x \in X \mid w_\lambda(x,x_0) < \infty\} = X,$$

because $w_\lambda(x,y) < \infty$ *for all* $x,y \in X$ *and* $\lambda > 0$.

Then we have three possibilities as explained below.

Case I. *Let* $x = (a_1,0), y = (a_2,0) \in A$. *Then*

$$\begin{aligned} M_\lambda(x,y) &= \max\{w_\lambda(x,y), w_\lambda(x,Tx), w_\lambda(y,Ty)\} \\ &= \max\{w_\lambda((a_1,0),(a_2,0)), w_\lambda((a_1,0),(0,\frac{a_1}{2})), w_\lambda((a_2,0),(0,\frac{a_2}{2}))\} \\ &= \max\{\frac{|a_1-a_2|}{\lambda}, \frac{a_1+\frac{a_1}{2}}{\lambda}, \frac{a_2+\frac{a_2}{2}}{\lambda}\} \\ &= \max\{\frac{|a_1-a_2|}{\lambda}, \frac{3a_1}{2\lambda}, \frac{3a_2}{2\lambda}\} \leq \frac{3}{2\lambda} \end{aligned}$$

and

$$w_{k\lambda}(Tx,Ty) = w_{k\lambda}\left(\left(0,\frac{a_1}{2}\right),\left(0,\frac{a_2}{2}\right)\right) = \frac{|a_1-a_2|}{2k\lambda} \leq \frac{1}{2k\lambda}.$$

If for any $\varepsilon > 0$

$$\varepsilon \leq M_\lambda(x,y) \leq \frac{3}{2\lambda} < \varepsilon + \delta,$$

then $2\lambda\varepsilon \leq 3$ *and hence,*

$$w_{k\lambda}(Tx,Ty) \leq \frac{1}{2k\lambda} < k\varepsilon$$

whenever $k^2 > \frac{1}{2\lambda\varepsilon} \geq \frac{1}{3}$. *Therefore, for* $\frac{1}{\sqrt{3}} < k < 1$ *the condition* (8) *holds.*

Case II. *Let* $x = (0,b_1), y = (0,b_2) \in B$. *This case mimics the case I, so, the condition* (8) *is satisfied for* $\frac{1}{\sqrt{3}} < k < 1$.

Case III. *Finally we take $x = (a, 0) \in A$, $y = (0, b) \in B$. Then*

$$\begin{aligned} M_\lambda(x,y) &= \max\{w_\lambda(x,y), w_\lambda(x,Tx), w_\lambda(y,Ty)\} \\ &= \max\{w_\lambda((a,0),(0,b)), w_\lambda((a,0),(0,\frac{a}{2})), w_\lambda((0,b),(\frac{b}{2},0))\} \\ &= \max\{\frac{a+b}{\lambda}, \frac{3a}{2\lambda}, \frac{3b}{2\lambda}\} \le \frac{2}{\lambda}. \end{aligned}$$

In this case we have

$$w_{k\lambda}(Tx,Ty) = w_{k\lambda}\left(\left(0,\frac{a}{2}\right),\left(\frac{b}{2},0\right)\right) = \frac{a+b}{2k\lambda} \le \frac{1}{k\lambda}.$$

If for any $\varepsilon > 0$

$$\varepsilon \le M_\lambda(x,y) \le \frac{2}{\lambda} < \varepsilon + \delta,$$

then $\lambda\varepsilon \le 2$ and hence,

$$w_{k\lambda}(Tx,Ty) \le \frac{1}{k\lambda} < k\varepsilon$$

whenever $k^2 > \frac{1}{\lambda\varepsilon} \ge \frac{1}{2}$. Therefore, the condition (8) holds for $\frac{1}{\sqrt{2}} < k < 1$.

Then, choosing k as $k > \max\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}\} = \frac{1}{\sqrt{2}}$ we conclude that for all $x, y \in X$, the conditions of Theorem 3.10 are satisfied for $\frac{1}{\sqrt{2}} < k < 1$ and hence, the mapping T has a unique fixed point which is $x = (0, 0)$.

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