Filomat 32:10 (2018), 3717–3724
https://doi.org/10.2298/FIL1810717M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Nonreversible Trees Having a Removable Edge

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Abstract. A relational structure is said to be reversible iff every bijective homomorphism (condensation) of that structure is an automorphism. In the case of a binary structure $\mathbb{X} = \langle X, \rho \rangle$, that is equivalent to the following statement: whenever we remove finite or infinite number of edges from $\mathbb{X}$, thus obtaining the structure $\mathbb{X}'$, we have that $\mathbb{X}' \ncong \mathbb{X}$. In this paper, we prove that if a nonreversible tree $\mathbb{X} = \langle X, \rho \rangle$ has a removable edge (i.e. if there is $\langle x, y \rangle \in \rho$ such that $\langle X, \rho \rangle \cong \langle X, \rho \setminus \{\langle x, y \rangle\} \rangle$, then it has infinitely many removable edges. We also show that the same is not true for arbitrary binary structure by constructing nonreversible digraphs having exactly n removable edges, for $n \in \mathbb{N}$.

1. Introduction

A relational structure $\mathbb{X}$ is said to be reversible iff every condensation (bijective homomorphism) $f : \mathbb{X} \to \mathbb{X}$ is an automorphism. If $L_b = \langle R \rangle$, $\text{ar}(R) = 2$, is the binary language, we have that an L_b-structure $\mathbb{X}$ is reversible iff whenever we remove finite or infinite number of edges from $\mathbb{X}$, thus obtaining the structure $\mathbb{X}'$, we have that $\mathbb{X}' \ncong \mathbb{X}$. The class of reversible structures includes linear orders, Boolean lattices (algebras), well founded posets with finite levels [3, 4], tournaments, Henson graphs [7], Henson digraphs [5] and monomorphic structures [6]. Reversible structures have *the Cantor-Schröder-Bernstein property for condensations*, i.e. whenever $\mathbb{X}$ is reversible and $\mathbb{Y}$ is arbitrary structure, and there are condensations $f : \mathbb{X} \to \mathbb{Y}$, and $g : \mathbb{Y} \to \mathbb{Y}$, we have $\mathbb{X} \cong \mathbb{Y}$.

We say that an edge $\langle x, y \rangle \in \rho$ is *removable* in the structure $\mathbb{X} = \langle X, \rho \rangle$ if and only if $\langle X, \rho \rangle \cong \langle X, \rho \setminus \{\langle x, y \rangle\} \rangle$. Then clearly $\mathbb{X}$ is not reversible. Not all nonreversible structures have a removable edge, for example nonreversible graphs trivially do not, because they are irreflexive and symmetric. Moreover, using the result of Dushnik and Miller [1] on the existance of embedding-rigid linear orders, or the result of Vopěnka, Pultr and Hedrlín [12] on the existance of endomorphism-rigid structures, one can easily construct nonreversible structure $\mathbb{X} = \langle X, \rho \rangle$ such that $\langle X, \rho \rangle \ncong \langle X, \rho \setminus \sigma \rangle$ for any finite $\sigma \subseteq \rho$. Among those nonreversible structures that have a removable edge, not all have infinitely many removable edges (see Example 2.2). The class of those nonreversible structures that have infinitely many removable edges is of particular interest, because it contains the class of all nonreversible structures that have the Cantor-Schröder-Bernstein property for condensations (the so called *weakly reversible* structures). For more information on weak reversibility, see [7].

2010 *Mathematics Subject Classification*. Primary 06A07; Secondary 05C40
Keywords. reversibility, binary relational structure, disconnected structure, partial order, tree.
Received: 10 April 2017; Accepted: 11 November 2017
Communicated by Vladimir Rakočević
Research supported by Project 174006 of the Ministry of Education, Science and Technological Development, Republic of Serbia
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In this article, we prove that if a tree has a removable edge, then it has infinitely many removable edges. In the following paragraph, we introduce notation that will be used throughout the paper. In Section 2 one can find definitions and facts about trees and disconnected binary structures, that play the central role in the paper. Section 3 contains some statements on the properties of the set of removable edges of a given tree. These results will be used in Section 4, that contains the main result of the paper.

Notation. If $\mathbb{X}$ and $\mathbb{Y}$ are relational structures, $\mathrm{Iso}(\mathbb{X},\mathbb{Y})$, $\mathrm{Cond}(\mathbb{X},\mathbb{Y})$, $\mathrm{Mono}(\mathbb{X},\mathbb{Y})$ and $\mathrm{Emb}(\mathbb{X},\mathbb{Y})$, will denote the set of all isomorphisms, condensations (bijective homomorphisms), monomorphisms and embeddings from $\mathbb{X}$ to $\mathbb{Y}$ respectively. $\mathrm{Iso}(\mathbb{X},\mathbb{X}) = \mathrm{Aut}(\mathbb{X})$ is the set of automorphisms of $\mathbb{X}$, instead of $\mathrm{Cond}(\mathbb{X},\mathbb{X})$ we shortly write $\mathrm{Cond}(\mathbb{X})$, etc.

For a function $f : X \to X$ and $\rho \subseteq X \times X$, instead of $(f \times f)[\rho]$ we write shortly $f[\rho]$. Thus, for binary structures $\mathbb{X} = \langle X, \rho\rangle$ and $\mathbb{Y} = \langle Y, \sigma\rangle$, we have:

$$\mathrm{Cond}(\mathbb{X},\mathbb{Y}) = \{f \in \mathrm{Sym}(X,Y) : f[\rho] \subseteq \sigma\}, \tag{1}$$

and

$$\mathrm{Iso}(\mathbb{X},\mathbb{Y}) = \{f \in \mathrm{Sym}(X,Y) : f[\rho] = \sigma\}. \tag{2}$$

For relational structures $\langle X, \rho\rangle$ and $\langle X, \sigma\rangle$, we often write shortly $\rho \cong \sigma$ instead of $\langle X, \rho\rangle \cong \langle X, \sigma\rangle$. According to that, for a binary structure $\mathbb{X} = \langle X, \rho\rangle$, we define the corresponding set of removable edges, i.e.

$$\rho^* := \Big\{\langle x, y\rangle \in \rho : \rho \cong \rho \setminus \{\langle x, y\rangle\}\Big\}. \tag{3}$$

In the sequel, when we write $\rho\restriction_A$ or $(\rho\restriction_A)\setminus\sigma$, we assume that the domain of the corresponding structure is the set A. According to that, we have

$$\rho\restriction_A \cong (\rho\restriction_B)\setminus\sigma \iff \langle A, \rho\restriction_A\rangle \cong \langle B, (\rho\restriction_B)\setminus\sigma\rangle,$$

and

$$(\rho\restriction_A)^* = \Big\{\langle x, y\rangle \in \rho\restriction_A : \langle A, \rho\restriction_A\rangle \cong \langle A, (\rho\restriction_A)\setminus\{\langle x, y\rangle\}\rangle\Big\}. \tag{4}$$

Also, if for a given structure $\mathbb{X} = \langle X, \rho\rangle$ we have $Y, Z \subseteq X$, then we shall often, by slightly abusing notation, instead of $\mathbb{X} \cong \langle Y, \rho\restriction_Y\rangle$ and $\langle Y, \rho\restriction_Y\rangle \cong \langle Z, \rho\restriction_Z\rangle$, write shortly $\mathbb{X} \cong Y$ and $Y \cong Z$, respectively. And if $\mathbb{X} = \langle X, \rho\rangle$ and $\mathbb{Y} = \langle Y, \sigma\rangle$ are trees, $x \in X$, $y \in Y$, instead of $\langle x\!\uparrow_{\mathbb{X}}, \rho\restriction_{x\uparrow_{\mathbb{X}}}\rangle \cong \langle y\!\uparrow_{\mathbb{Y}}, \sigma\restriction_{y\uparrow_{\mathbb{Y}}}\rangle$ we shortly write $x\!\uparrow_{\mathbb{X}} \cong y\!\uparrow_{\mathbb{Y}}$ (see the next section).

2. Preliminaries

Trees. Let $\mathbb{X} = \langle X, \rho\rangle \in \mathrm{Mod}_{L_b}$ be a tree, i.e. $\rho = <$ is irreflexive and transitive, and for each $x \in X$ the set $x\!\downarrow_{\mathbb{X}}$ is well-ordered, where for $x \in X$:

$$x\!\downarrow_{\mathbb{X}} := \{y \in X : y < x\}, \qquad x\!\uparrow_{\mathbb{X}} := \{y \in X : x < y\},$$

$$x\,\bar{\downarrow}_{\mathbb{X}} := \{y \in X : y \leq x\}, \qquad x\,\bar{\uparrow}_{\mathbb{X}} := \{y \in X : x \leq y\},$$

$$x\!\updownarrow_{\mathbb{X}} := x\,\bar{\downarrow}_{\mathbb{X}} \cup x\,\bar{\uparrow}_{\mathbb{X}}.$$

For $Y \subseteq X$ we have $Y\!\downarrow_{\mathbb{X}} := \bigcup_{x\in Y} x\!\downarrow_{\mathbb{X}}$, and similarly for $Y\!\uparrow_{\mathbb{X}}$, $Y\,\bar{\downarrow}_{\mathbb{X}}$ and $Y\,\bar{\uparrow}_{\mathbb{X}}$. If the tree $\mathbb{X}$ is clear from the context, we write only $x\!\downarrow$ instead of $x\!\downarrow_{\mathbb{X}}$, etc.

For $x \in X$, the height function of x is given by $\mathrm{ht}_{\mathbb{X}}(x) := \mathrm{otp}(x\!\downarrow) \in \mathrm{Ord}$, and for $\alpha \in \mathrm{Ord}$ the corresponding level is defined by

$$\mathrm{Level}_{\mathbb{X}}(\alpha) := \{x \in X : \mathrm{ht}_{\mathbb{X}}(x) = \alpha\},$$

where $\mathrm{Root}_{\mathbb{X}} := \mathrm{Level}_{\mathbb{X}}(0)$. Also,

$$\mathrm{Level}'_{\mathbb{X}}(\alpha) := \left\{x \in \mathrm{Level}_{\mathbb{X}}(\alpha) : |\{y \in \mathrm{Level}_{\mathbb{X}}(\alpha) : x\uparrow \cong y\uparrow\}| \geq \omega\right\}, \tag{5}$$

and $\mathrm{Root}'_{\mathbb{X}} := \mathrm{Level}'_{\mathbb{X}}(0)$. If $\mathrm{ht}_{\mathbb{X}}(x)$ is a successor ordinal, then $\mathrm{pred}_{\mathbb{X}}(x) := \max(x\downarrow)$. For $\alpha \in \mathrm{Ord}$ we define

$$X_{<\alpha} := \bigcup_{\beta<\alpha} \mathrm{Level}_{\mathbb{X}}(\beta). \tag{6}$$

The fragments $X_{\leq\alpha}$, $X_{>\alpha}$ and $X_{\geq\alpha}$ are defined in the similar way.

The equivalence relation $\sim_{\mathrm{v}}$ on X is defined in the following way:

$$x \sim_{\mathrm{v}} y \Leftrightarrow x\downarrow = y\downarrow .$$

The corresponding equivalence class of x will be denoted by $[x]_{\sim_{\mathrm{v}}}$.

Disconnected binary structures. If $\mathbb{X} = \langle X, \rho\rangle$ is a binary structure, then the transitive closure ρ_{rst} of the relation $\rho_{rs} = \Delta_X \cup \rho \cup \rho^{-1}$ (given by $x\rho_{rst}y$ iff there are $n \in \mathbb{N}$ and $z_0 = x, z_1, \dots, z_n = y$ such that $z_i\ \rho_{rs}\ z_{i+1}$, for each $i < n$) is the minimal equivalence relation on X containing ρ. The corresponding equivalence classes $[x]$, $x \in X$, are called the *connectivity components* of $\mathbb{X}$ and the structure $\mathbb{X}$ is called *connected* iff $|X/\rho_{rst}| = 1$. For example, the connectivity components of a tree $\mathbb{X}$ are $x\uparrow$, $x \in \mathrm{Root}_{\mathbb{X}}$. A tree $\mathbb{X}$ is connected, if and only if $|\mathrm{Root}_{\mathbb{X}}| = 1$ (the so called *rooted trees*).

If $\mathbb{X}_i = \langle X_i, \rho_i\rangle$, $i \in I$, are connected binary structures and $X_i \cap X_j = \emptyset$, for different $i, j \in I$, then the structure $\bigcup_{i\in I} \mathbb{X}_i = \langle \bigcup_{i\in I} X_i, \bigcup_{i\in I} \rho_i\rangle$ is the *disjoint union* of the structures $\mathbb{X}_i$, $i \in I$, and the structures $\mathbb{X}_i$, $i \in I$, are its components.

The following characterization of nonreversible disconnected binary structures was obtained in [9].

Proposition 2.1. *Let $\mathbb{X}_i$, $i \in I$, be pairwise disjoint and connected L_b-structures, and let $\mathbb{X} = \langle X, \rho\rangle = \bigcup_{i\in I} \mathbb{X}_i$. Then the following conditions are equivalent:*

(a) ρ is not reversible, i.e. there exists $g \in \mathrm{Cond}(\mathbb{X})$ such that $g[\rho] \subsetneq \rho$;

(b) There exist a surjection $f : I \to I$, and $g_i \in \mathrm{Mono}(\mathbb{X}_i, \mathbb{X}_{f(i)})$, for $i \in I$, where $\{g_i[X_i] : i \in f^{-1}[\{j\}]\}$ is a partition of X_j for all $j \in I$, such that: f is not an injection, or $g_i \notin \mathrm{Iso}(\mathbb{X}_i, \mathbb{X}_{f(i)})$ for some $i \in I$.

Then, $g = \bigcup_{i\in I} g_i$.

Example 2.2. *Let $\mathbb{Y}_n := \langle \mathbb{Z}, \sigma_n\rangle$, where $\sigma_n := \{\langle k, k+1\rangle : k \in \mathbb{Z}\} \cup \{\langle nl, nl\rangle : l \in \omega\}$, for $n \in \mathbb{N}$. The structures $\mathbb{Y}_n$ are connected, rigid, nonreversible, and we have that $\sigma_n^* = \{\langle 0, 0\rangle\}$, for all $n \in \mathbb{N}$. Then for the disjoint union $\mathbb{X}_m = \langle X_m, \rho_m\rangle = \bigcup_{n=1}^m \mathbb{Y}_n$ we have that $|\rho_m^*| = m$, for all $m \in \mathbb{N}$. We see that there exist both connected and disconnected nonreversible structures that have finitely many removable edges.*

3. Properties of the set of removable edges of a tree

Proposition 3.1. *Let $\mathbb{X} = \langle X, \rho\rangle \in \mathrm{Mod}_{L_b}$ be a tree.*

(a) If $\langle x, y\rangle \in \rho^*$, then $\mathrm{ht}_{\mathbb{X}}(y) = \mathrm{ht}_{\mathbb{X}}(x) + 1$ and $y \in \mathrm{Max}\,\mathbb{X}$;

(b) For every $\alpha \in \mathrm{Ord}$ we have $\rho^* \restriction_{X_{\geq\alpha}} \subseteq (\rho \restriction_{X_{\geq\alpha}})^*$;

(c) If $|\mathrm{Root}_{\mathbb{X}}| = 1$ then $\rho^* = (\rho \restriction_{X_{\geq 1}})^*$;

(d) For all $x \in X$ we have $(\rho \restriction_{[x]_{\sim_{\mathrm{v}}}\uparrow})^* \subseteq \rho^*$;

(e) For all $x \in X$ we have $(\rho \restriction_{x\uparrow})^* \subseteq \rho^*$.

Proof.

(a) If $\langle x, y\rangle \in \rho^* \subseteq \rho$, then $\mathrm{ht}_{\mathbb{X}}(x) < \mathrm{ht}_{\mathbb{X}}(y)$, that is $\mathrm{ht}_{\mathbb{X}}(x)+1 \leq \mathrm{ht}_{\mathbb{X}}(y)$. If we assume that $\mathrm{ht}_{\mathbb{X}}(x)+1 < \mathrm{ht}_{\mathbb{X}}(y)$, then there is $z \in y\downarrow$ such that $\mathrm{ht}_{\mathbb{X}}(z) = \mathrm{ht}_{\mathbb{X}}(x) + 1$. Then $x < z < y$, and we conclude that $\rho \setminus \{\langle x, y\rangle\}$ is not transitive, which is a contradiction. Let us now assume that $y \notin \mathrm{Max}\,\mathbb{X}$, and let $y < w$, for some $w \in X$. Then $x, y \in w\downarrow_{\mathbb{X}'}$ are incomparable in $\mathbb{X}' := \langle X, \rho \setminus \{\langle x, y\rangle\}\rangle$, which is also a contradiction.

(b) Take $\langle x, y\rangle \in \rho^* \upharpoonright_{X_{\geq\alpha}}$. Then $\mathbb{X} = \langle X, \rho\rangle \cong \langle X, \rho \setminus \{\langle x, y\rangle\}\rangle = \mathbb{X}'$, which implies that

$$\langle X_{\geq\alpha}, \rho \upharpoonright_{X_{\geq\alpha}}\rangle \cong \langle (X_{\geq\alpha})_{\mathbb{X}'}, (\rho \setminus \{\langle x, y\rangle\}) \upharpoonright_{(X_{\geq\alpha})_{\mathbb{X}'}}\rangle.$$

Since $(X_{\geq\alpha})_{\mathbb{X}'} = X_{\geq\alpha}$ and $(\rho \setminus \{\langle x, y\rangle\}) \upharpoonright_{X_{\geq\alpha}} = (\rho \upharpoonright_{X_{\geq\alpha}}) \setminus \{\langle x, y\rangle\}$, we have proven (b).

(c) If $|\mathrm{Root}_{\mathbb{X}}| = 1$, then by (b) we have $\rho^* = \rho^* \upharpoonright_{X_{\geq 1}} \subseteq (\rho \upharpoonright_{X_{\geq 1}})^*$. Now take $\langle x, y\rangle \in (\rho \upharpoonright_{X_{\geq 1}})^*$. Then $\langle X_{\geq 1}, \rho \upharpoonright_{X_{\geq 1}}\rangle \cong \langle X_{\geq 1}, (\rho \upharpoonright_{X_{\geq 1}}) \setminus \{\langle x, y\rangle\}\rangle$, that is $\langle X_{\geq 1}, \rho \upharpoonright_{X_{\geq 1}}\rangle \cong \langle X_{\geq 1}, (\rho \setminus \{\langle x, y\rangle\}) \upharpoonright_{X_{\geq 1}}\rangle$. Since $|\mathrm{Root}_{\mathbb{X}}| = 1$, this implies that $\langle X, \rho\rangle \cong \langle X, \rho \setminus \{\langle x, y\rangle\}\rangle$, which proves the other inclusion.

(d) Take $\langle y, z\rangle \in (\rho \upharpoonright_{[x]_{\sim_v}\uparrow})^*$. Then there is

$$g \in \mathrm{Iso}\Big(\Big\langle [x]_{\sim_v}\uparrow, \rho \upharpoonright_{[x]_{\sim_v}\uparrow}\Big\rangle, \Big\langle [x]_{\sim_v}\uparrow, (\rho \upharpoonright_{[x]_{\sim_v}\uparrow}) \setminus \{y, z\}\Big\rangle\Big).$$

Since for any $u, v \in [x]_{\sim_v}$ we have $u\downarrow = v\downarrow$, we easily conclude that

$$f := g \cup \mathrm{id}_{X\setminus[x]_{\sim_v}\uparrow} \in \mathrm{Iso}\Big(\Big\langle X, \rho\Big\rangle, \Big\langle X, \rho \setminus \{\langle y, z\rangle\}\Big\rangle\Big),$$

which means that $\langle y, z\rangle \in \rho^*$.

(e) If $x\uparrow = \{x\}$, this is trivial. And if $y \in \mathrm{Min}(x\uparrow\setminus\{x\})$, then $(x\uparrow)_{\geq 1} = [y]_{\sim_v}\uparrow$, and (e) follows from (c) and (d). □

Lemma 3.2. *Let $\mathbb{X} = \langle X, \rho\rangle \in \mathrm{Mod}_{L_b}$ be a tree. Then we have:*

$$\rho^* = \bigcup_{x\in\mathrm{Root}_{\mathbb{X}}} (\rho \upharpoonright_{x\uparrow})^* \ \dot\cup\ (\rho' \cup \rho''), \tag{7}$$

where

$$\rho' := \bigcup_{x\in\mathrm{Root}_{\mathbb{X}}} \Big\{\langle x, y\rangle : y \in \mathrm{Max}(x\uparrow) \cap \mathrm{Level}_{\mathbb{X}}(1) \ \wedge\ \exists z \in \mathrm{Root}'_{\mathbb{X}}\ z\uparrow \cong \{y\} \ \wedge\ x\uparrow \cong x\uparrow\setminus\{y\}\Big\},$$

and

$$\rho'' := \bigcup_{x\in\mathrm{Root}'_{\mathbb{X}}} \Big\{\langle \mathrm{pred}_{\mathbb{X}}(y), y\rangle \notin (\rho \upharpoonright_{x\uparrow})^* : y \in \mathrm{Max}(x\uparrow) \ \wedge\ \exists Z \in [\mathrm{Root}'_{\mathbb{X}}]^{\leq 2}\ (\rho \upharpoonright_{x\uparrow}) \setminus \{\langle \mathrm{pred}_{\mathbb{X}}(y), y\rangle\} \cong \rho \upharpoonright_{Z\uparrow}\Big\}.$$

Proof. Since by Proposition 3.1 (c) $(\rho \upharpoonright_{x\uparrow})^* = (\rho \upharpoonright_{x\uparrow\setminus\{x\}})^*$, we conclude that the first union is disjoint.

($\supseteq$) From Proposition 3.1 (e) it follows that $\bigcup_{x\in\mathrm{Root}_{\mathbb{X}}}(\rho \upharpoonright_{x\uparrow})^* \subseteq \rho^*$. Now take $\langle x^*, y^*\rangle \in \rho'$. Then we have that, by removing the single edge $\langle x^*, y^*\rangle$ from $\mathbb{X}$, the connectivity component $x^*\uparrow$ ends up split into two components: $\{y^*\}$ and $x^*\uparrow \setminus \{y^*\}$. Since $|\{z \in \mathrm{Root}_{\mathbb{X}} : z\uparrow = \{z\}\}| \geq \omega$, and since $x^*\uparrow \cong x^*\uparrow\setminus\{y^*\}$, we conclude that the structures $\langle X, \rho\rangle$ and $\langle X, \rho \setminus \{\langle x^*, y^*\rangle\}\rangle$ have isomorphic components. Hence, $\rho \cong \rho \setminus \{\langle x^*, y^*\rangle\}$, which means that $\langle x^*, y^*\rangle \in \rho^*$. And if

$$\langle x^*, y^*\rangle \in \Big\{\langle \mathrm{pred}_{\mathbb{X}}(y), y\rangle \notin (\rho \upharpoonright_{x_0\uparrow})^* : y \in \mathrm{Max}(x_0\uparrow) \ \wedge\ \exists Z \in [\mathrm{Root}'_{\mathbb{X}}]^{\leq 2}\ (\rho \upharpoonright_{x_0\uparrow}) \setminus \{\langle \mathrm{pred}_{\mathbb{X}}(y), y\rangle\} \cong \rho \upharpoonright_{Z\uparrow}\Big\},$$

for some $x_0 \in \mathrm{Root}_{\mathbb{X}'}$, we have two possibilities:

1. By removing the single edge $\langle x^*, y^*\rangle$ from $\mathbb{X}$, the connectivity component $x_0\uparrow$ remains connected. Then we have that there exists $Z = \{z\} \in [\mathrm{Root}'_{\mathbb{X}}]^1$ such that $(\rho \upharpoonright_{x_0\uparrow}) \setminus \{\langle x^*, y^*\rangle\} \cong \rho \upharpoonright_{Z\uparrow}$, and since $\langle x^*, y^*\rangle \notin (\rho \upharpoonright_{x_0\uparrow})^*$, we have that $\rho \upharpoonright_{x_0\uparrow} \not\cong (\rho \upharpoonright_{x_0\uparrow}) \setminus \{\langle x^*, y^*\rangle\}$. But $x_0 \in \mathrm{Root}'_{\mathbb{X}}$, which implies that the structures $\langle X, \rho\rangle$ and $\langle X, \rho \setminus \{\langle x^*, y^*\rangle\}\rangle$ have isomorphic components. Therefore, $\rho \cong \rho \setminus \{\langle x^*, y^*\rangle\}$, which means that $\langle x^*, y^*\rangle \in \rho^*$.

2. By removing the single edge $\langle x^*, y^*\rangle$ from $\mathbb{X}$, the connectivity component $x_0\uparrow$ ends up split into two components. Then $x_0 = x^*$, and those two components are $\{y^*\}$ and $x^*\uparrow \setminus \{y^*\}$. Then we have that there exists $Z = \{z_1, z_2\} \in [\mathrm{Root}'_{\mathbb{X}}]^2$ such that $(\rho \upharpoonright_{x^*\uparrow}) \setminus \{\langle x^*, y^*\rangle\} \cong \rho \upharpoonright_{Z\uparrow}$, and since $x^* \in \mathrm{Root}'_{\mathbb{X}}$, we conclude that, regardless of whether $x^*\uparrow \cong x^*\uparrow\setminus\{y^*\}$ or not, we have that the structures $\langle X, \rho\rangle$ and $\langle X, \rho \setminus \{\langle x^*, y^*\rangle\}\rangle$ have isomorphic components. Therefore, $\rho \cong \rho \setminus \{\langle x^*, y^*\rangle\}$, which means that $\langle x^*, y^*\rangle \in \rho^*$.

($\subseteq$) Take $\langle u, v\rangle \in \rho^*$. Then $\rho \cong \rho \setminus \{\langle u, v\rangle\}$, that is $g[\rho] = \rho \setminus \{\langle u, v\rangle\}$ for some $g \in \mathrm{Cond}(\mathbb{X})$. By Proposition 2.1 there are a surjection $f : \mathrm{Root}_{\mathbb{X}} \to \mathrm{Root}_{\mathbb{X}}$, and $g_x \in \mathrm{Mono}(x\uparrow, f(x)\uparrow)$, for $x \in \mathrm{Root}_{\mathbb{X}}$, where

$$\{g_y[y\uparrow] : y \in f^{-1}[\{x\}]\} \text{ is a partition of } x\uparrow \text{ for all } x \in \mathrm{Root}_{\mathbb{X}}, \tag{8}$$

such that

$$g = \bigcup_{x\in \mathrm{Root}_{\mathbb{X}}} g_x, \tag{9}$$

and such that: f is not an injection, or $g_x \notin \mathrm{Iso}(x\uparrow, f(x)\uparrow)$ for some $x \in \mathrm{Root}_{\mathbb{X}}$.

1. If f is not an injection, then $\{x \in \mathrm{Root}_{\mathbb{X}} : |f^{-1}[\{x\}]| > 1\} = \{x^*\}$, and $|f^{-1}[\{x^*\}]| = 2$, because otherwise, we would have by (8), (9), and since $x\uparrow$ is connected that $|\rho \setminus g[\rho]| > 1$. We conclude that in this case, by removing the edge $\langle u, v\rangle$ from $\mathbb{X}$, the connectivity component $x^*\uparrow$ was split into two components. This is possible if and only if

$$u = x^* \in \mathrm{Root}_{\mathbb{X}} \text{ and } v = y^* \in \mathrm{Max}(x^*\uparrow) \cap \mathrm{Level}_{\mathbb{X}}(1), \tag{10}$$

and in that case:

$$\text{those two components in } \left\langle X, \rho \setminus \{\langle x^*, y^*\rangle\}\right\rangle \text{ are } \{y^*\} \text{ and } x^*\uparrow \setminus \{y^*\}. \tag{11}$$

We have two possibilities:

• $f^{-1}[\{x^*\}] \cap \{f^n(x^*) : n \in \omega\} \neq \emptyset$. Let $f^{-1}[\{x^*\}] = \{z^*, w^*\}$, and let $m \in \omega$ be the smallest number such that $f^m(x^*) = w^*$. Then there is the sequence $\langle z_n : n \in \omega\rangle$ of different elements from $\mathrm{Root}_{\mathbb{X}} \setminus \{f^n(x^*) : n \in \omega\}$ such that $z_0 = z^*$ and $f^{-1}[\{z_k\}] = \{z_{k+1}\}$ for all $k \in \omega$. Since $|\rho \setminus g[\rho]| = 1$, by (8) and (9) we conclude that $g_{z_{k+1}} \in \mathrm{Iso}\left(z_{k+1}\uparrow, z_k\uparrow\right)$, for all $k \in \omega$, which means that

$$\{z_k : k \in \omega\} \subseteq \mathrm{Root}'_{\mathbb{X}}. \tag{12}$$

Similarly, $g_{f^n(x^*)} \in \mathrm{Iso}(f^n(x^*)\uparrow, f^{n+1}(x^*)\uparrow)$ for $n < m$, and, in particular, $x^*\uparrow \cong w^*\uparrow$. By (8) and (11), we have $\left\{g_{w^*}[w^*\uparrow], g_{z^*}[z^*\uparrow]\right\} = \left\{x^*\uparrow \setminus \{y^*\}, \{y^*\}\right\}$. Since $|w^*\uparrow| = |x^*\uparrow| > 1$, we have that $g_{w^*}[w^*\uparrow] = x^*\uparrow \setminus \{y^*\}$, and that $g_{z^*}[z^*\uparrow] = \{y^*\}$, whence $z^*\uparrow = \{z^*\}$. Since $|\rho \setminus g[\rho]| = 1$, we conclude that $g_{w^*} \in \mathrm{Iso}(w^*\uparrow, x^*\uparrow \setminus \{y^*\})$, i.e. that $x^*\uparrow \cong w^*\uparrow \cong x^*\uparrow \setminus \{y^*\}$. Now, from (10) and (12) it follows that $\langle u, v\rangle = \langle x^*, y^*\rangle \in \rho'$.

• $f^{-1}[\{x^*\}] \cap \{f^n(x^*) : n \in \omega\} = \emptyset$. Then there are sequences $\langle x_n : n \in \omega\rangle$, $\langle y_n : n \in \omega\rangle$, and $\langle z_n : n \in \omega\rangle$ of different elements from $\mathrm{Root}_{\mathbb{X}}$, such that $x_0 = y_0 = z_0 = x^*$, and such that $f^{-1}[\{x_{k+1}\}] = \{x_k\}$, for $k \in \omega$, $f^{-1}[\{x_0\}] = \{y_1, z_1\}$, and $f^{-1}[\{y_k\}] = \{y_{k+1}\}$, $f^{-1}[\{z_k\}] = \{z_{k+1}\}$, for all $k \in \mathbb{N}$. Since $|\rho \setminus g[\rho]| = 1$, by (8) and (9) we conclude that $g_{x_k} \in \mathrm{Iso}(x_k\uparrow, x_{k+1}\uparrow)$, for all $k \in \omega$, and that $g_{y_{k+1}} \in \mathrm{Iso}(y_{k+1}\uparrow, y_k\uparrow)$, $g_{z_{k+1}} \in \mathrm{Iso}(z_{k+1}\uparrow, z_k\uparrow)$, for all $k \in \mathbb{N}$, i.e.

$$\{x_k : k \in \omega\} \cup \{y_k : k \in \mathbb{N}\} \cup \{z_k : k \in \mathbb{N}\} \subseteq \mathrm{Root}'_{\mathbb{X}}. \tag{13}$$

By (8) and (11) we have $\left\{g_{y_1}[y_1\uparrow], g_{z_1}[z_1\uparrow]\right\} = \left\{\{y^*\}, x^*\uparrow \setminus \{y^*\}\right\}$. Without loss of generality, let $g_{y_1}[y_1\uparrow] = \{y^*\}$ and $g_{z_1}[z_1\uparrow] = x^*\uparrow \setminus \{y^*\}$. This implies that $y_1\uparrow = \{y_1\}$, and since $|\rho \setminus g[\rho]| = 1$, we get that $g_{z_1} \in \mathrm{Iso}(z_1\uparrow, x^*\uparrow \setminus \{y^*\})$, i.e. that $(\rho\restriction_{x^*\uparrow}) \setminus \{\langle x^*, y^*\rangle\} \cong \rho\restriction_{Z\uparrow}$, where $Z = \{y_1, z_1\} \in [\mathrm{Root}'_{\mathbb{X}}]^{\leq 2}$. Now, from (10), (13) and Proposition 3.1 (c), it follows that $\langle u, v\rangle = \langle x^*, y^*\rangle \in \rho''$.

2. f is bijective, but $g_{x^\star} \notin \mathrm{Iso}(x^\star\uparrow, f(x^\star)\uparrow)$ for some $x^\star \in \mathrm{Root}_{\mathbb{X}}$. We have two possibilities:

• $x^\star \notin \{f^n(x^\star) : n \in \mathbb{N}\}$. Then the sequence $\langle x_n : n \in \mathbb{Z}\rangle$, where $x_n := f^n(x^\star)$ for $n \in \mathbb{Z}$, consists of different elements from $\mathrm{Root}_{\mathbb{X}}$, and since $|\rho \setminus g[\rho]| = 1$, by (8) and (9) we conclude that $g_{x_n} \in \mathrm{Iso}\left(x_n\uparrow, x_{n+1}\uparrow\right)$, for all $n \in \mathbb{Z} \setminus \{0\}$, which means that

$$\{x_n : n \in \mathbb{Z}\} \subseteq \mathrm{Root}'_{\mathbb{X}}. \tag{14}$$

Since $g_{x_0} \in \mathrm{Cond}(\rho \restriction_{x_0\uparrow}, \rho \restriction_{x_1\uparrow}) \setminus \mathrm{Iso}(\rho \restriction_{x_0\uparrow}, \rho \restriction_{x_1\uparrow})$, by (8) and (9) we have $\emptyset \neq \rho \restriction_{x_1\uparrow} \setminus g_{x_0}[\rho \restriction_{x_0\uparrow}] \subseteq \rho \setminus g[\rho] = \{\langle u, v\rangle\}$. Hence, $\rho \restriction_{x_1\uparrow} \setminus \{\langle u, v\rangle\} \cong \rho \restriction_{Z\uparrow}$, where $Z = \{x_0\} \in [\mathrm{Root}'_{\mathbb{X}}]^{\leq 2}$. Now, from Proposition 3.1 (a) and (14) it follows that $\langle u, v\rangle \in \bigcup_{x\in\mathrm{Root}_{\mathbb{X}}} (\rho \restriction_{x\uparrow})^* \cup \rho''$.

• $x^\star \in \{f^n(x^\star) : n \in \mathbb{N}\}$. Let $m \in \mathbb{N}$ be the smallest number such that $f^m(x^\star) = x^\star$. If we put $x_n := f^n(x^\star)$, for $n \in \omega$, then $x^\star = x_0 = x_m$. Now for every $k < m$ we have that $f^{-1}[\{x_{k+1}\}] = \{x_k\}$, and since $|\rho \setminus g[\rho]| = 1$, by (8) and (9) we conclude that $g_{x_k} \in \mathrm{Iso}(x_k\uparrow, x_{k+1}\uparrow)$ for $0 < k < m$, that is

$$g_{x_k}[\rho \restriction_{x_k\uparrow}] = \rho \restriction_{x_{k+1}\uparrow}, \quad \text{for } 0 < k < m. \tag{15}$$

Since $g_{x_0} \in \mathrm{Cond}(\rho \restriction_{x_0\uparrow}, \rho \restriction_{x_1\uparrow}) \setminus \mathrm{Iso}(\rho \restriction_{x_0\uparrow}, \rho \restriction_{x_1\uparrow})$, as above, we have that $g_{x_0}[\rho \restriction_{x_0\uparrow}] = \rho \restriction_{x_1\uparrow} \setminus \{\langle u, v\rangle\}$. Now, by (15) we have

$$g_{x_m} \circ \cdots \circ g_{x_2} \circ g_{x_1}[\rho \restriction_{x_1\uparrow}] = \cdots = g_{x_m}[\rho \restriction_{x_m\uparrow}] = g_{x_0}[\rho \restriction_{x_0\uparrow}] = \rho \restriction_{x_1\uparrow} \setminus \{\langle u, v\rangle\},$$

that is $\rho \restriction_{x_1\uparrow} \cong \rho \restriction_{x_1\uparrow} \setminus \{\langle u, v\rangle\}$, and hence $\langle u, v\rangle \in (\rho \restriction_{x_1\uparrow})^* \subseteq \bigcup_{x\in\mathrm{Root}_{\mathbb{X}}} (\rho \restriction_{x\uparrow})^*$. □

Proposition 3.3. *Let $\mathbb{X} = \langle X, \rho\rangle \in \mathrm{Mod}_{L_b}$ be a tree. Then we have:*
(a) If the set $\rho^* \setminus \bigcup_{x\in\mathrm{Root}_{\mathbb{X}}} (\rho \restriction_{x\uparrow})^*$ is nonempty, then it is infinite;
(b) If $|\mathrm{Root}_{\mathbb{X}}| = 1$ and the set $\rho^* \setminus \bigcup_{x\in\mathrm{Level}_{\mathbb{X}}(1)} (\rho \restriction_{x\uparrow})^*$ is nonempty, then it is infinite.

Proof.
(a) Since for $y \in \mathrm{Max}(x\uparrow) \cap \mathrm{Level}_{\mathbb{X}}(1)$ we have

$$x\uparrow \cong x\uparrow\setminus\{y\} \iff |\mathrm{Max}(x\uparrow) \cap \mathrm{Level}_{\mathbb{X}}(1)| \geq \omega, \tag{16}$$

we conclude that, for any $x \in \mathrm{Root}_{\mathbb{X}}$, the set

$$\Big\{\langle x, y\rangle : y \in \mathrm{Max}(x\uparrow) \cap \mathrm{Level}_{\mathbb{X}}(1) \ \wedge\ \exists z \in \mathrm{Root}'_{\mathbb{X}}\ z\uparrow \cong \{y\} \ \wedge\ x\uparrow \cong x\uparrow\setminus\{y\}\Big\},$$

is clearly either empty or infinite. The set

$$\bigcup_{x\in\mathrm{Root}'_{\mathbb{X}}} \Big\{\langle \mathrm{pred}_{\mathbb{X}}(y), y\rangle \notin (\rho \restriction_{x\uparrow})^* : y \in \mathrm{Max}(x\uparrow) \ \wedge\ \exists Z \in [\mathrm{Root}'_{\mathbb{X}}]^{\leq 2}\ (\rho \restriction_{x\uparrow}) \setminus \{\langle \mathrm{pred}_{\mathbb{X}}(y), y\rangle\} \cong \rho \restriction_{Z\uparrow}\Big\},$$

is, by (5), also either empty or infinite. The statement now follows from Lemma 3.2.
(b) This follows from (a) and Proposition 3.1 (c) □

4. Main result

Theorem 4.1. *Let $\mathbb{X} = \langle X, \rho\rangle \in \mathrm{Mod}_{L_b}$ be a tree. If the tree $\mathbb{X}$ has a removable edge, then it has infinitely many removable edges.*

Proof. We shall prove by induction, that for any tree $\mathbb{X} = \langle X, \rho\rangle$, and any $n \in \mathbb{N}$,

$$|\rho^*| \geq n \Rightarrow |\rho^*| > n. \tag{17}$$

We first assume that $|\rho^*| = 1$, that is $\rho^* = \{\langle x^*, y^*\rangle\}$. Then, by Proposition 3.1 (a), we have that $\mathrm{ht}_{\mathbb{X}}(y^*)$ is a successor ordinal. Let $y^*\downarrow = \{x_\alpha : \alpha < \mathrm{ht}_{\mathbb{X}}(y^*)\}$ be the increasing enumeration of $y^*\downarrow$, such that $\mathrm{ht}_{\mathbb{X}}(x_\alpha) = \alpha$.

Next we prove, by transfinite induction, that

$$(\rho \restriction_{x_\alpha\uparrow})^* = \{\langle x^*, y^*\rangle\}, \quad \text{for every } \alpha < \mathrm{ht}_{\mathbb{X}}(y^*). \tag{18}$$

By Proposition 3.3 (a) and Proposition 3.1 (e), it is $\rho^* = \bigcup_{x\in\mathrm{Root}_{\mathbb{X}}} (\rho \restriction_{x\uparrow})^*$, which implies that $(\rho \restriction_{x_0\uparrow})^* = \{\langle x^*, y^*\rangle\}$. Let us now assume, for some $\gamma < \mathrm{ht}_{\mathbb{X}}(y^*)$, that for all $\beta < \gamma$ we have $(\rho \restriction_{x_\beta\uparrow})^* = \{\langle x^*, y^*\rangle\}$. It is possible:

1. $\gamma = \delta + 1$. Since $\mathrm{Level}_{x_\delta\uparrow}(1) = [x_\gamma]_{\sim_v}$, by Proposition 3.1 (c) and the induction hypothesis, we have that $(\rho \restriction_{[x_\gamma]_{\sim_v}\uparrow})^* = (\rho \restriction_{x_\delta\uparrow})^* = \{\langle x^*, y^*\rangle\}$. Now, by Proposition 3.3 (a) and Proposition 3.1 (e), we have that $(\rho \restriction_{[x_\gamma]_{\sim_v}\uparrow})^* = \bigcup_{x\in[x_\gamma]_{\sim_v}} (\rho \restriction_{x\uparrow})^*$, and consequently $(\rho \restriction_{x_\gamma\uparrow})^* = \{\langle x^*, y^*\rangle\}$.

2. γ is a limit ordinal. Then, since $|\rho^*| = 1$, and since $(\rho \restriction_{x_\beta\uparrow})^* = \{\langle x^*, y^*\rangle\}$ for all $\beta < \gamma$, we conclude, by Proposition 3.1 (e), that:

$$\text{for every } \beta < \gamma, \text{ and for every } x \in [x_\beta]_{\sim_v} \setminus \{x_\beta\}, \text{ we have } x\uparrow \ncong x_\beta\uparrow . \tag{19}$$

Since $\langle X, \rho\rangle = \mathbb{X} \cong \mathbb{X}' = \langle X, \rho'\rangle = \langle X, \rho \setminus \{\langle x^*, y^*\rangle\}\rangle$, there is $f \in \mathrm{Iso}(\mathbb{X}, \mathbb{X}')$, and for such f we have

$$f[\mathrm{Level}_{\mathbb{X}}(\beta)] = \mathrm{Level}_{\mathbb{X}'}(\beta) = \mathrm{Level}_{\mathbb{X}}(\beta), \quad \text{for } \beta < \gamma < \mathrm{ht}_{\mathbb{X}}(y^*).$$

Next we prove, by transfinite induction, that

$$f(x_\beta) = x_\beta, \quad \text{for all } \beta < \gamma < \mathrm{ht}_{\mathbb{X}}(y^*). \tag{20}$$

Since $\langle x^*, y^*\rangle \in (\rho \restriction_{x_0\uparrow})^*$, we have that $x\uparrow \cong x\uparrow_{\mathbb{X}'}$, for all $x \in \mathrm{Root}_{\mathbb{X}} = \mathrm{Root}_{\mathbb{X}'}$. Hence, by (19), it is

$$x\uparrow_{\mathbb{X}'} \ncong x_0\uparrow_{\mathbb{X}'} \text{ for all } x \in \mathrm{Root}_{\mathbb{X}'} \setminus \{x_0\}. \tag{21}$$

Since $f : \mathbb{X} \to \mathbb{X}'$ is isomorphism, we have $x_0\uparrow_{\mathbb{X}'} \cong x_0\uparrow \cong f[x_0\uparrow] = f(x_0)\uparrow_{\mathbb{X}'}$, which, together with (21), implies $f(x_0) = x_0$. Assume now, for some $\delta < \gamma$, that for all $\zeta < \delta$ we have $f(x_\zeta) = x_\zeta$. Then $x_\delta\downarrow = f[x_\delta\downarrow] = f(x_\delta)\downarrow_{\mathbb{X}'} = f(x_\delta)\downarrow$, which means that $x_\delta \sim_v f(x_\delta)$, that is

$$f(x_\delta) \in [x_\delta]_{\sim_v} = [x_\delta]_{\sim_{v_{\mathbb{X}'}}}. \tag{22}$$

Since $\langle x^*, y^*\rangle \in (\rho \restriction_{x_\delta\uparrow})^*$, we have that $x\uparrow \cong x\uparrow_{\mathbb{X}'}$, for all $x \in [x_\delta]_{\sim_v} = [x_\delta]_{\sim_{v_{\mathbb{X}'}}}$. Hence, by (19), it is

$$x\uparrow_{\mathbb{X}'} \ncong x_\delta\uparrow_{\mathbb{X}'} \text{ for all } x \in [x_\delta]_{\sim_{v_{\mathbb{X}'}}} \setminus \{x_\delta\}. \tag{23}$$

Now we have $x_\delta\uparrow_{\mathbb{X}'} \cong x_\delta\uparrow \cong f[x_\delta\uparrow] = f(x_\delta)\uparrow_{\mathbb{X}'}$, which, together with (22) and (23), implies that $f(x_\delta) = x_\delta$. We have proven (20).

By (20), we have $f(x)\downarrow_{\mathbb{X}'} = f[x\downarrow] = f[x_\gamma\downarrow] = x_\gamma\downarrow = x_\gamma\downarrow_{\mathbb{X}'}$, for any $x \in [x_\gamma]_{\sim_v}$. Hence, $f(x) \in [x_\gamma]_{\sim_{v_{\mathbb{X}'}}}$, that is $f\Big[[x_\gamma]_{\sim_v}\Big] \subseteq [x_\gamma]_{\sim_{v_{\mathbb{X}'}}}$. Since $f^{-1} \in \mathrm{Iso}(\mathbb{X}', \mathbb{X})$, we analogously prove that $f^{-1}\Big[[x_\gamma]_{\sim_{v_{\mathbb{X}'}}}\Big] \subseteq [x_\gamma]_{\sim_v}$, i.e. $[x_\gamma]_{\sim_{v_{\mathbb{X}'}}} \subseteq f\Big[[x_\gamma]_{\sim_v}\Big]$. Hence, $f\Big[[x_\gamma]_{\sim_v}\Big] = [x_\gamma]_{\sim_{v_{\mathbb{X}'}}}$, and thus $f\Big[[x_\gamma]_{\sim_v}\uparrow\Big] = [x_\gamma]_{\sim_{v_{\mathbb{X}'}}}\uparrow_{\mathbb{X}'}$. It is easy to see that $[x_\gamma]_{\sim_v}\uparrow = [x_\gamma]_{\sim_{v_{\mathbb{X}'}}}\uparrow_{\mathbb{X}'}$, hence

$$f\Big[[x_\gamma]_{\sim_v}\uparrow\Big] = [x_\gamma]_{\sim_v}\uparrow . \tag{24}$$

Since $f : \mathbb{X} \to \mathbb{X}'$ is an isomorphism, this implies that

$$\Big\langle [x_\gamma]_{\sim_v}\uparrow, \rho \restriction_{[x_\gamma]_{\sim_v}\uparrow}\Big\rangle \cong \Big\langle f\Big[[x_\gamma]_{\sim_v}\uparrow\Big], \rho' \restriction_{f[[x_\gamma]_{\sim_v}\uparrow]}\Big\rangle = \tag{25}$$

$$\Big\langle [x_\gamma]_{\sim_v}\uparrow, \rho' \restriction_{[x_\gamma]_{\sim_v}\uparrow}\Big\rangle = \Big\langle [x_\gamma]_{\sim_v}\uparrow, (\rho \restriction_{[x_\gamma]_{\sim_v}\uparrow}) \setminus \{\langle x^*, y^*\rangle\}\Big\rangle,$$

which means that $\langle x^*, y^*\rangle \in (\rho \restriction_{[x_\gamma]_{\sim_v}\uparrow})^*$. By Proposition 3.1 (d), we have that $(\rho \restriction_{[x_\gamma]_{\sim_v}\uparrow})^* = \{\langle x^*, y^*\rangle\}$. Now, by Proposition 3.3 (a) and Proposition 3.1 (e), we have that $(\rho \restriction_{[x_\gamma]_{\sim_v}\uparrow})^* = \bigcup_{x\in[x_\gamma]_{\sim_v}} (\rho \restriction_{x\uparrow})^*$, and consequently, $(\rho \restriction_{x_\gamma\uparrow})^* = \{\langle x^*, y^*\rangle\}$. We have proven (18).

Since $x^* = x_{\mathrm{ht}_{\mathbb{X}}(x^*)} \in y^*\downarrow$, from (18) it follows that $(\rho \restriction_{x^*\uparrow})^* = \{\langle x^*, y^*\rangle\}$, and that is a contradiction with Proposition 3.1 (c). Therefore, $|\rho^*| \neq 1$, and thus we have proven that $|\rho^*| \geq 1 \Rightarrow |\rho^*| > 1$, which is the basis of the main induction.

Let us now assume that, for any tree $\mathbb{X} = \langle X, \rho\rangle$ we have

$$|\rho^*| \geq m \Rightarrow |\rho^*| > m, \quad \text{for } m \in \{1, 2, \ldots, n-1\}, \tag{26}$$

and let us assume that $|\rho^*| = n$. Let $\xi := \min\left(\mathrm{ht}_{\mathbb{X}}\left[\pi_2[\rho^*]\right]\right)$, and let $\langle x^*, y^*\rangle \in \rho^*$, such that $\mathrm{ht}_{\mathbb{X}}(y^*) = \xi$. Let $y^*\downarrow = \{x_\alpha : \alpha < \mathrm{ht}_{\mathbb{X}}(y^*)\}$ be the increasing enumeration of $y^*\downarrow$, such that $\mathrm{ht}_{\mathbb{X}}(x_\alpha) = \alpha$.

Next we prove, by transfinite induction, that

$$(\rho \upharpoonright_{x_\alpha\Uparrow})^* = \rho^*, \quad \text{for every } \alpha < \mathrm{ht}_{\mathbb{X}}(y^*). \tag{27}$$

By Proposition 3.3 (a) and Proposition 3.1 (e), it is $\rho^* = \bigcup_{x\in \mathrm{Root}_{\mathbb{X}}}(\rho \upharpoonright_{x\Uparrow})^*$, which, together with (26), implies that $(\rho \upharpoonright_{x_0\Uparrow})^* = \rho^*$. Let us now assume, for some $\gamma < \mathrm{ht}_{\mathbb{X}}(y^*)$, that for all $\beta < \gamma$ we have $(\rho \upharpoonright_{x_\beta\Uparrow})^* = \rho^*$. It is possible:

1. $\gamma = \delta + 1$. Since $\mathrm{Level}_{x_\delta\Uparrow}(1) = [x_\gamma]_{\sim_v}$, by Proposition 3.1 (c) and the induction hypothesis, we have that $(\rho \upharpoonright_{[x_\gamma]_{\sim_v}\Uparrow})^* = (\rho \upharpoonright_{x_\delta\Uparrow})^* = \rho^*$. Now, by Proposition 3.3 (a) and Proposition 3.1 (e), we have that $(\rho \upharpoonright_{[x_\gamma]_{\sim_v}\Uparrow})^* = \bigcup_{x\in[x_\gamma]_{\sim_v}}(\rho \upharpoonright_{x\Uparrow})^*$, and therefore, by (26), it is $(\rho \upharpoonright_{x_\gamma\Uparrow})^* = \rho^*$.

2. γ is a limit ordinal. Then, since $|\rho^*| = n$, and since $(\rho \upharpoonright_{x_\beta\Uparrow})^* = \rho^*$ for all $\beta < \gamma$, we conclude, by Proposition 3.1 (e), that:

$$\text{for every } \beta < \gamma, \text{ and for every } x \in [x_\beta]_{\sim_v} \setminus \{x_\beta\}, \text{ we have } x\Uparrow \not\cong x_\beta\Uparrow. \tag{28}$$

Take any $\langle x^*, y^*\rangle \in \rho^*$. Since $\langle X, \rho\rangle = \mathbb{X} \cong \mathbb{X}' = \langle X, \rho'\rangle = \langle X, \rho \setminus \{\langle x^*, y^*\rangle\}\rangle$, there is $f \in \mathrm{Iso}(\mathbb{X}, \mathbb{X}')$, and for such f we analogously as with (20) and (24), by transfinite induction, prove that

$$f(x_\beta) = x_\beta, \quad \text{for all } \beta < \gamma < \mathrm{ht}_{\mathbb{X}}(y^*),$$

and that

$$f\Big[[x_\gamma]_{\sim_v}\Uparrow\Big] = [x_\gamma]_{\sim_v}\Uparrow .$$

Since f is an isomorphism, as in (25) we conclude that $\langle x^*, y^*\rangle \in (\rho \upharpoonright_{[x_\gamma]_{\sim_v}\Uparrow})^*$. Since $\langle x^*, y^*\rangle \in \rho^*$ was arbitrary, we have that $\rho^* \subseteq (\rho \upharpoonright_{[x_\gamma]_{\sim_v}\Uparrow})^*$, and hence, by Proposition 3.1 (d), we have $(\rho \upharpoonright_{[x_\gamma]_{\sim_v}\Uparrow})^* = \rho^*$. Now, by Proposition 3.3 (a) and Proposition 3.1 (e), we have that $(\rho \upharpoonright_{[x_\gamma]_{\sim_v}\Uparrow})^* = \bigcup_{x\in[x_\gamma]_{\sim_v}}(\rho \upharpoonright_{x\Uparrow})^*$. This, together with (26), implies that $(\rho \upharpoonright_{x_\gamma\Uparrow})^* = \rho^*$. We have proven (27).

Since $x^* = x_{\mathrm{ht}_{\mathbb{X}}(x^*)} \in y^*\downarrow$, from (27) it follows that $\langle x^*, y^*\rangle \in \rho^* = (\rho \upharpoonright_{x^*\Uparrow})^*$, and that is a contradiction with Proposition 3.1 (c). Therefore, $|\rho^*| \neq n$, which together with (26) gives us

$$|\rho^*| \geq n \Rightarrow |\rho^*| > n.$$

□

Acknowledgments. This research was supported by the Ministry of Education, Science and Technological Development of the Republic of Serbia (Project 174006).

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