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# A Compactification of an Orbit Space

**Dünya Karapınar**[a]

[a]*Department of Mathematics, Faculty of Science, Karadeniz Technical University, 61080 Trabzon, Turkey*

**Abstract.** Let $X$ be a Tychonoff $G-$space, $G$ be a finite discrete group and $A$ be a dense and invariant subspace of $X$. In this paper, by means of Gelfand's method, we construct a compactification of the orbit space $A/G$. As an application, we show that the set of maximal ideals of even function ring with Stone topology is a compactification of non-negative rationals.

## 1. Introduction

By a topological transformation group, we mean a triple $(X, G, \theta)$ where $G$ is a topological group, $X$ is a Tychonoff space and $\theta$ is a continuous action of $G$ on $X$. In this case, $X$ will be called a $G-$space. Using the notation $\theta_g(x) = \theta(g, x)$ for each $(g, x) \in G \times X$, we have $\theta_e = 1_X$ ($e$ denotes the identity element in $G$) and $\theta_g \circ \theta_h = \theta_{gh}$. So $g \to \theta_g$ determines a homomorphism of $G$ into the group of homeomorphisms of $X$.

A compactification $\gamma X$ of $X$ is called a $G-$compactification, if the action of $G$ on $X$ extends to $\gamma X$. $X$ may not have a $G-$compactification. For example, Megrelishvili [3] established a Tychonoff $G-$space admitting no compact Hausdorff extension. But there are some partial results for sufficient conditions for existing $G-$compactification. For instance, R. Palais [4] showed that the Alexandroff compactification for locally compact $G-$space $X$ is its $G-$compactification, and J. de Vries [7] proved that if $G$ is a locally compact group, then every Tychonoff $G-$space $X$ has a $G-$compactification and also proved that a $G-$space $X$ has a $G-$compactification if and only if the action is bounded. If $X$ has a $G-$ compactification, then it has a largest one(in the usual order of compactifications), denoted by $\beta_G X$.

The following problems in the theory of $G-$spaces are well-known:

(1) the problem of existence of a $G-$compactification, say $\gamma X$, of a Tychonoff $G-$space $X$ and a compactification $\gamma(X/G)$ of its orbit space $X/G$.

(2) in case of existence of these compactifications, the question of how $\gamma(X/G)$ is related with the orbit space $\gamma X/G$.

Srivastava [6] proved that $\beta_G X = \beta X$ and $\beta(X/G) = \beta X/G$ for finite group $G$. ($\beta X$ and $\beta(X/G)$ are Stone-Cech compactifications of $X$ and $X/G$.)

In this paper, for finite $G$, we give some useful description of the compactification of the orbit space $A/G$ for dense and invariant subspace $A$ of $X$. Among different methods for constructing compactifications, we use Gelfand's method.

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*Email address:* dunyakarapinar@ktu.edu.tr (Dünya Karapınar)

## 2. Preliminaries

In this section, we shall state a few definitions and facts about transformation groups and Gelfand's method for compactifications. We refer the reader to [1,5] for more details.

**Definition 2.1.** A subspace $A$ of a $G-$space $X$ is called invariant, if

$$\theta(G \times A) = A.$$

**Definition 2.2.** If $X$ is a $G-$space and $x \in X$, the subspace

$$G(x) = \{\theta(g, x) = gx : g \in G\}$$

is called the orbit of $x$. Let $X/G$ denote the set of all orbits $G(x)$ of a $G-$space $X$ and $\pi : X \to X/G$ denote the orbit map taking $x$ to $G(x)$. Then $X/G$ endowed with the quotient topology relative to $\pi$ is called the orbit space of $X$. The orbit map is open and continuous.

**Definition 2.3.** An action $\theta$ of a group $G$ on a space $X$ is called trivial, if $G(x) = \{x\}$ for all $x \in X$.

It is easy to see that the induced action of $G$ on $X/G$ is trivial. Now, we state some basic definitions and theorems about Gelfand's method for compactifications. We will denote continuous and bounded real-valued function rings by $C^*(X)$.

**Definition 2.4.** Let $X$ be a topological space. A subcollection $\mathcal{B}$ of subsets of $X$ is called a closed base for $X$, if each closed subset of $X$ can be written as an intersection of sets belonging to $\mathcal{B}$.

**Definition 2.5.** Let $\Omega$ be a subring of $C^*(X)$ which contains all constant functions and $M_\Omega X$ denotes the set of all maximal ideals of $\Omega$. For each $f \in \Omega$, define $S(f) = \{M \in M_\Omega X : f \in M\}$. It is easy to see that the family $\{S(f) : f \in \Omega\}$ is closed base for a topology on $M_\Omega X$ which is called the Stone topology.

**Theorem 2.6.** *$M_\Omega X$ with the Stone topology is a compact and Haussdorff space.*

*Proof.* See [5, Theorem 4.5.j]. □

**Definition 2.7.** A complete subring $\Omega$ of $C^*(X)$ with respect to the sup-norm metric is called regular, if contains all constant functions and $Z(\Omega) = \{Z(f) : f \in \Omega\}$ is a closed base for $X$ where $Z(f)$ is the zero-set of $f$.

If $x \in X$ and $\Omega$ is a regular subring of $C^*(X)$, then $M_x = \{f \in \Omega : f(x) = 0\} \in M_\Omega X$ (see[5]) Thus, we can define a continuous function $\lambda : X \to M_\Omega X$ by $\lambda(x) = M_x$.

The proof of the following theorem is given in [5].

**Theorem 2.8.** (Gelfand, [2]) *If $\Omega$ be a regular subring of $C^*(X)$ for a space $X$, then $\lambda : X \to M_\Omega X$ is a dense embedding.*

**Definition 2.9.** A compactification $\gamma X$ of a Tychonoff space $X$ is called a Gelfand compactification, if for some regular subring $\Omega$ of $C^*(X)$, $\gamma X$ and $M_\Omega X$ are equivalent compactifications of $X$ which is denoted by $\gamma X \equiv_X M_\Omega X$.

**Theorem 2.10.** *Let $X$ be a Tychonoff space and $\gamma X$ be a compactification of $X$. Then $\gamma X \equiv_X M_\Omega X$ for some regular subring $\Omega$ of $C^*(X)$.*

*Proof.* See [5, Theorem 4.5.o]. □

Thus, a compactification of a Tychonoff space is a Gelfand compactification. For example for Stone-Cech compactification, $\beta X$, of $X$, we have $\beta X \equiv_X M_\Omega X$ where $\Omega = C^*(X)$.

## 3. Main Result

From now on $X$ will be a Tychonoff $G-$space where $G$ is a finite discrete group and $A$ will be a dense and $G-$invariant subspace of $X$.

We shall prove the following lemma.

**Lemma 3.1.** *If $\mathcal{A} = \{f|_A : f \in C^*(X)\}$, then $\mathcal{A}$ is a regular subring of $C^*(A)$. Hence $M_{\mathcal{A}}A$ is a compactification of $A$.*

*Proof.* Since $Z(f) \cap A = Z(f|_A)$ for all $f \in C^*(X)$, it is easy to see that $Z(\mathcal{A})$ is a closed base for $A$. Thus it suffices to show that the subring $\mathcal{A}$ is complete with respect to the sup-norm metric. Let $(f_n|_A)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}$ and $\epsilon > 0$. There is an $N \in \mathbb{N}$ such that

$$sup\{|f_n(x) - f_m(x)| : x \in A\} < \epsilon$$

for $N \leq n \leq m$. Let $x \in X$. Since $A$ is dense subset of $X$, there is a net $(x_\lambda)$ in $A$ such that $\lim x_\lambda = x$. It can be easily seen that

$$| f_n(x) - f_m(x) |= \lim | (f_n - f_m)(x_\lambda) |\leq sup\{|f_n(x) - f_m(x)| : x \in A\} < \epsilon$$

for $N \leq n \leq m$.

Thus $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^*(X)$. Since $C^*(X)$ is complete, there exist $f \in C^*(X)$ such that $\lim f_n = f$ and also $\lim(f_n|_A) = f|_A$. So this implies that $\mathcal{A}$ is a regular subring of $C^*(A)$. □

**Proposition 3.2.** *The rings $\mathcal{A}$ and $C^*(X)$ are naturally isomorphic and hence the compactification $M_{\mathcal{A}}A$ can be identified with $\beta X$.*

*Proof.* Clearly, the map $C^*(X) \to \mathcal{A}$ given by $f \to f|_A$ preserves the ring operations and surjective by the definitions of $\mathcal{A}$. On the other hand if $f|_A = g|_A$, then the closed set $B = \{x \in X : f(x) = g(x)\}$ contains $A$ and hence $B = X$, that is, $f = g$. This proves that the ring $\mathcal{A}$ is isomorphic to $C^*(X)$.

The isomorphism of the rings $\mathcal{A} \to C^*(X)$ induces the homeomorphism $\mu : M_{\mathcal{A}}A \to M_{C^*(X)}X = \beta X$ Since the composition $A \overset{\lambda}{\hookrightarrow} M_{\mathcal{A}}A \overset{\mu}{\to} M_{C^*(X)}X$ coincides with the composition $A \hookrightarrow X \overset{\lambda}{\hookrightarrow} M_{C^*(X)}X$, we conclude that the compactification $A \overset{\lambda}{\hookrightarrow} M_{\mathcal{A}}A$ is equivalent to the compactification of $A$ represented by $A \hookrightarrow X \hookrightarrow \beta X$. In other words, the compactification $M_{\mathcal{A}}A$ can be identified with $\beta X$. □

It follows from the above proposition 3.2 that the proof of the following proposition can be regarded as another proof of the known result that $\beta X$ is a $G-$compactification, if $G$ is finite ([6]).

**Proposition 3.3.** *$M_{\mathcal{A}}A$ is a $G-$compactification of $A$.*

*Proof.* Let $P$ be a maximal ideal of $\mathcal{A}$ and for $g \in G$, define the set $gP = \{f|_A \circ \theta_{g^{-1}} : f|_A \in P\}$ where $\theta_{g^{-1}} : A \to A$ is defined by $\theta_{g^{-1}}(x) = g^{-1}x$.

First we show that $gP$ is a maximal ideal of $\mathcal{A}$. If $f|_A \circ \theta_{g^{-1}}, h|_A \circ \theta_{g^{-1}} \in gP$, then

$$f|_A \circ \theta_{g^{-1}} - h|_A \circ \theta_{g^{-1}} = (f - h)|_A \circ \theta_{g^{-1}} \in gP$$

On the other hand, let $f|_A \circ \theta_{g^{-1}} \in gP$ and $h|_A \in \mathcal{A}$. Define $h' = h \circ \theta_g$. Then it is easy to see that

$$(h|_A)(f|_A \circ \theta_{g^{-1}}) = (h'|_A f|_A) \circ \theta_{g^{-1}} \in gP$$

Now, let $I$ be an ideal of $\mathcal{A}$ such that $gP \subseteq I \subseteq \mathcal{A}$. Then the ideal

$$g^{-1}I = \{f|_A \circ \theta_g : f|_A \in I\}$$

of $\mathcal{A}$ satisfies the relation

$$P \subseteq g^{-1}I \subseteq \mathcal{A}$$

Since $P$ is a maximal ideal of $\mathcal{A}$,

$$g^{-1}I = P \text{ or } g^{-1}I = \mathcal{A}$$

This implies that

$$I = gP \text{ or } I = \mathcal{A}$$

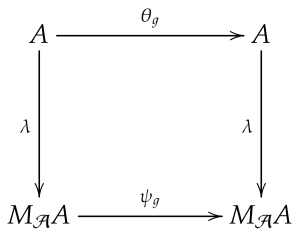
Hence $gP$ is a maximal ideal of $\mathcal{A}$.

An action $\psi$ of $G$ on $M_{\mathcal{A}}A$ is defined by $\psi(g, P) = gP$. Clearly $eP = P$ and $g(hP) = (gh)P$ where $e$ is the identity in $G$ and $g, h \in G$.

Since

$$\psi^{-1}(S(f)) = \{(g, P) : gP \in S(f)\} = \bigcup_{g \in G} \{g\} \times S(gf)$$

which is closed, the action $\psi$ is continuous. On the other hand, it can be checked that the following diagram is commutative




This implies that $M_{\mathcal{A}}A$ is a $G-$compactification of $A$. □

**Remark 3.4.** $M_{\mathcal{A}}A$ may be different from $\beta A$. For example, take $X = \mathbb{R}$ and $A = \mathbb{Q}$. Then $\mathcal{A} \neq C^*(\mathbb{Q})$ and $M_{\mathcal{A}}\mathbb{Q} \neq \beta\mathbb{Q}$.

Note that, in view of the proposition 3.2, $M_{\mathcal{A}}\mathbb{Q} = \beta\mathbb{R}$. Thus $M_{\mathcal{A}}\mathbb{Q} \neq \beta\mathbb{Q}$ corresponds to $\beta\mathbb{R} \neq \beta\mathbb{Q}$.

**Lemma 3.5.** *Let $\mathcal{A}' = \{f \in \mathcal{A} : f$ takes a constant value for each orbit$\}$. Then $\mathcal{A}'$ is a complete subring of $C^*(A)$.*

*Proof.* Let $(f_n|_A)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}'$. It follows, by Lemma 3.1, that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C^*(X)$ and from the completeness of $\mathcal{A}$ there exists $f \in \mathcal{A}$ such that $\lim f_n|_A = f$. Since each $f_n|_A$ has constant value on orbits, $f_n(gx) = f_n(hx)$ for each $x \in A$ and $g, h \in G$. Therefore

$$f(gx) = \lim f_n(gx) = \lim f_n(hx) = f(hx)$$

which implies completeness of $\mathcal{A}'$. □

**Remark 3.6.** Observe that $Z(\mathcal{A}')$ can be a closed base for $A$ only in the case of trivial action of $G$.

Indeed, every set of $Z(\mathcal{A}')$ is a $G-$invariant subspace of $A$ and hence, if $Z(\mathcal{A}')$ is a closed base for $A$, every closed subset of invariant, in particular, every one point set is invariant, that is the action of $G$ is trivial.

Since the orbit space $A/G$ is dense subspace of the orbit space $X/G$, by the Lemma 3.1, the space $M_{\mathcal{B}}(A/G)$ with the Stone topology is a compactification of the orbit space $A/G$ where

$$\mathcal{B} = \{f\,|_{A/G} : f \in C^*(X/G)\}$$

**Lemma 3.7.** *The rings $\mathcal{B}$ and $\mathcal{A}'$ are naturally isomorphic.*

*Proof.* Since $f|_{G(x)}$ is constant for each $f \in \mathcal{A}'$ and for each $x \in A$, there exists a unique $h_f \in C^*(A/G)$ such that $f = h_f \circ \pi$ where $\pi$ is the orbit map (i.e. $h_f(G(x)) = f(x)$ for each $x \in A$). Since $A$ is dense subspace of $X$, for each $x \in X$ there exists a net $(x_\lambda)$ in $A$ such that $\lim x_\lambda = x$. Furthermore the nets $(f(gx_\lambda))$ and $(f(hx_\lambda))$ are equal for each $g, h \in G$. So if we have $f = \widetilde{f}|_A$ for some $\widetilde{f} \in C^*(X)$, then

$$\widetilde{f}(gx) = \lim f(gx_\lambda) = \lim f(hx_\lambda) = \widetilde{f}(hx)$$

which implies that $\widetilde{f}$ takes a constant value on each orbit as well as $f$; consequently, $\widetilde{f}$ induces a unique map $f' \in C^*(X/G)$ and $h_f = f'|_{A/G}$. This implies that the map $\varphi : \mathcal{A}' \to \mathcal{B}$ given by $f \to h_f$ is well defined and also obviously preserves the ring operations. It is easy to see that $\varphi$ is isomorphism because it has inverse $\psi : \mathcal{B} \to \mathcal{A}'$ is given by $f \to f \circ \pi$ □

After these preparations, we are going to prove the following main theorem.

**Theorem 3.8.** *$M_{\mathcal{B}}(A/G)$ is homeomorphic to $M_{\mathcal{A}'}A$ .*

*Proof.* The above ring isomorphism $\varphi : \mathcal{A}' \to \mathcal{B}$ induces, $\overline{\varphi} : M_{\mathcal{A}'}A \to M_{\mathcal{B}}(A/G)$, defined by $\overline{\varphi}(P) = \varphi(P) = \{h_f : f \in P\}$. Since

$$\overline{\varphi}^{-1}(S(f)) = \{P : \varphi(P) \in S(f)\} = \{P : f \in \varphi(P)\} = \{P : f \circ \pi \in P\} = S(f \circ \pi),$$

We conclude that $\overline{\varphi}$ is continuous.

Similarly, the inverse isomorphism $\psi : \mathcal{B} \to \mathcal{A}'$ induces $\overline{\psi} : M_{\mathcal{B}}(A/G) \to M_{\mathcal{A}'}A$ defined by

$$\overline{\psi}(P) = \psi(P) = \{f \circ \pi : f \in P\}$$

Moreover for each $f \in \mathcal{A}'$

$$\overline{\psi}^{-1}(S(f)) = \{P \in M_{\mathcal{B}}(A/G) : \psi(P) \in S(f)\} = \{P \in M_{\mathcal{B}}(A/G) : h_f \in P\} = S(h_f)$$

which implies the continuity of $\overline{\psi}$ and it is easily checked that

$$\overline{\varphi}(\overline{\psi}(P)) = P \text{ for each } P \in M_{\mathcal{B}}(A/G) \text{ and } \overline{\psi}(\overline{\varphi}(P)) = P \text{ for each } P \in M_{\mathcal{A}'}A.$$

□

**Corollary 3.9.** *If we take $A = X$, we have $\beta(X/G) = M_{X'}X$ where*

$$X' = \{f \in C^*(X) : f \text{ takes a constant value for each orbit}\}$$

The next application of Theorem 3.8 shows the set of maximal ideals of even function ring with Stone topology is a compactification of non-negative rationals.

**Example 3.10.** The antipodal map on $\mathbb{R}$ viewed as an action of the group $G = \mathbb{Z}_2$ on $\mathbb{R}$. If $A = \mathbb{Q}$ (rationals), then

$$\mathcal{A}' = \{f|_{\mathbb{Q}} : f \in C^*(\mathbb{R}) \text{ and } f \text{ is even function}\}$$

Thus

$$M_{\mathcal{A}'}\mathbb{Q} = \{P : P \text{ is maximal ideal of } \mathcal{A}'\}$$

is a compactification of the orbit space $A/G = \mathbb{Q}/\mathbb{Z}_2 = \mathbb{Q}^+$ (non-negative rationals.)

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