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Some Common Fixed Point Theorems for Contractive Maps and Applications

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Abstract. The aim of this paper is to obtain some new fixed point theorems for single valued mappings and also for hybrid pair of mappings. Our results extend and generalize well known results due to Aamri and El Moutawakil, Sintunavarat and Kumam, Kadelburg et al., and Kamran; and also generalize and rectify some recent results due to Bisht [On Existence of common fixed points under Lipschitz- Type Mapping pairs with Applications, Numer. Func. Anal. Optim. 38(11), 2017, 1446-1457].

1. Introduction

Generalizing Banach contraction principle, Jungck [7] initiated the study of common fixed points for a pair of commuting self-mappings. In 1982, Sessa [18] introduced the concept of weakly commuting maps. In order to generalize the concept of weak commutative, Jungck [8] defined the notion of compatible maps. Jungck further weakened the notion of compatibility by introducing the notion of weak compatibility [9]. Over the last two decades, several authors have proved common fixed point theorems for a pair of mappings under different contractive conditions using compatibility and its weaker versions (see [1, 3, 5, 14, 15, 19] and references therein).

Pant [12–15] initiated the study of non-compatible maps and introduced the notion of pointwise R-weakly commuting mappings. It is well known that for single valued mappings pointwise R- weakly commuting is equivalent to weak compatibility. Further, Aamri and El Moutawakil [1] defined (E. A) property for self-maps and obtained some fixed point theorems for such mapping under strict contractive conditions.

In 1989, Singh et al. [19] extended the notion of compatible mappings and obtained some coincidence and common fixed point theorems for nonlinear hybrid contractions. Afterwards, Pathak [17] generalized the concept of compatibility by defining weak compatibility for hybrid pairs of mappings (including single-valued case) and utilized the same to prove common fixed point theorems. Inspired by the work of Aamri and Moutawakil [1], Kamran [11] extended the notion of property (E.A) for a hybrid pair of mappings.

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In 2011, Sintunavarat and Kumam [20] introduced the notion of common limit range property for single-valued mappings and showed its superiority over property (E.A). Motivated by Sintunavarat and Kumam [20], Imdad et al. [6] established common limit range property for a hybrid pair of mappings and proved some fixed point results.

Recently, Bisht [4] and Kadelburg [10] investigated the usefulness of the notion CLR_g in fixed point considerations under contractive conditions.

It may be pointed out that Theorem 2.6 and Theorem 2.7 of a recent paper due to Bisht [4] do not hold. We rectify these theorems and generalize them for both single valued as well as hybrid pair of mappings. Our results extend the results of Aamri and El Moutawakil [1], Kamran [11], Sintunavarat and Kumam [21] and Kadelburg et al. [10].

2. Preliminaries

Let f and g be self-maps of a metric space (X, d). Following [4], we denote $C(f, g)$ and $PC(f, g)$ for the set of coincidence points and the set of points of coincidence of f and g respectively, i.e., $C(f, g) = \{x \in X : fx = gx\}$ and $PC(f, g) = \{y \in X : y = fx = gx, \text{ for some } x \in X\}$ respectively.

Definition 2.1. *Let (X, d) be a metric space with $f, g : X \to X$. A pair of mappings (f, g) is said to be:*

(1) *Commuting on X if $fgx = gfx$ for all $x \in X$.*
(2) *Weakly Commuting [18] on X if $d(fgx, gfx) \leq d(fx, gx)$ for all $x \in X$.*
(3) *R-weakly commuting [12] if there exists some positive real number R such that $d(fgx, gfx) \leq Rd(fx, gx)$ for all $x \in X$.*
(4) *Pointwise R- weakly commuting [13] if given $x \in X$ there exist some positive real number R such that $d(fgx, gfx) \leq Rd(fx, gx)$.*
(5) *Compatible [8] if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.*
(6) *Noncompatible [14] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent.*
(7) *Property (E. A.) [1] if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.*
(8) *Weakly compatible [9] if $fx = gx$ implies that $fgx = gfx$.*
(9) *Occasionally weakly compatible [2] if, for some $x \in X$, $fx = gx$ implies that $fgx = gfx$.*
(10) *Conditionally commuting [16] if $C(f, g) \neq \phi$ implies that there exists $\phi \neq Y \subseteq C(f, g)$ such that, for all $y \in Y$, $fgy = gfy$.*

Let (X, d) be a metric space. Then, the following definitions will be needed.

(1) $CL(X) = \{A : A$ is a nonempty closed subset of $X\}$,
(2) $CB(X) = \{A : A$ is a nonempty closed and bounded subset of $X\}$,
(3) $d(x, A) = \inf_{a\in A} d(x, a)$,
(4) $H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\}$, where $A, B \in CB(X)$. Generally, H is called Hausdorff metric.

Definition 2.2. *Let (X, d) be a metric space with $f : X \to X$ and $T : X \to CB(X)$.*

(1) *A point $x \in X$ is a fixed point of f (resp. T) if $x = fx$ (resp. $x \in Tx$). The set of all fixed points of f (resp. T) is denoted by $F(f)$ (resp. $F(T)$).*
(2) *A point $x \in X$ is a coincidence point of f and T if $fx \in Tx$. The set of all coincidence points of f and T is denoted by $C(f, T)$.*

(3) *A point $x \in X$ is a common fixed point of f and T if $x = fx \in Tx$. The set of all common fixed points of f and T is denoted by $F(f, T)$.*

Definition 2.3. *[10] Let (X, d) be a metric space with $f : X \to X$ and $T : X \to CB(X)$. A hybrid pair of mappings (f, T) is said*

(1) *to be commuting on X if $fTx \subseteq Tfx$ for all $x \in X$,*
(2) *to be weakly commuting on X if $H(fTx, Tfx) \leq d(fx, Tx)$ for all $x \in X$,*
(3) *to be compatible if $fTx \in CB(X)$ for all $x \in X$ and $\lim_n H(Tfx_n, fTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $Tx \to A \in CB(X)$ and $fx \to t \in A$, as $n \to \infty$,*
(4) *to be noncompatible if there exists at least one sequence $\{x_n\}$ in X such that $Tx \to A \in CB(X)$ and $fx_n \to t \in A$, as $n \to \infty$, but $\lim_n H(Tfx_n, fTx_n)$ is either nonzero or nonexistent,*
(5) *to be weakly compatible if $fTx = Tfx$ for each $x \in C(f, T)$,*
(6) *to be occasionally weakly compatible if $fTx \subseteq Tfx$ for some $x \in C(f, T)$,*
(7) *to be coincidentally idempotent if $ffv = fv$ for every $v \in C(f, T)$; that is, f is idempotent at the coincidence points of f and T,*
(8) *to be occasionally coincidentally idempotent if $ffv = fv$ for some $v \in C(f, T)$.*
(9) *to satisfy Property (E. A.) if there exists a sequence $\{x_n\}$ in X, some $t \in X$ and $A \in CB(X)$ such that $\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n$.*

Definition 2.4. *[20] Two self-mapppings f and g of a metric space (X, d) are said to satisfy the common limit in range of g property if there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gt$, for some $t \in X$.*

Definition 2.5. *[6] Let (X, d) be a metric space with $f : X \to X$ and $T : X \to CB(X)$. Then the hybrid pair of mappings (f, T) is said to satisfy the common limit range property with respect to the mapping f if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = ft \in A = \lim_{n\to\infty} Tx_n$, for some $t \in X$ and $A \in CB(X)$.*

3. Main Results

In a recent work, Bisht [4] obtained the following results.

Theorem 3.1 (Theorem 2.6 of [4]). *Let f and g be weakly compatible self-mappingsof a metric space (X, d) satisfying*

(i) $d(fx, fy) < max\Big\{d(gx, gy), d(fx, gx), d(fy, gy), \frac{[d(fx,gy)+d(fy,gx)]}{2}\Big\}$ (1)

whenever the right-hand side is nonzero. If f and g satisfy the common limit range property with respect to g (CLR_g), then f and g have a unique common fixed point.

Theorem 3.2 (Theorem 2.7 of [4]). *Let f and g be weakly compatible self-mappings of a metric space (X, d) satisfying*

(i) *gX is a complete subspace of X or fX is a complete subspace with $fX \subset gX$;*
(ii) $d(fx, fy) < max\Big\{d(gx, gy), d(fx, gx), d(fy, gy), \frac{[d(fx,gy)+d(fy,gx)]}{2}\Big\}$;

whenever the right hand side is non-zero. If f and g satisfy the (E. A) property, then f and g have a unique common fixed point.

It was claimed in the proof of Theorem 2.6 of Bisht[4] that

$$d(fx_n, ft) < \max\left\{d(gx_n, gt), d(fx_n, gx_n), d(ft, gt), \frac{d(fx_n, gt) + d(ft, gx_n)}{2}\right\}$$

yields a contradiction unless $ft = gt$.

However, the above inequality does not lead to a contradiction since it yields

$$d(gt, ft) \leq d(gt, ft)$$

on letting $n \to \infty$, which is not a contradiction. We now give an example which shows that Theorem 2.6 of Bisht [4] does not hold.

Example 3.3. *Let $X = (2, 6]$ equipped with the usual metric d on X. Let $f, g : X \to X$ be defined as*

$$f(x) = \frac{x+10}{4} \text{ if } 2 < x < 6, \quad f(6) = 3; \quad g(x) = \frac{14-x}{2} \text{ if } 2 < x \leq 6.$$

Clearly, f and g satisfy all the conditions of Theorem 3.1, but f and g do not possess a common fixed point (or coincidence point). If we consider the sequence $\{6-\frac{1}{n}\}$ in X then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 4 = g(6) \in X$; this shows that mappings f and g satisfy CLR_g property. Further, if we consider inequality (1) used in the proof of Theorem 2.6 of [4], then we get

$$d(fx_n, f6) < \max\left\{d(gx_n, g6), d(fx_n, gx_n), d(f6, g6), \frac{d(fx_n, g6) + d(f6, gx_n)}{2}\right\}.$$

On letting $n \to \infty$, this yields $d(g6, f6) \leq d(g6, f6) = 1$, which is not a contradiction.

We consider two approaches for rectifying Theorem 2.6 and 2.7 of Bisht [4].

(1) Replacing conditions (i) of Theorem 3.1 and (ii) of Theorem 3.2 by a stronger contractive condition, say:

$$(i') \qquad d(fx, fy) < max\left\{d(gx, gy), \frac{d(fx, gx) + d(fy, gy)}{2}, \frac{[d(fx, gy) + d(fy, gx)]}{2}\right\},$$

(2) Replacing conditions (i) of Theorem 3.1 and (ii) of Theorem 3.2 by some φ-contractive condition, say:

$$(i'') \qquad d(fx, fy) \leq \varphi\Big(\max\left\{d(gx, gy), d(fx, gx), d(fy, gy), \frac{[d(fx, gy) + d(fy, gx)]}{2}\right\}\Big)$$

where $\varphi : R_+ \to R_+$ denotes a function such that $\varphi(t) < t$ for each $t > 0$.

If we adopt the first approach, we get the following:

Theorem 3.4. *Let f and g be weakly compatible self-mappings of a metric space (X, d) satisfying*

$$(i') \qquad d(fx, fy) < max\left\{d(gx, gy), \frac{d(fx, gx) + d(fy, gy)}{2}, \frac{[d(fx, gy) + d(fy, gx)]}{2}\right\}$$

whenever the right-hand side is nonzero. If f and g satisfy the common limit range property with respect to g (CLR_g), then f and g have a unique common fixed point.

Proof. Since f and g satisfy the CLR_g property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gt$, for some $t \in X$. By condition (i') we get

$$d(fx_n, ft) < \max\left\{d(gx_n, gt), \frac{d(fx_n, gx_n) + d(ft, gt)}{2}, \frac{d(fx_n, gt) + d(ft, gx_n)}{2}\right\}.$$

On letting $n \to \infty$, we have

$$\begin{aligned} d(gt, ft) &\leq \max\left\{d(gt, gt), \frac{d(gt, gt) + d(ft, gt)}{2}, \frac{d(gt, gt) + d(ft, gt)}{2}\right\}, \\ &= \max\left\{0, \frac{d(ft, gt)}{2}, \frac{d(ft, gt)}{2}\right\} = \frac{d(ft, gt)}{2} \end{aligned}$$

This yields $ft = gt$. Thus t is coincidence point of f and g.

Further, weak compatibility of f and g implies that f and g commute at t, i.e., $fgt = gft$. This implies $fft = fgt = gft = ggt$. If $ft \neq fft$, then using (i'), we get

$$\begin{aligned} d(ft, fft) &< max\left\{d(gt, gft), \frac{d(ft, gt) + d(fft, gft)}{2}, \frac{d(ft, gft) + d(fft, gt)}{2}\right\} \\ &= d(ft, fft) \end{aligned}$$

a contradiction. Hence $ft = fft = gft$ and ft is a common fixed point of f and g. Uniqueness of the common fixed point follows from (i'). □

Remark 3.5. *Theorem 3.4 gives a proper generalization of the main result due to Aamri and El Moutawakil [1] as our result does not require the following conditions assumed by Aamri and El Moutawakil [1]:*
(a) $g(X) \subset f(X)$
(b) completeness of $g(X)$ *or* $f(X)$.

If we consider the second approach for modifying Theorem 2.6 of Bisht [4], then we obtain the following theorem:

Theorem 3.6. *Let f and g be weakly compatible self-mappings of a metric space (X, d) satisfying*

$$(i'') \qquad d(fx, fy) \leq \varphi\Big(\max\Big\{d(gx, gy), d(fx, gx), d(fy, gy), \frac{[d(fx, gy) + d(fy, gx)]}{2}\Big\}\Big)$$

whenever the right-hand side is nonzero, where $\varphi : R_+ \to R_+$ is a function such that $\varphi(t) < t$ for each $t > 0$. If f and g satisfy the common limit range property with respect to g (CLR_g), then f and g have a unique common fixed point.

Proof. Since f and g satisfy the CLR_g property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gt$, for some $t \in X$. By condition (i″) we get

$$d(fx_n, ft) \leq \varphi\Big(max\Big\{d(gx_n, gt), d(fx_n, gx_n), d(ft, gt), \frac{d(fx_n, gt) + d(ft, gx_n)}{2}\Big\}\Big).$$

Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gt$, if $ft \neq gt$ then for sufficiently large n we get

$$d(fx_n, ft) \leq \varphi(d(ft, gt)).$$

On taking limit as $n \to \infty$ this yields $d(ft, gt) \leq \varphi(d(ft, gt)) < d(ft, gt)$, a contradiction. Hence $ft = gt$. Thus t is coincidence point of f and g.

Further, weak compatibility of f and g implies that f and g commute at t, i.e., $fgt = gft$. This implies $fft = fgt = gft = ggt$. If $ft \neq fft$, then using (i″), we get

$$\begin{aligned} d(ft, fft) &\leq \varphi\Big(max\Big\{d(gt, gft), d(ft, gt), d(fft, gft), \frac{d(ft, gft) + d(fft, gt)}{2}\Big\}\Big) \\ &< max\Big\{d(gt, gft), d(ft, gt), d(fft, gft), \frac{d(ft, gft) + d(fft, gt)}{2}\Big\} \\ &= d(ft, fft), \end{aligned}$$

a contradiction. Hence $ft = fft = gft$ and ft is a common fixed point of f and g. Uniqueness of the common fixed point follows from (i″). □

We now generalize Theorem 3.6 by assuming a slightly stronger condition on φ.

Theorem 3.7. *Let f and g be weakly compatible self-mappings of a metric space (X, d) satisfying*

(ii″) $d(fx, fy) \leq \varphi(max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\})$

whenever the right-hand side is nonzero, where $\varphi : R_+ \to R_+$ is an upper semi-continuous function such that $\varphi(t) < t$ for each $t > 0$. If f and g satisfy the common limit range property with respect to g (CLR_g), then f and g have a unique common fixed point.

Proof. Since f and g satisfy the CLR_g property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gt$, for some $t \in X$. By condition (ii″) we get

$$d(fx_n, ft) \leq \varphi(max\{d(gx_n, gt), d(fx_n, gx_n), d(ft, gt), d(fx_n, gt), d(ft, gx_n)\}).$$

If $ft \neq gt$ then on letting $n \to \infty$ we get

$$\begin{aligned} d(gt, ft) &\leq \varphi(max\{d(gt, gt), d(gt, gt), d(ft, gt), d(gt, gt), d(ft, gt)\}), \\ &= \varphi(max\{0, 0, d(ft, gt), 0, d(ft, gt)\}) = \varphi(d(ft, gt)) < d(ft, gt), \end{aligned}$$

a contradiction. Thus $ft = gt$ and t is a coincidence point of f and g.

Further, weak compatibility of f and g implies that f and g commute at t, i.e., $fgt = gft$. This implies $fft = fgt = gft = ggt$. If $ft \neq fft$, then using (ii″), we get

$$\begin{aligned} d(ft, fft) &\leq \varphi(max\{d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(fft, gt)\}) \\ &< max\{d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(fft, gt)\} \\ &= d(ft, fft), \end{aligned}$$

a contradiction. Hence $ft = fft = gft$ and ft is a common fixed point of f and g. Uniqueness follows from (ii″). □

Remark 3.8. *Theorem 3.7 generalizes the result due to Aamri and El Moutawakil [1].*

We now give an example to illustrate Theorem 3.7.

Example 3.9. *Let $X = \{0, 0.5, 0.6, 1\}$ be equipped with the usual metric d on X. Define mapping f and $g : X \to X$ by*

$$\begin{aligned} f(x) &= 0, \text{ if } x \in \{0, 0.5\}, \quad f(x) = 0.5, \text{ if } x \in \{0.6, 1\}; \\ g(x) &= x \quad \forall \quad x \in X \end{aligned}$$

Then f and g satisfy all the conditions of Theorem 3.7 with $\varphi(t) = 0.9t$, and have a unique common fixed point $x = 0$. It is easy to verify in this example that f and g satisfy the common limit range property in g. To see this, we can consider the constant sequence $\{x_n = 0\}$. Then $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0 = g0$. Also, f and g are weakly compatible, since they commute at their coincidence point $x = 0$. However f and g do not satisfy the contractive condition used in the main result of Aamri and El Moutawakil [1]. To see this, let $x = 0.5$ and $y = 0.6$, then $d(fx, fy) = 0.5$ and $max\Big\{d(gx, gy), \frac{d(fx,gx)+d(fy,gy)}{2}, \frac{[d(fx,gy)+d(fy,gx)]}{2}\Big\} = 0.3$. This shows that the main result of Aamri and El Moutawakil [1] is a particular case of Theorem 3.7.

We now prove a common fixed theorem for a hybrid pair of maps.

Theorem 3.10. *Let f be a self-mapping of a metric space (X, d) and let T be a mapping from X into $CB(X)$ satisfying*

$$(iii'') \quad H(Tx, Ty) \leq \varphi(max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty), d(fy, Tx)\})$$

whenever the right hand side is non-zero, where $\varphi : R_+ \to R_+$ is an upper semi-continuous function such that $\varphi(t) < t$ for each $t > 0$. If f and T satisfy the common limit range property with respect to f (CLR_f), then f and T have a coincidence point. Moreover, if the pair (f, T) is occasionally coincidentally idempotent, then f and T have a common fixed point.

Proof. Since f and T satisfy the CLR_f property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = ft \in A = \lim_{n\to\infty} Tx_n$, for some $t \in X$ and $A \in CB(X)$.
We claim that $ft \in Tt$.
By condition (iii″) we get

$$H(Tx_n, Tt) \leq \varphi(max\{d(fx_n, ft), d(fx_n, Tx_n), d(ft, Tt), d(fx_n, Tt), d(ft, Tx_n)\}).$$

On letting $n \to \infty$, we get

$$H(A, Tt) \leq \varphi(max\{0, 0, d(ft, Tt), d(ft, Tt), 0\}) \leq \varphi(d(ft, Tt)).$$

Since $ft \in A$, if ft is not in Tt, then we get

$$d(ft, Tt) \leq H(A, Tt) \leq \varphi(d(ft, Tt)) < d(ft, Tt),$$

a contradiction. Hence $ft \in Tt$ and t is coincidence point of f and T.

If the mappings f and T are occasionally coincidentally idempotent, two cases arise:
Case I: f and T may be coincidentally idempotent at t, then we have $fft = ft \in Tt$. Now we show that $Tt = Tft$. If not, using condition (iii″), we get

$$\begin{aligned} H(Tft, Tt) &\leq \varphi(max\{d(fft, ft), d(fft, Tft), d(ft, Tt), d(fft, Tt), d(ft, Tft)\}) \\ &= \varphi(d(ft, Tft)) < d(ft, Tft). \end{aligned}$$

Since $ft \in Tt$, we have

$$d(ft, Tft) \leq H(Tft, Tt) < d(ft, Tft),$$

which is a contradiction. Hence $Tt = Tft$. This implies $ft = fft \in Tt = Tft$ and, hence, ft is a common fixed point of the mappings f and T.
Case II: If f and T are not coincidentally idempotent at t, then by virtue of occasionally coincidentally idempotent property of f and T, there exists a coincidence point $t' \in X$ of f and T at which f and T are coincidentally idempotent, i.e., $fft' = ft'$. The rest of the proof is similar to that in Case I. □

Corollary 3.11. *Let f be a self mapping of a metric space (X, d) and let T be a mapping from X into $CB(X)$ satisfying condition*

$$(iv'') \qquad H(Tx, Ty) \leq \varphi\Big(\max\Big\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{d(fx, Ty) + d(fy, Tx)}{2}\Big\}\Big),$$

where $\varphi : R_+ \to R_+$ is an upper semi-continuous function such that $\varphi(t) < t$ for each $t > 0$. Suppose that the pair (f, T) satisfies the common limit range property with respect to the mapping f. Then the mappings f and T have a coincidence point, i.e., $C(f, T) \neq \phi$. Moreover, if the pair (f, T) is occasionally coincidentally idempotent, then f and T have a common fixed point.

Theorem 3.10 extends the results of Kamran [11], Sintunavarat et al. [21] and Zoran Kadelburg et al. [10]. We now give an example to illustrate Theorem 3.10.

Example 3.12. *Let $X = \{0, \frac{1}{2}, \frac{3}{5}, 1\}$ be equipped with the standard Hausdorff metric d on X. Consider the mappings $f : X \to X$ and $T : X \to CB(X)$ given by $f(x) = x$ for every $x \in X$ and*

$$Tx = \{0\}, \text{ if } x = 0, \frac{1}{2}; \quad Tx = \Big\{\frac{1}{2}\Big\}, \text{ if } x = \frac{3}{5}, 1.$$

Here, T and f satisfy all the conditions of Theorem 3.10 with $\varphi(t) = 0.9t$, and have a unique common fixed point $x = 0$. It is easy to verify in this example that f and T satisfy the CLR_f property. To see this, we can consider the constant sequence $\{x_n = 0\}$. Then $\lim_{n\to\infty} fx_n = 0 = f(0) \in A = \lim_{n\to\infty} Tx_n$.

Also, T and f are occasionally coincidentally idempotent, since $ff0 = f0$ for $0 \in C(f, T)$. However, f and T do not satisfy the contractive conditions used in the main results of Kamran [11], Sintunavarat et al. [21], Kadelburg et al. [10]. To see this, let $x = \frac{1}{2}$ and $y = \frac{3}{5}$. Then $H(Tx, Ty) = 0.5$ and $\max\Big\{d(fx, fy), \frac{d(fx,Tx)+d(fy,Ty)}{2}, \frac{d(fx,Ty)+d(fy,Tx)}{2}\Big\} = 0.3$.

Remark 3.13. *We also point out that in Theorem 3.5, 3.6 and 3.7, we cannot use conditional commutativity in place of weak compatibility since for single- valued mappings contractive conditions exclude the possibility of more than one coincidence point [4]. Hence, conditional commutativity reduces to weak compatibility or pointwise R-weakly commuting and no real generalization is obtained by assuming conditional commutativity.*

4. Applications

We now give an application of Theorem 3.7 to solve an eigen value problem for operators defined on a normed space. Our application rectifies the application given by Bisht [4] since the application given in [4] is based on Theorem 2.6 [4] which does not hold. Let X be a normed space. A real number λ is said to be an eigenvalue of a mapping $f : X \to X$ if there exists a point $x \neq 0 \in X$ such that $\lambda x = fx$.

Theorem 4.1. *Let X be a normed space and f be a self-mapping of X with $f(0) \neq 0$, satisfying the following conditions:*

(i) there exists a sequence $\{x_m\}$ such that

$$\lim_{m\to\infty} f_n x_m = \lim_{n\to\infty} x_m = t \text{ for some } t \in X,$$

where $f_n = (1 - \frac{1}{n})f, n = 2, 3, \ldots;$

(ii) either X or fX is complete;

(iii) $\|fx - fy\| \leq \max\{\|x - y\|, \|f_n x - x\|, \|f_n y - y\|, \|f_n x - y\|, \|f_n y - x\|\}$.

Then $M_n = \frac{1}{(1-\frac{1}{n})}$ is an eigenvalue of f for each $n > 1$.

Proof. Let I denote the identity mapping on X. Then, clearly, f_n and I are weakly compatible mappings satisfying the common limit range property. Further, $\|f_n x - f_n y\| = (1 - \frac{1}{n})\|fx - fy\|$ for each $n > 1$. By using (iii), we have $\|f_n x - f_n y\| \leq (1 - \frac{1}{n}) \max\{\|x - y\|, \|f_n x - x\|, \|f_n y - y\|, \|f_n x - y\|, \|f_n y - x\|\} \leq \varphi(\max\{\|Ix - Iy\|, \|f_n x - Ix\|, \|f_n y - Iy\|, \|f_n x - Iy\|, \|f_n y - Ix\|\})$, where $\varphi(t) = (1 - \frac{1}{n})t$. Thus conditions of Theorem 3.7 are satisfied for f_n and I, for all $x, y \in X$, and each $n > 1$. Thus, there exists x_n in X such that $x_n = f_n x_n$ for each $n > 1$. This implies $x_n = (1 - \frac{1}{n})fx_n$, i.e, $fx_n = \frac{1}{(1-\frac{1}{n})}x_n = M_n x_n$, say, for each $n > 1$. Since $f(0) \neq 0$, we get $x_n \neq 0$ for each $n > 1$. Thus x_n is eigenvector corresponding to eigen value M_n for f. □

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