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μ-contractions in ordered metric spaces endowed with a w_0-distance

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Abstract.
We introduce in the setting of ordered metric spaces a new contractive condition called ordered μ-contraction. We use such a condition in order to provide new and more general results of existence and uniqueness of fixed point. We remark that from our main result one can easily deduce the Banach contraction principle, the Boyd-Wong result and other known results of fixed point in the existing literature.

1. Introduction

In [10] Ran and Reurings established a result similar to Banach contraction principle in the setting of metric sets endowed with a partial order. Motivated by this, several authors recently studied fixed point problems that involve monotone mappings defined on partially ordered metric spaces. Moreover, we remark that Nieto and Rodríguez-López extended the main fixed point theorem of [10] to ordered metric spaces (see [9]). Further, they used such a result in order to solve problems of integro-differential type.

The aim of this paper is to provide new and more general results of existence and uniqueness of fixed point in the setting of ordered metric spaces. In order to do this, following Jleli *et al.* (see [8]), we introduce a new contractive notion which involves two suitable families of functions. We stress that applying our main theorem we can easily deduce some of the most known results of fixed point in the existing literature as the Banach contraction principle (see [2]) and the Boyd-Wong result (see [3]).

The paper is organized as follows. Section 2 is dedicated to the mathematical background. Precisely, we recall the notion of w_0-distance and its properties. Furthermore, we collect some notions related to ordered metric spaces that we use throughout the paper. In Section 3, we introduce a new type of contraction which we call ordered μ-contraction and we establish our main result (see Theorem 3.3). Section 4 is aimed to point out that the notion of ordered μ-contraction includes different contractive conditions in the existing literature (see Corollaries 4.1, 4.2, 4.3 and 4.4). In Sections 5 and 6 we use our main theorem in order to establish fixed point results for cyclic mappings and mappings that verify a contractive condition of integral type on ordered metric spaces (see Theorem 5.3 and Theorems 6.1, 6.2, respectively). Finally, in Section 7 we give a result of existence and uniqueness for the solution of a first-order periodic differential problem (see Theorem 7.1).

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2. Preliminaries

We work in the setting of ordered metric spaces endowed with a w_0-distance. The notion of w_0-distance was recently introduced, therefore for convenience of the reader we recall it and its properties. Further, we collect the notions related to ordered metric spaces that we use in the following.

Kostić *et al.* in [7] revised the definition of w-distance introduced in the setting of metric spaces by Kada *et al.* in [4]. They supposed in addition the lower semicontinuity with respect to both variables and gave the following definition.

Definition 2.1. *Let (X, d) be a metric space. A function $\sigma : X \times X \to [0, +\infty[$ is called a w_0-distance on X if the following three conditions are verified:*

(σ_1) *$\sigma(a, c) \leq \sigma(a, b) + \sigma(b, c)$ for all $a, b, c \in X$;*

(σ_2) *the functions $\sigma(b, \cdot), \sigma(\cdot, b) : X \to [0, +\infty[$ are lower semicontinuous for any $b \in X$;*

(σ_3) *for any $\epsilon > 0$ there exists $\delta > 0$ such that $\sigma(a, b) \leq \delta$ and $\sigma(a, c) \leq \delta$ imply $d(b, c) \leq \epsilon$.*

The main properties of a w-distance (and so of a w_0-distance) are provided by the following lemma.

Lemma 2.2 (see [4]). *Let (X, d) be a metric space and let σ be a w-distance on X. Let $\{a_m\}$ and $\{b_m\}$ be sequences in X, let $\{\alpha_m\}$ and $\{\beta_m\}$ be sequences in $[0, +\infty[$ converging to 0 and let $a, b, c \in X$. Then the following hold:*

(i) *If $\sigma(a_m, b) \leq \alpha_m$ and $\sigma(a_m, c) \leq \beta_m$ for any $m \in \mathbb{N}$, then $b = c$. In particular, if $\sigma(a, b) = 0$ and $\sigma(a, c) = 0$, then $b = c$.*

(ii) *If $\sigma(a_m, b_m) \leq \alpha_m$ and $\sigma(a_m, c) \leq \beta_m$ for any $m \in \mathbb{N}$, then b_m converges to c.*

(iii) *If $\sigma(a_k, a_m) \leq \alpha_m$, for any $k, m \in \mathbb{N}$ with $k > m$, then $\{a_m\}$ is a Cauchy sequence.*

(iv) *If $\sigma(b, a_m) \leq \alpha_m$, for any $m \in \mathbb{N}$, then $\{a_m\}$ is a Cauchy sequence.*

Let (X, d) be a metric space and σ be a w_0-distance on X. Let us denote by $\mu : X \times X \to [0, +\infty[$ the function defined by

$$\mu(b, c) = \max\{\sigma(b, c), \sigma(c, b)\} \quad \text{for any } b, c \in X.$$

We stress that by Definition 2.1 and Lemma 2.2 we can easily deduce the following properties of μ:

(μ_1) $\mu(b, c) = 0 \Rightarrow b = c$ for any $b, c \in X$;

(μ_2) μ is symmetric, that is, $\mu(b, c) = \mu(c, b)$ for any $b, c \in X$;

(μ_3) μ satisfies the triangle inequality, that is, $\mu(a, c) \leq \mu(a, b) + \mu(b, c)$ for any $a, b, c \in X$;

(μ_4) $\mu(a, c) \leq \liminf_{m \to +\infty} \mu(a, a_m)$ whenever $a_m \to c$ as $m \to +\infty$, that is, μ is lower semicontinuous with respect to the second variable;

(μ_5) $\mu(c, a) \leq \liminf_{m \to +\infty} \mu(a_m, a)$ whenever $a_m \to c$ as $m \to +\infty$, that is, μ is lower semicontinuous with respect to the first variable.

We also remark that, following [8] and [15], we use a contractive notion which involves two suitable families of functions. We denote such families with $\mathcal{H}$ and $\mathcal{S}$. In particular, $\mathcal{H}$ is the family of functions $H : [0, +\infty[^3 \to [0, +\infty[$ satisfying the following conditions (see [8]):

(H_1) $\max\{\alpha, \beta\} \leq H(\alpha, \beta, \gamma)$, for all $\alpha, \beta, \gamma \in [0, +\infty[$;

(H_2) $H(0, 0, 0) = 0$;

(H_3) H is continuous.

Instead, $\mathcal{S}$ is the family of functions $S : [0, +\infty[^2 \to \mathbb{R}$ satisfying the following conditions (see [1, 5]):

(S_1) $S(\alpha, \beta) < \beta - \alpha$ for all $\alpha, \beta > 0$;

(S_2) if $\{\alpha_m\}$, $\{\beta_m\}$ are sequences in $]0, +\infty[$ such that $\lim_{m\to+\infty} \alpha_m = \lim_{m\to+\infty} \beta_m = \ell \in]0, +\infty[$ then

$$\limsup_{m\to+\infty} S(\alpha_m, \beta_m) < 0.$$

Below, we give some examples of functions $H : [0, +\infty[^3 \to [0, +\infty[$ and $S : [0, +\infty[^2 \to \mathbb{R}$ belonging to the families $\mathcal{H}$ and $\mathcal{S}$, respectively.

(i) $H(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$, for all $\alpha, \beta, \gamma \in [0, +\infty[$, belongs to $\mathcal{H}$;

(ii) $H(\alpha, \beta, \gamma) = \max\{\alpha, \beta\} + \gamma$, for all $\alpha, \beta, \gamma \in [0, +\infty[$, belongs to $\mathcal{H}$;

(iii) $S(\alpha, \beta) = h\beta - \alpha$, for all $\alpha, \beta \in [0, +\infty[$ where $h \in [0, 1[$, belongs to $\mathcal{S}$;

(iv) $S(\alpha, \beta) = \beta - \nu(\beta) - \alpha$, for all $\alpha, \beta \in [0, +\infty[$ where $\nu : [0, +\infty[\to [0, +\infty[$ is a lower semicontinuous function such that $\nu(\beta) = 0$ if and only if $\beta = 0$, belongs to $\mathcal{S}$;

(v) $S(\alpha, \beta) = \beta\nu(\beta) - \alpha$, for all $\alpha, \beta \in [0, +\infty[$ where $\nu : [0, +\infty[\to [0, 1[$ is such that $\lim_{\beta\to r^+} \nu(\beta) < 1$ for all $r > 0$, belongs to $\mathcal{S}$.

We conclude this section with some remarks on ordered metric spaces. Let (X, d) be a metric space and $(X, \preceq)$ be a partially ordered set. Here, we call $(X, d, \preceq)$ an ordered metric space. We recall that two elements $b, c \in X$ are comparable if $b \preceq c$ or $c \preceq b$. A mapping $f : (X, \preceq) \to (X, \preceq)$ is nondecreasing if $fb \preceq fc$ whenever $b \preceq c$. Further, a sequence $\{a_m\}$ is nondecreasing if $a_{m-1} \preceq a_m$ for all $m \in \mathbb{N}$. In addition, we recall that an ordered metric space $(X, d, \preceq)$ is regular if for every nondecreasing sequence $\{a_m\} \subset X$ such that $a_m \to c \in X$, we have $a_{m-1} \preceq c$ for all $m \in \mathbb{N}$. Moreover, X has the property (A) if for each pair of non comparable elements $b, c \in X$ there exists $u \in X$ such that $b \preceq u$ and $c \preceq u$.

Finally, given a function $f : X \to X$ and a point $a_0 \in X$, we call the sequence $\{a_m\}$ defined by $a_m = fa_{m-1}$, for all $m \in \mathbb{N}$, a sequence of Picard starting at a_0.

3. Ordered μ-contractions

In this section, we start introducing a new type of contraction which we call ordered μ-contraction. Next, we give an auxiliar result and, finally, we state and prove our main result.

Definition 3.1. *Let $(X, d, \preceq)$ be an ordered metric space and $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X. A mapping $f : X \to X$ is an ordered μ-contraction if there exist three functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and $\eta : X \to [0, +\infty[$, such that*

$$S(H(\mu(fb, fc), \eta(fb), \eta(fc)), H(\mu(b, c), \eta(b), \eta(c))) \geq 0 \quad \text{for all } b, c \in X,\ b \preceq c. \tag{1}$$

The following technical lemma is useful in order to establish our main result.

Lemma 3.2. *Let $(X, d, \preceq)$ be an ordered metric space and $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X. Further, let $f : X \to X$ be a nondecreasing ordered μ-contraction with respect to the functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and $\eta : X \to [0, +\infty[$. Then any sequence $\{a_m\}$ of Picard starting at a point $a_0 \in X$ such that $a_0 \preceq fa_0$ is a Cauchy sequence whenever $a_{m-1} \neq a_m$ for all $m \in \mathbb{N}$.*

Proof. We consider a sequence $\{a_m\}$ of Picard starting at a point $a_0 \in X$ such that $a_0 \preceq fa_0$. Further, we assume that the sequence $\{a_m\}$ is such that $a_{m-1} \neq a_m$ for all $m \in \mathbb{N}$. This assures $\mu(a_{m-1}, a_m) > 0$ for all $m \in \mathbb{N}$ (we recall that $\mu(b, c) = 0$ implies $b = c$). Hence, thanks to property (H_1), we have also

$$H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m)) \geq \mu(a_{m-1}, a_m) > 0 \quad \text{for all } m \in \mathbb{N}.$$

Now, we show that

$$\lim_{m\to+\infty} \mu(a_{m-1}, a_m) = 0 \quad \text{and} \quad \lim_{m\to+\infty} \eta(a_m) = 0. \tag{2}$$

We recall that f is nondecreasing and this implies $a_{m-1} \preceq a_m$ for all $m \in \mathbb{N}$. Therefore, using (1) with $b = a_{m-1}$ and $c = a_m$ and the property (S_1), we get

$$\begin{aligned} 0 &\leq S(H(\mu(a_m, a_{m+1}), \eta(a_m), \eta(a_{m+1})), H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m))) \\ &< H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m)) - H(\mu(a_m, a_{m+1}), \eta(a_m), \eta(a_{m+1})) \end{aligned}$$

for all $m \in \mathbb{N}$. From the previous inequality we deduce that

$$H(\mu(a_m, a_{m+1}), \eta(a_m), \eta(a_{m+1})) < H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m)) \quad \text{for all } m \in \mathbb{N}$$

and hence, we can affirm that $\{H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m))\}$ is a decreasing sequence of positive real numbers. Then, there exists some $\ell \geq 0$ such that

$$\lim_{m\to+\infty} H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m)) = \ell.$$

Further, we can affirm that $\ell = 0$. In fact, if we assume $\ell > 0$, choosing

$$\alpha_m = H(\mu(a_m, a_{m+1}), \eta(a_m), \eta(a_{m+1})) \quad \text{and} \quad \beta_m = H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m)),$$

by (S_2) we get

$$0 \leq \limsup_{m\to+\infty} S\left(H(\mu(a_m, a_{m+1}), \eta(a_m), \eta(a_{m+1})), H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m))\right) < 0$$

and thus we conclude that $\ell = 0$. Finally, thanks to the property (H_1), we have

$$\max\{\mu(a_{m-1}, a_m), \eta(a_{m-1})\} \leq H(\mu(a_{m-1}, a_m), \eta(a_{m-1}), \eta(a_m)) \quad \text{for all } m \in \mathbb{N}$$

and hence

$$\lim_{m\to+\infty} \mu(a_{m-1}, a_m) = 0 \quad \text{and} \quad \lim_{m\to+\infty} \eta(a_{m-1}) = 0.$$

Now, we prove that $\{a_m\}$ is a Cauchy sequence. We observe that, thanks to Lemma 2.2 (iii), it is sufficient to prove that for any $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that

$$\mu(a_n, a_m) < \epsilon \quad \text{for all } m > n \geq n(\epsilon). \tag{3}$$

So, we suppose for way of contradiction that (3) does not hold, that is, we suppose that there exist a positive real number ϵ_0 and two sequences $\{m_k\}$ and $\{n_k\}$ such that $m_k > n_k \geq k$ and $\mu(a_{n_k}, a_{m_k}) \geq \epsilon_0 > \mu(a_{n_k}, a_{m_k-1})$ for all $k \in \mathbb{N}$. Hence, by using the first limit of (2), we infer that

$$\lim_{k\to+\infty} \mu(a_{n_k}, a_{m_k}) = \lim_{k\to+\infty} \mu(a_{n_k-1}, a_{m_k-1}) = \epsilon_0.$$

Taking into account that $\mu(a_{n_k}, a_{m_k}) > 0$ and that we can assume $\mu(a_{n_k-1}, a_{m_k-1}) > 0$ for all $k \in \mathbb{N}$, we have

$$H(\mu(a_{n_k}, a_{m_k}), \eta(a_{n_k}), \eta(a_{m_k})) > 0$$

and

$$H(\mu(a_{n_k-1}, a_{m_k-1}), \eta(a_{n_k-1}), \eta(a_{m_k-1})) > 0$$

for all $k \in \mathbb{N}$. We notice that, since H is a continuous function, we have also

$$\begin{aligned}\lim_{k\to+\infty} H(\mu(a_{n_k-1}, a_{m_k-1}), \eta(a_{n_k-1}), \eta(a_{m_k-1})) &= \lim_{k\to+\infty} H(\mu(a_{n_k}, a_{m_k}), \eta(a_{n_k}), \eta(a_{m_k})) \\ &= H(\epsilon_0, 0, 0) \geq \epsilon_0 > 0.\end{aligned}$$

Now, taking into account that f is nondecreasing and $n_k < m_k$ for all k, we get $a_{n_k-1} \preceq a_{m_k-1}$ for all $k \in \mathbb{N}$. Then we can use (1) with $b = a_{n_k-1}$ and $c = a_{m_k-1}$ and the property (S_2) with

$$\alpha_k = H(\mu(a_{n_k}, a_{m_k}), \eta(a_{n_k}), \eta(a_{m_k})) \text{ and } \beta_k = H(\mu(a_{n_k-1}, a_{m_k-1}), \eta(a_{n_k-1}), \eta(a_{m_k-1}))$$

in order to obtain that

$$0 \leq \limsup_{k\to+\infty} S\left(H(\mu(a_{n_k}, a_{m_k}), \eta(a_{n_k}), \eta(a_{m_k}), H(\mu(a_{n_k-1}, a_{m_k-1}), \eta(a_{n_k-1}), \eta(a_{m_k-1}))\right) < 0.$$

Clearly, this is a contradiction and thus, we conclude that for any $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that (3) holds, that is, the sequence $\{a_m\}$ is Cauchy. □

Now, we can state and prove our main result.

Theorem 3.3. *Let $(X, d, \preceq)$ be an ordered complete metric space and $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X. Further, we assume that $f : X \to X$ is a nondecreasing ordered μ-contraction with respect to the functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and the lower semicontinuous function $\eta : X \to [0, +\infty[$. If there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$, X has the property (A) and is regular, then f has a unique fixed point a such that $\eta(a) = 0$.*

Proof. We start by proving the existence of a fixed point for f. Let a_0 be a point of X such that $a_0 \preceq fa_0$. We consider a sequence of Picard $\{a_m\}$ starting at a_0. We observe that if $a_k = a_{k+1}$ for some $k \in \mathbb{N}$ then a_k is a fixed point of f, that is, $a_k = fa_k$. Further, we notice that $\eta(a_k) = 0$. In fact, by $\eta(a_k) > 0$ it follows that

$$0 < \eta(a_k) \leq H(\mu(a_k, a_k), \eta(a_k), \eta(a_k)).$$

We stress that $a_k = a_{k+1}$ implies $a_m = a_k$ for all $m \geq k$, $m \in \mathbb{N}$. So, since $a_k \preceq a_k$ we can use (1) with $b = a_k$ and $c = a_k$ and the property (S_1), in order to infer that

$$\begin{aligned}0 &\leq S(H(\mu(a_k, a_k), \eta(a_k), \eta(a_k)), H(\mu(a_k, a_k), \eta(a_k), \eta(a_k))) \\ &< H(\mu(a_k, a_k), \eta(a_k), \eta(a_k)) - H(\mu(a_k, a_k), \eta(a_k), \eta(a_k)) = 0.\end{aligned}$$

Obviously, this is a contradiction and hence we conclude that $\eta(a_k) = 0$.

Then, we can assume that $a_m \neq a_{m+1}$ for every $m \in \mathbb{N}$. We recall that Lemma 3.2 assures that $\{a_m\}$ is Cauchy. Further, since $(X, d, \preceq)$ is complete, there exists some $a \in X$ such that

$$\lim_{m\to+\infty} a_m = a.$$

Firstly, we show that $\eta(a) = 0$. We stress that by the proof of Lemma 3.2 we deduce that for every $k \in \mathbb{N}$ there exists $m(k) \in \mathbb{N}$ such that

$$\mu(a_{m(k)}, a_m) < \frac{1}{k} \quad \text{for all } m > m(k). \tag{4}$$

Now, taking into account that μ is semicontinuous with respect to the second variable (see property (μ_4)), from (4), we get

$$\mu(a_{m(k)}, a) \leq \liminf_{m\to+\infty} \mu(a_{m(k)}, a_m) \leq \frac{1}{k}.$$

Hence, we infer that there exists a subsequence $\{a_{m(k)}\}$ of $\{a_m\}$ such that

$$\lim_{k\to+\infty}\mu(a_{m(k)},a)=0. \tag{5}$$

Finally, taking into account that η is a lower semicontinuous function, thanks to (2), we conclude that

$$0\le\eta(a)\le\liminf_{m\to+\infty}\eta(a_m)=0,$$

that is, $\eta(a)=0$.

Secondarily, we prove that a is a fixed point of f. We notice that a is a fixed point of f if there exists a subsequence a_{m_j} of a_m such that $a_{m_j}=a$ or $fa_{m_j}=fa$, for all $j\in\mathbb{N}$. If a such subsequence there is not, we can assume that $a_m\neq a$ and $fa_m\neq fa$ for all $m\in\mathbb{N}$. Hence, it follows that $H(\mu(fa_m,fa),\eta(fa_m),\eta(fa))\ge\mu(fa_m,fa)>0$ and $H(\mu(a_m,a),\eta(a_m),\eta(a))\ge\mu(a_m,a)>0$. Now, taking into account that X is regular and so $a_m\preceq a$ for all $m\in\mathbb{N}$, we can use (1) with $b=a_m$ and $c=a$ and the property (S_1), in order to get that

$$\begin{aligned}0&\le S(H(\mu(fa_m,fa),\eta(fa_m),\eta(fa)),H(\mu(a_m,a),\eta(a_m),\eta(a)))\\&<H(\mu(a_m,a),\eta(a_m),\eta(a))-H(\mu(fa_m,fa),\eta(fa_m),\eta(fa))\end{aligned}$$

and so

$$H(\mu(fa_m,fa),\eta(fa_m),\eta(fa))<H(\mu(a_m,a),\eta(a_m),\eta(a))\quad\text{for all } n\in\mathbb{N}.$$

Taking into account that μ is semicontinuous with respect to the first variable (see property (μ_5)) and H is continuous, by using (5), we get

$$\begin{aligned}\mu(a,fa)&\le\liminf_{k\to+\infty}\mu(a_{m(k)+1},fa)=\liminf_{k\to+\infty}\mu(fa_{m(k)},fa)\\&\le\liminf_{k\to+\infty}H(\mu(fa_{m(k)},fa),\eta(fa_{m(k)}),\eta(fa))\\&\le\liminf_{k\to+\infty}H(\mu(a_{m(k)},a),\eta(a_{m(k)}),\eta(a))\\&=H(0,0,0)=0.\end{aligned}$$

Hence, we conclude that $\mu(a,fa)=0$. This assures that $a=fa$, that is, a is a fixed point of f.

Now, we prove the uniqueness of the fixed point. We suppose by way of contradiction that f has two fixed points $a,b\in X$ with $a\neq b$. Taking into account that $a\neq b$, we can affirm that a and b are not comparable. Hence, by property (A), there exists $u\in X$ such that $a\preceq u$ and $b\preceq u$. Let $\{u_m\}$ be the sequence of Picard starting at the point $u_0=u$. We notice that, since f is nondecreasing, $a\preceq u_0$ and $b\preceq u_0$ imply $a\preceq u_{m-1}$ and $b\preceq u_{m-1}$ for all $m\in\mathbb{N}$. Further, since a and b are not comparable, we have that $a\neq u_{m-1}$ and $b\neq u_{m-1}$ for all $m\in\mathbb{N}$. Now, $a\preceq u_{m-1}$ permits to use (1) in order to get

$$\begin{aligned}0&\le S(H(\mu(fa,fu_{m-1}),\eta(fa),\eta(fu_{m-1})),H(\mu(a,u_{m-1}),\eta(a),\eta(u_{m-1})))\\&<H(\mu(a,u_{m-1}),\eta(a),\eta(u_{m-1}))-H(\mu(a,u_m),\eta(a),\eta(u_m)).\end{aligned}$$

From the previous inequality, we easily infer that the sequence $\{H(\mu(a,u_{m-1}),\eta(a),\eta(u_{m-1}))\}\subset[0,+\infty[$ is decreasing and so there exists $\ell\in[0,+\infty[$ such that

$$\lim_{m\to+\infty}H(\mu(a,u_{m-1}),\eta(a),\eta(u_{m-1}))=\ell.$$

Now, if we assume $\ell>0$, using (1) and (S_2), we get that

$$0\le\limsup_{m\to+\infty}S(H(\mu(fa,fu_{m-1}),\eta(fa),\eta(fu_{m-1})),H(\mu(a,u_{m-1}),\eta(a),\eta(u_{m-1})))<0,$$

and hence we deduce that $\ell=0$. Taking into account this, by property (H_1), we infer

$$\lim_{m\to+\infty}\mu(a,u_{m-1})=0.$$

In a similar way, we can also deduce that

$$\lim_{m\to+\infty} \mu(b, u_{m-1}) = 0.$$

Hence, taking into account that

$$\mu(a, b) \leq \mu(a, u_{m-1}) + \mu(u_{m-1}, b),$$

letting $m \to +\infty$, we get $\mu(a, b) = 0$. This implies that $a = b$ and so f has an unique fixed point. □

We remark that the contractive condition (1) does not ensure that the mapping f is continuous. Therefore, in the previous theorem we use the regularity of the ordered metric space $(X, d, \preceq)$ in order to conclude that the limit of a convergent Picard sequence is a fixed point of f. We also notice that if f is continuous we can immediately conclude that the limit of a convergent Picard sequence is a fixed point of f. Taking into account this, following the proof of Theorem 3.3, we in addition obtain the following result.

Theorem 3.4. *Let $(X, d, \preceq)$ be an ordered complete metric space and $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X. Further, we assume that $f : X \to X$ is a nondecreasing ordered μ-contraction with respect to the functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and the lower semicontinuous function $\eta : X \to [0, +\infty[$. If there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$, f is continuous and X has the property (A), then f has a unique fixed point a such that $\eta(a) = 0$.*

Finally, we stress that the uniqueness of the fixed point of f in Theorems 3.3 and 3.4 follows because we assume that X has the property (A). Hence, if we do not ask that X has the property (A), from the proof of Theorems 3.3, we get the following result.

Theorem 3.5. *Let $(X, d, \preceq)$ be an ordered complete metric space and $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X. Further, we assume that $f : X \to X$ is a nondecreasing ordered μ-contraction with respect to the functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and the lower semicontinuous function $\eta : X \to [0, +\infty[$. In addition, we assume that there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$. If f is continuous or X is regular, then f has a fixed point a such that $\eta(a) = 0$.*

Example 3.6. *Let $X = [0, 2]$ endowed with the usual metric $d(b, c) = |b - c|$ for all $b, c \in X$. Further, we endow X with the w_0-distance $\sigma : X \times X \to [0, +\infty[$ defined by $\sigma(b, c) = d(b, c)$ for all $b, c \in X$. In addition, we consider on X the partial order $\preceq$ given by*

$$b, c \in X, \quad b \preceq c \text{ if } b = c \quad \text{or} \quad (b \leq c,\ b, c \in [0, 1]).$$

We notice that $(X, d, \preceq)$ is a regular ordered complete metric space and $\mu : X \times X \to [0, +\infty[$ is given by $\mu(b, c) = d(b, c)$ for all $b, c \in X$. Let $f : X \to X$ be the function defined by

$$fb = \begin{cases} 0 & \text{if } b \in [0, 2[, \\ 2 & \text{if } b = 2. \end{cases}$$

Clearly, f satisfies the contractive condition of Theorem 3.3 with respect to the functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and the lower semicontinuous function $\eta : X \to [0, +\infty[$ defined by $S(\alpha, \beta) = k\beta - \alpha$ for all $\alpha, \beta \in [0, +\infty[$ with $k \in [0, 1[$, $H(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$ for all $\alpha, \beta, \gamma \in [0, +\infty[$ and $\eta(b) = 0$ for all $b \in X$, respectively. Let $b, c \in X$ with $b \preceq c$, then we have

$$H(\mu(fb, fc), \eta(fb), \eta(fc)) = 0 \quad \text{and} \quad H(\mu(b, c), \eta(b), \eta(c)) = c - b$$

and hence

$$S(H(\mu(fb, fc), \eta(fb), \eta(fc)), H(\mu(b, c), \eta(b), \eta(c))) = k(c - b) \geq 0.$$

Taking into account that all the conditions of Theorem 3.5 are satisfied (we recall that X is regular and f is nondecreasing), we can affirm that f has a fixed point in X. In addition, we notice that $a = 0$ and $a = 2$ are two fixed points of f sucht that $\eta(a) = 0$.

4. Consequences

In this section, we formulate and easily prove some corollaries, thanks to Theorem 3.3. Such corollaries are aimed to show that the notion of ordered μ-contraction includes different contractive conditions in the existing literature (see, for example, [3, 8, 11, 12]).

Corollary 4.1 (see [11]). *Let $(X, d, \preceq)$ be a complete ordered metric space, $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X and let $f : X \to X$ be a nondecreasing mapping. Further, we assume that there exist a function $H \in \mathcal{H}$, a function $\nu : [0, +\infty[\to [0, 1[$ with $\limsup_{t\to r^+} \nu(t) < 1$ for all $r > 0$ and a lower semicontinuous function $\eta : X \to [0, +\infty[$ such that*

$$H(\mu(fb, fc), \eta(fb), \eta(fc)) \leq \nu(H(\mu(b, c), \eta(b), \eta(c)))\, H(\mu(b, c), \eta(b), \eta(c)) \quad \text{for all } b, c \in X, b \preceq c.$$

If there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$, X has the property (A) and is regular, then f has a unique fixed point a such that $\eta(a) = 0$.

Proof. The claim follows by Theorem 3.3 taking $S \in \mathcal{S}$ given by $S(\alpha, \beta) = \beta\, \nu(\beta) - \alpha$, for all $\alpha, \beta \geq 0$. □

Next, we give a result of Rhoades type (see [12]) and a result of Jleli *et al.* type (see [8], Theorem 2.1).

Corollary 4.2. *Let $(X, d, \preceq)$ be a complete ordered metric space, $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X and let $f : X \to X$ be a nodecreasing mapping. Further, we suppose that there exist a function $H \in \mathcal{H}$ and two lower semicontinuous functions $\nu : [0, +\infty[\to [0, +\infty[$ with $\nu^{-1}(0) = \{0\}$ and $\eta : X \to [0, +\infty[$ such that*

$$H(\mu(fb, fc), \eta(fb), \eta(fc)) \leq H(\mu(b, c), \eta(b), \eta(c)) - \nu(H(\mu(b, c), \eta(b), \eta(c))) \ \text{ for all } b, c \in X,\ b \preceq c.$$

If there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$, X has the property (A) and is regular, then f has a unique fixed point a such that $\eta(a) = 0$.

Proof. We obtain the claim by Theorem 3.3 if we take $S \in \mathcal{S}$ given by $S(\alpha, \beta) = \beta - \nu(\beta) - \alpha$, for all $\alpha, \beta \geq 0$. □

Corollary 4.3. *Let $(X, d, \preceq)$ be a complete ordered metric space, $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X and let $f : X \to X$ be a nondecreasing mapping. Further, we assume that there exist $h \in [0, 1[$, a function $H \in \mathcal{H}$ and a lower semicontinuous function $\eta : X \to [0, +\infty[$ such that*

$$H(\mu(fb, fc), \eta(fb), \eta(fc)) \leq h\, H(\mu(b, c), \eta(b), \eta(c)) \quad \text{for all } b, c \in X,\ b \preceq c.$$

If there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$, X has the property (A) and is regular, then f has a unique fixed point a such that $\eta(a) = 0$.

Proof. We get the claim thanks to Theorem 3.3 if we choose $S \in \mathcal{S}$ given by $S(\alpha, \beta) = h\,\beta - \alpha$ for all $\alpha, \beta \geq 0$. □

Finally, we give a result of Boyd-Wong type (see [3]).

Corollary 4.4. *Let $(X, d, \preceq)$ be a complete metric space, $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X and let $f : X \to X$ be a nondecreasing mapping. Suppose that there exist a function $H \in \mathcal{H}$, an upper semicontinuous function $\tau : [0, +\infty[\to [0, +\infty[$ with $\tau(t) < t$ for all $t > 0$ and $\tau(0) = 0$ and a lower semicontinuous function $\eta : X \to [0, +\infty[$ such that*

$$H(\mu(fb, fc), \eta(fb), \eta(fc)) \leq \tau(H(\mu(b, c), \eta(b), \eta(c))) \quad \text{for all } b, c \in X,\ b \preceq c.$$

If there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$, X has the property (A) and is regular, then f has a unique fixed point a such that $\eta(a) = 0$.

Proof. By appling Theorem 3.3 and by choosing $S \in \mathcal{S}$ given by $S(\alpha, \beta) = \tau(\beta) - \alpha$, for all $\alpha, \beta \geq 0$, we have the claim. □

Remark 4.5. *If we take $\sigma = d$, $H(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$ for all $\alpha, \beta, \gamma \in [0, +\infty[$ and $\eta(b) = 0$ for all $b \in X$, then Corollary 4.3 provides the Banach contraction principle and Corollary 4.4 provides the Boyd-Wong result in the setting of ordered metric spaces.*

Remark 4.6. *We notice that if we replace the hypothesis "X is regular" with the hypothesis "f is continuous" then the Corollaries 4.1-4.4 are yet valid.*

5. Ordered cyclic μ-contractions

In this section, we remind the notion of cyclic representation introduced in the setting of metric spaces by Kirk *et al.* in [6] (see also [14]). Following [16], we combine such notion with one of ordered μ-contraction and so we establish a new fixed point result for cyclic mappings on ordered metric spaces.

Definition 5.1 (see [6, 14]). *Let (X, d) be a metric space, r be a positive integer and $f : X \to X$ be a mapping. We say that $X = \cup_{i=1}^{r} B_i$ is a cyclic representation of X with respect to f if*

(i) B_i is a nonempty closed set for each $i = 1, 2, \dots, r$;

(ii) $f(B_i) \subset B_{i+1}$ for each $i = 1, 2, \dots, r$, where $B_{r+1} = B_1$.

Starting by the previous definition, we introduce the notion of ordered cyclic μ-contraction as follows.

Definition 5.2. *Let $(X, d, \preceq)$ be an ordered metric space and $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X. Further, let r be a positive integer, $B_1, \dots, B_r$ be nonempty closed subsets of X and $Y = \cup_{i=1}^{r} B_i$. We say that a mapping $f : Y \to Y$ is an ordered cyclic μ-contraction if*

(i) $Y = \bigcup_{i=1}^{r} B_i$ is a cyclic representation of Y with respect to f;

(ii) there exist three functions $S \in \mathcal{S}$, $H \in \mathcal{H}$ and $\eta : Y \to [0, +\infty[$ such that

$$S\left(H(\mu(fb, fc), \eta(fb), \eta(fc)), H(\mu(b, c), \eta(b), \eta(c))\right) \geq 0$$

for every $b \in B_i$, $c \in B_{i+1}$, $i = 1, 2, \dots, r$ and $b \preceq c$.

Now, we are ready to formulate a new fixed point result for cyclic mappings on ordered metric spaces. We remark that such a result generalizes the Kirk *et al.*'s cyclic fixed point theorems (see [6], Theorems 1.3, 2.3 and 2.4). Further, it is an extension to ordered metric spaces of Theorem 4.3 of [16].

Theorem 5.3. *Let $(X, d, \preceq)$ be an ordered complete metric space and $\sigma : X \times X \to [0, +\infty[$ be a w_0-distance on X. Moreover, let r be a positive integer, $B_1, \dots, B_r$ be nonempty closed subsets of X, $Y = \cup_{i=1}^{r} B_i$ and $f : Y \to Y$ be a nondecreasing ordered cyclic μ-contraction. If there exists a point $a_1 \in B_1$ such that $a_1 \preceq fa_1$, Y has the property (A) and is regular and, in addition, η is a lower semicontinuous function, then f has a unique fixed point a such that $\eta(a) = 0$.*

Proof. We stress that in order to have the claim it is sufficient to show that $\cap_{i=1}^{r} B_i \neq \emptyset$. In fact, taking into account that B_i is closed for each $i = 1, 2, \dots, r$, if $\cap_{i=1}^{r} B_i \neq \emptyset$ then $\cap_{i=1}^{r} B_i$ is an ordered complete metric space with respect to $(d, \preceq)$. Furthermore, since $f : Y \to Y$ is a nondecreasing ordered cyclic μ-contraction, we have that $f(\cap_{i=1}^{r} B_i) \subset \cap_{i=1}^{r} B_i$. This assures that $f : \cap_{i=1}^{r} B_i \to \cap_{i=1}^{r} B_i$ is a nondecreasing ordered μ-contraction on $\cap_{i=1}^{r} B_i$. So, we can apply Theorem 3.3 and conclude that f has a unique fixed point a in $\cap_{i=1}^{r} B_i \subset Y$ such that $\eta(a) = 0$.

Then, we show that $\cap_{i=1}^{r} B_i \neq \emptyset$. We consider a point $a_1 \in B_1$ such that $a_1 \preceq fa_1$. Let $\{a_m\}$ be a sequence of Picard starting at a_1. We notice that $a_{mr+i} \in B_i$ for all $i = 1, \dots, r$ and $m \in \mathbb{N} \cup \{0\}$. In fact, $Y = \cup_{i=1}^{r} B_i$ is a cyclic representation of Y with respect to f. Hence, we deduce that if $a_k = a_{k+1}$ for some $k \in \mathbb{N}$ then $a_m = a_k$ for all $m \geq k$ and so $a_k \in B_i$ for each $i = 1, \dots, r$. Clearly, this assures that $\cap_{i=1}^{r} B_i \neq \emptyset$. So, we suppose that $a_m \neq a_{m+1}$ for every $m \in \mathbb{N}$. Taking into account that $(Y, d, \preceq)$ is complete and the sequence $\{a_m\} \subset Y$ is Cauchy (by Lemma 3.2), we can affirm that there exists $a \in X$ such that

$$\lim_{m \to +\infty} a_m = a.$$

Further, we can affirm that $a \in \cap_{i=1}^{r} B_i$. In fact, the set B_i is closed for each $i = 1, \dots, r$ and $a_{mr+i} \to a$ as $m \to +\infty$. This assures that $\cap_{i=1}^{r} B_i \neq \emptyset$ and, hence, we have the claim. $\square$

6. Contractions of integral type

In this section, we introduce a new contractive condition of integral type. In order to do this, we consider suitable functions of the family $\mathcal{S}$.

Let $\nu : [0,+\infty[\to [0,+\infty[$ be an upper semicontinuous function such that $\nu(\alpha) < \alpha$ for all $\alpha > 0$ and $\nu(0) = 0$. We affirm that the function $S : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ defined by $S(\alpha,\beta) = \nu(\beta) - \alpha$ belongs to $\mathcal{S}$. Firstly, we notice that $S(\alpha,\beta) = \nu(\beta) - \alpha < \beta - \alpha$ for each $\alpha,\beta > 0$ and so S satisfies the property (S_1). Further, if $\{\alpha_m\}$, $\{\beta_m\}$ in $]0,+\infty[$ are two sequences such that $\lim_{m\to+\infty}\alpha_m = \lim_{m\to+\infty}\beta_m = \ell \in]0,+\infty[$ then, taking into account that ν is upper semicontinuous, we have

$$\limsup_{m\to+\infty} S(\alpha_m,\beta_m) \leq \limsup_{m\to+\infty} \nu(\beta_m) - \ell \leq \nu(\ell) - \ell < \ell - \ell = 0.$$

Clearly, this assures that S also satisfies the property (S_2).

In a similar way, we deduce that given a lower semicontinuous function $\nu : [0,+\infty[\to [0,+\infty[$ such that $\nu(\alpha) > \alpha$ for all $\alpha > 0$ and $\nu(0) = 0$, the function $S : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ defined by $S(\alpha,\beta) = \beta - \nu(\alpha)$ belongs to $\mathcal{S}$.

Next, let $\tau : [0,+\infty[\to [0,+\infty[$ be a function that is Lebesgue integrable in every interval $[0,t]$ with $t > 0$. Thanks to the previous considerations, we can affirm that

- $S : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ defined by $S(\alpha,\beta) = \int_0^\beta \tau(s)ds - \alpha$, for all $\alpha,\beta \in [0,+\infty[$, belongs to $\mathcal{S}$ if $\int_0^t \tau(s)\,ds < t$ for all $t > 0$;
- $S : [0,+\infty[\times[0,+\infty[\to [0,+\infty[$ defined by $S(\alpha,\beta) = \beta - \int_0^\alpha \tau(s)ds$, for all $\alpha,\beta \in [0,+\infty[$, belongs to $\mathcal{S}$ if $\int_0^t \tau(s)\,ds > t$ for all $t > 0$.

Finally, we can establish two new results of fixed point that involve a contractive condition of integral type.

Theorem 6.1. *Let $(X,d,\preceq)$ be an ordered complete metric space, $\sigma : X\times X \to [0,+\infty[$ be a w_0-distance on X and let $f : X \to X$ be a nondecreasing mapping. Further, we assume that there exist a function $H \in \mathcal{H}$, a lower semicontinuous function $\eta : X \to [0,+\infty[$ and a function $\tau : [0,+\infty[\to [0,+\infty[$ Lebesgue integrable in every interval $[0,t]$, $t > 0$, such that*

$$H(\mu(fb,fc),\eta(fb),\eta(fc)) \leq \int_0^{H(\mu(b,c),\eta(b),\eta(c))} \tau(s)ds \quad \text{for all } b,c \in X,\ b \preceq c.$$

If there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$, X has the property (A), X is regular and further $\int_0^t \tau(s)ds < t$ for all $t > 0$, then f has a unique fixed point a such that $\eta(a) = 0$.

Proof. We have immediately the claim by Theorem 3.3 if we choose $S \in \mathcal{S}$ given by $S(\alpha,\beta) = \int_0^\beta \tau(s)ds - \alpha$, for all $\alpha,\beta \in [0,+\infty[$. □

Theorem 6.2. *Let $(X,d,\preceq)$ be an ordered complete metric space, $\sigma : X\times X \to [0,+\infty[$ be a w_0-distance on X and let $f : X \to X$ be a nondecreasing mapping. Moreover, we suppose that there exist a function $H \in \mathcal{H}$, a lower semicontinuous function $\eta : X \to [0,+\infty[$ and a function $\tau : [0,+\infty[\to [0,+\infty[$ Lebesgue integrable in every interval $[0,t]$, $t > 0$, such that*

$$\int_0^{H(\mu(fb,fc),\eta(fb),\eta(fc))} \tau(s)ds \leq H(\mu(b,c),\eta(b),\eta(c)) \quad \text{for all } b,c \in X,\ b \preceq c.$$

If there exists a point $a_0 \in X$ such that $a_0 \preceq fa_0$, X has the property (A), X is regular and further $\int_0^t \tau(s)ds > t$ for all $t > 0$, then f has a unique fixed point a such that $\eta(a) = 0$.

Proof. The claim follows by Theorem 3.3 if we choose $S \in \mathcal{S}$ given by $S(\alpha,\beta) = \beta - \int_0^{\alpha} \tau(s)ds$, for all $\alpha, \beta \in [0, +\infty[$. $\square$

Next, we give an example which shows the userfulness of our contractive condition. In addition, such an example is aimed to remark that the function η has a crucial role in enlarging the class of self mappings which are μ-contractions.

Example 6.3. *Let $X = [0, 2]$. We endow X with the usual metric $d(b, c) = |b - c|$ for all $b, c \in X$ and, in addition, we consider on X the w_0-distance $\sigma : X \times X \to [0, +\infty[$ defined by $\sigma(b, c) = c$ for all $b, c \in X$. Further, we define on X a partial order $\preceq$ given by*

$$b, c \in X, \quad b \preceq c \text{ if } b = c, \quad (b \le c,\ b, c \in [0, \tfrac{39}{22}]) \quad \text{or} \quad (b \in [0, 2[\text{ and } c = 2).$$

We notice that $(X, d, \preceq)$ is an ordered complete metric space. Moreover, $\mu : X \times X \to [0, +\infty[$ is given by $\mu(b, c) = \max\{b, c\}$ for all $b, c \in X$. Let $f : X \to X$ be the function defined by

$$fb = \begin{cases} hb & \text{if } b \in [0, \frac{39}{22}] \text{ and } 0 \le h \le 1/3, \\ \frac{3}{2} & \text{if } b \in]\frac{39}{22}, 2]. \end{cases}$$

Clearly, f satisfies the contractive condition of Theorem 6.2 with respect to the function $H \in \mathcal{H}$ defined by $H(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$ for all $\alpha, \beta, \gamma \in [0, +\infty[$, the lower semicontinuous function $\eta : X \to [0, +\infty[$ defined by $\eta(b) = b$ for all $b \in X$ and the function $\tau : [0, +\infty[\to [0, +\infty[$ given by

$$\tau(s) = 1 + \frac{1}{(s+1)^2} \quad \text{for all } s \in [0, +\infty[.$$

Let $b, c \in X$ with $b \preceq c$, then we have

$$H(\mu(fb, fc), \eta(fb), \eta(fc)) = 2fc + fb \quad \text{and} \quad H(\mu(b, c), \eta(b), \eta(c)) = 2c + b.$$

If $b \le c$ and $b, c \in [0, \frac{39}{22}]$, then $2fc + fb \le 3hc$ and hence

$$\begin{aligned} \int_0^{H(\mu(fb,fc),\eta(fb),\eta(fc))} \tau(s)ds &\le \int_0^{3hc} \tau(s)ds \\ &= \frac{3hc+2}{3hc+1} 3hc \le 2c \le 2c + b \quad (\text{if } 0 \le h \le \tfrac{1}{3}) \\ &= H(\mu(b, c), \eta(b), \eta(c)). \end{aligned}$$

If $b \in [0, \frac{39}{22}]$ and $c = 2$, then $2fc + fb = 3 + hb$ and hence

$$\begin{aligned} \int_0^{H(\mu(fb,fc),\eta(fb),\eta(fc))} \tau(s)ds &= \int_0^{3+hb} \tau(s)ds = \frac{5+hb}{4+hb}(3+hb) \\ &\le 4 + b = H(\mu(b, c), \eta(b), \eta(c)). \end{aligned}$$

If $b \in]\frac{39}{22}, 2]$ and $c = 2$, then $2fc + fb = \frac{9}{2}$ and hence

$$\begin{aligned} \int_0^{H(\mu(fb,fc),\eta(fb),\eta(fc))} \tau(s)ds &= \int_0^{\frac{9}{2}} \tau(s)ds = \frac{117}{22} \\ &\le 4 + \frac{39}{22} < 4 + b \\ &= H(\mu(b, c), \eta(b), \eta(c)). \end{aligned}$$

If $b = c \in]\frac{39}{22}, 2[$, then $2fc + fb = \frac{9}{2}$ and hence

$$\begin{aligned}\int_0^{H(\mu(fb,fc),\eta(fb),\eta(fc))} \tau(s)ds &= \int_0^{\frac{9}{2}} \tau(s)ds = \frac{117}{22}\\ &= 3\frac{39}{22} < 3c\\ &= H(\mu(b,c),\eta(b),\eta(c)).\end{aligned}$$

Now, taking into account that all the conditions of Theorem 6.2 are satisfied, we can affirm that f has a unique fixed point $a = 0 = \eta(a)$ in X.

We remark that if we choose the w_0-distance $\sigma = d$ and $\eta(b) = 0$ for all $b \in X$ then from $d(f0, f2) = 3/2$ and $d(0, 2) = 2$ it follows that

$$\int_0^{d(f0,f2)} \tau(s)ds = \frac{21}{10} \geq 2 = d(0,2).$$

This assures that Theorem 27 of [13] cannot be used in order to affirm that f has a fixed point with respect to the contractive condition of Theorem 6.2 associated to the function τ.

7. Application to differential equations

In this section, we provide an application of our results to ordinary differential equations. Precisely, we use Theorem 3.5 in order to prove the existence of a unique solution for a first-order periodic differential problem. We follow the general approach, that is, we convert such a problem into a integral equation which describes exactly a fixed point of a mapping.

Here, we work in $\mathbb{R}^2$ where we consider the partial order $\leq$ given by:

$$(x,y),(z,w) \in \mathbb{R}^2, \quad (x,y) \leq (z,w) \quad \text{if and only if } x \leq z \text{ and } y \leq w.$$

Let $\mathbb{R}^2_+ = \{u \in \mathbb{R}^2 : 0 \leq u\}$ and let $\|(x,y)\| = \max\{|x|,|y|\}$ for all $(x,y) \in \mathbb{R}^2$. In addition, let τ be a positive real number and let $I = [0,\tau]$. Let us denote by $C(I,\mathbb{R}^2)$ the space of continuous functions $b : I \to \mathbb{R}^2$. We recall that $C(I,\mathbb{R}^2)$ is a complete metric space with respect to the metric $d : C(I,\mathbb{R}^2) \times C(I,\mathbb{R}^2) \to [0,+\infty[$ given by

$$d(b,c) = \sup_{t\in I} \|b(t) - c(t)\| \quad \text{for all } b,c \in C(I,\mathbb{R}^2).$$

We endow $C(I,\mathbb{R}^2)$ with the w_0-distance $\sigma : C(I,\mathbb{R}^2) \times C(I,\mathbb{R}^2) \to \mathbb{R}^+_0$ defined by $\sigma(b,c) = d(0,c)$ for all $b,c \in C(I,\mathbb{R}^2)$. This implies that $\mu(b,c) = \max\{d(0,b), d(0,c)\}$ for all $b,c \in C(I,\mathbb{R}^2)$. Moreover, we define on $C(I,\mathbb{R}^2)$ the partial order $\leq$ given by

$$b \leq c \quad \text{if } b(t) \leq c(t) \text{ for all } t \in I.$$

We notice that the ordered metric space $(C(I,\mathbb{R}^2), d, \leq)$ is regular and has the property (A).

Next, we consider the first-order periodic problem

$$\begin{cases} b'(t) = g(t,b(t)), & t \in I,\\ b(0) = b(\tau). \end{cases} \tag{6}$$

where $g : I \times \mathbb{R}^2 \to \mathbb{R}^2$ is a continuous function. We know that the problem (6) is equivalent to the integral equation

$$b(t) = \int_0^\tau G(t,s)[g(s,b(s)) + \gamma b(s)]ds,$$

where γ is a real number and the Green function $G : I \times I \to \mathbb{R}$ is defined by

$$G(t,s) = \begin{cases} \dfrac{e^{\gamma(\tau+s-t)}}{e^{\gamma\tau}-1} & \text{if } 0 \le s \le t \le \tau, \\[2ex] \dfrac{e^{\gamma(s-t)}}{e^{\gamma\tau}-1} & \text{if } 0 \le t < s \le \tau. \end{cases}$$

By the definition of G, we notice that

$$\int_0^\tau G(t,s)\,ds = \frac{1}{\gamma} \quad \text{for each } t \in I.$$

Therefore, we can associate to the problem (6) the integral operator $f : C(I,\mathbb{R}^2) \to C(I,\mathbb{R}^2)$ defined by

$$(fb)(t) = \int_0^\tau G(t,s)[g(s,b(s)) + \gamma b(s)]ds \quad \text{for all } b \in C(I,\mathbb{R}^2).$$

Now, we stress that $b \in C(I,\mathbb{R}^2)$ is a solution of the problem (6) if and only if b is a fixed point of f.

If the function g is nondecreasing in the second variable and $\gamma > 0$, then for all $b, c \in C(I,\mathbb{R}^2)$ with $b \preceq c$, we have that

$$g(s,b(s)) + \gamma b(s) \preceq g(s,c(s)) + \gamma c(s) \quad \text{for all } s \in I.$$

Since $G(t,s) \ge 0$ for each $t,s \in I$, we have also

$$\begin{aligned}(fb)(t) &= \int_0^\tau G(t,s)[g(s,b(s)) + \gamma b(s)]ds \\ &\preceq \int_0^\tau G(t,s)[g(s,c(s)) + \gamma c(s)]ds \\ &=(fc)(t) \quad \text{for all } t \in I\end{aligned}$$

and hence it follows that $fb \preceq fc$.

Now, we formulate the following technical assumption (such a assumption is needed in order to use one among of our results of fixed point):

There exists $\gamma > 0$ such that for all $c \in C(I,\mathbb{R}^2)$, we have

$$\|f(s,c(s)) + \gamma c(s)\| \le \gamma \frac{\|c(s)\|}{1+\|c(s)\|} \quad \text{for all } s \in I. \tag{7}$$

Then for all $b, c \in C(I,\mathbb{R}^2)$ such that $b \preceq c$, we deduce that

$$\begin{aligned}\mu(fb,fc) &= \sup_{t\in I} \|(fc)(t)\| \\ &\le \sup_{t\in I} \int_0^\tau G(t,s)\|g(s,c(s)) + \gamma c(s)\|\,ds \\ &\le \sup_{t\in I} \int_0^\tau G(t,s)\gamma \frac{\|c(s)\|}{1+\|c(s)\|}ds.\end{aligned} \tag{8}$$

Taking into account that the function $t \to \frac{t}{1+t}$ is nondecreasing, we have that

$$\frac{\|c(s)\|}{1+\|c(s)\|} \le \frac{\sup_{s\in I}\|c(s)\|}{1+\sup_{s\in I}\|c(s)\|} = \frac{d(0,c)}{1+d(0,c)}. \tag{9}$$

So, using (8) and (9), we obtain

$$\mu(fb, fc) \leq \gamma \left(\sup_{t \in I} \int_0^\tau G(t,s)\, ds \right) \frac{d(0,c)}{1 + d(0,c)} = \frac{\mu(b,c)}{1 + \mu(b,c)}$$

for all $b, c \in C(I, \mathbb{R}^2)$ with $b \preceq c$ (we recall that $d(0,b) \leq d(0,c)$ since $b \preceq c$).

Now, let $h \in C(I, \mathbb{R}^2)$, we say that h is a lower solution of problem (6) if h satisfies

$$\begin{cases} h'(t) \preceq g(t, h(t)) & \text{for all } t \in I, \\ h(0) \preceq h(\tau). \end{cases}$$

We can easily see that

$$h(t) \preceq \int_0^\tau G(t,s)[g(s,h(s)) + \gamma h(s)]\, ds = (fh)(t) \quad \text{for all } t \in I,$$

that is, $h \preceq fh$. Then, taking into account that the function $S : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, given by

$$S(\alpha, \beta) = \frac{\beta}{1 + \beta} - \alpha,$$

belongs to the family $\mathcal{S}$, we can use Theorem 3.5 with S as above, $H(\alpha, \beta, \gamma) = \alpha + \beta + \gamma$ for all $\alpha, \beta, \gamma \in [0, +\infty[$ and $\eta(b) = 0$ for all $b \in C(I, \mathbb{R}^2)$. In this way, we obtain the following result.

Theorem 7.1. *For the Problem* (6) *with $g : I \times \mathbb{R}^2 \to \mathbb{R}^2$ continuous and nondecreasing with respect to the second variable, the existence of a lower solution provides the existence of a unique solution if there exists a positive real numbers γ such that* (7) *holds for all $c \in C(I, \mathbb{R}^2)$.*

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