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On Anti-Invariant Semi-Riemannian Submersions from Lorentzian Para-Sasakian Manifolds

Morteza Faghfouri[a], **Sahar Mashmouli**[a]

[a] *Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran.*

Abstract. In this paper, we study a semi-Riemannian submersion from Lorentzian almost (para) contact manifolds and find necessary and sufficient conditions for the characteristic vector field to be vertical or horizontal. We also obtain decomposition theorems for anti-invariant semi-Riemannian submersions from Lorentzian para-Sasakian manifolds onto Lorentzian manifolds.

1. Introduction

Semi-Riemannian submersions between semi-Riemannian manifolds were studied by O'Neill [19, 20] and Gray [9]. Moreover, B. Ṣahin in [22, 23] introduced anti-invariant Riemannian submersions and slant submersions from almost Hermitian manifold onto Riemannian manifolds. Also, anti-invariant Riemannian submersions were studied in [2, 6, 7, 14, 15, 18]. The theory of Lorentzian submersion was introduced by Magid and Falcitelli *et al* in [16] and [17], respectively. In [13] Kaneyuki and Williams defined the almost paracontact structure on pseudo-Riemannian manifold. Recently, Gündüzalp and Şahin studied paracontact structures in [10–12].

In this paper, we studied anti-invariant semi-Riemannian submersions from Lorentzian almost (para) contact manifolds. In Sect. 3, we introduced anti-invariant semi-Riemannian submersions from Lorentzian almost (para) contact manifolds and presented three examples. Also we find necessary and sufficient conditions for the characteristic vector field to be vertical or horizontal. In sect. 4, we studied anti-invariant semi-Riemannian submersions from Lorentzian (para) Sasakian manifolds onto a Riemannian manifold such that the characteristic vector field is vertical and investigated the geometry of leaves of the distributions. In sect. 5, we studied anti-invariant semi-Riemannian submersions from Lorentzian (para) Sasakian manifolds onto Lorentzian manifolds such that the characteristic vector field is a horizontal vector field and we obtained decomposition theorems for it.

2. Preliminaries

In this section, we recall some necessary details background on Lorentzian almost contact manifold, Lorentzian almost para contact manifold, semi-Riemannian submersion and harmonic maps.

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Email addresses: faghfouri@tabrizu.ac.ir (Morteza Faghfouri), s_mashmouli91@ms.tabrizu.ac.ir (Sahar Mashmouli)

2.1. Lorentzian almost (para)contact manifold

Let (M, g) be a $(2n+1)$-dimensional Lorentzian manifold with a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η which satisfies

$$\phi^2 X = \varepsilon X + \eta(X)\xi, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2}$$

$$\eta(X) = \varepsilon g(X, \xi), \tag{3}$$

$$\eta(\xi) = -\varepsilon, \tag{4}$$

for any vector fields X, Y tangent to M, it is called Lorentzian almost contact manifold or Lorentzian almost para contact manifold for $\varepsilon = -1$ or $\varepsilon = 1$, respectively[1]. In this case, (1) and (4) imply that $\phi\xi = 0, \eta\circ\phi = 0$, and rank $\phi = 2n$. However, for any vector fields X, Y in $\Gamma(TM)$,

$$g(\phi X, Y) = \varepsilon g(X, \phi Y). \tag{5}$$

Let Φ be the 2-form in M given by $\Phi(X, Y) = g(X, \phi Y)$. Then, M is called Lorentzian metric contact manifold if $d\eta(X, Y) = \Phi(X, Y)$. So, if the manifold satisfies $[\phi, \phi] + 2d\eta \otimes \xi = 0$, then M is called normal Lorentzian almost contact manifold. If ξ is a Killing tensor vector field, then the (para) contact structure is called K-(para) contact. In such a case, we have

$$\nabla_X \xi = \varepsilon\phi X, \tag{6}$$

where ∇ denotes the Levi-Civita connection of g. A Lorentzian almost contact manifold or Lorentzian almost para contact manifold M is called Lorentzian Sasakian (LS) or Lorentzian para Sasakian (LPS) if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X. \tag{7}$$

Now we will introduce a well known Sasakian manifold example on $\mathbb{R}^{2n+1}$.

Example 2.1 ([3]). *Let $\mathbb{R}^{2n+1} = \{(x^1, \ldots, x^n, y^1, \ldots, y^n, z) | x^i, y^i, z \in \mathbb{R}, i = 1, \ldots, n\}$. Consider $\mathbb{R}^{2n+1}$ with the following structure:*

$$\phi_\epsilon\left(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}\right) = -\epsilon\sum_{i=1}^{n} Y_i\frac{\partial}{\partial x^i} - \sum_{i=1}^{n} X_i\frac{\partial}{\partial y^i} + \sum_{i=1}^{n} Y_i y_i\frac{\partial}{\partial z}, \tag{8}$$

$$g = -\eta\otimes\eta + \frac{1}{4}\sum_{i=1}^{n}(dx^i\otimes dx^i + dy^i\otimes dy^i), \tag{9}$$

$$\eta_\epsilon = -\frac{\epsilon}{2}\left(dz - \sum_{i=1}^{n} y^i dx^i\right), \tag{10}$$

$$\xi = 2\frac{\partial}{\partial z}. \tag{11}$$

Then, $(\mathbb{R}^{2n+1}, \phi_\epsilon, \xi, \eta_\epsilon, g)$ is a Lorentzian Sasakian manifold if $\epsilon = -1$ and Lorentzian para Sasakian manifold if $\epsilon = 1$. The vector fields $E_i = 2\frac{\partial}{\partial y^i}, E_{n+i} = 2(\frac{\partial}{\partial x^i} + y_i\frac{\partial}{\partial z})$ and ξ form a ϕ-basis for the contact metric structure.

2.2. Semi-Riemannian submersion

Let (M, g_M) and (N, g_N) be semi-Riemannian manifolds. A semi-Riemannian submersion $F : M \to N$ is a submersion of semi-Riemannian manifolds such that:

1. The fibers $F^{-1}(q), q \in N$ are semi-Riemannian submanifolds of M.
2. F_* preserves scalar products of vectors normal to fibers.

For each $q \in N$, $F^{-1}(q)$ is a submanifold of M of dimension $\dim M - \dim N$. The submanifolds $F^{-1}(q), q \in N$ are called fibers, and a vector field on M is vertical if it is always tangent to fibers, horizontal if it is always orthogonal to fibers. A vector field X on M is called basic if X is a horizontal vector field and F-related to a vector field X_* on N. Every vector field X_* on N has a unique horizontal lift X to M, and X is basic. For a semi-Riemannian submersion $F : M \to N$, let $\mathcal{H}$ and $\mathcal{V}$ denote the projections of the tangent spaces of M onto the subspaces of horizontal and vertical vectors, respectively. In the other words, $\mathcal{H}$ and $\mathcal{V}$ are the projection morphisms on the distributions $(\ker F_*)^\perp$ and $\ker F_*$, respectively [20].

Lemma 2.2 ([19]). *Let $F : M \to N$ be semi-Riemannian submersion between a semi-Riemannian manifolds and X, Y are basic vector fields on M. Then*

a) $g_M(X, Y) = g_N(X_*, Y_*) \circ F$.
b) *the horizontal part $\mathcal{H}[X, Y]$ of $[X, Y]$ is a basic vector field and corresponds to $[X_*, Y_*]$, i.e., $F_*(\mathcal{H}[X, Y]) = [X_*, Y_*]$.*
c) *$[V, X]$ is vertical vector field for any vector field V of* $\ker F_*$.
d) *$\mathcal{H}(\nabla^M_X Y)$ is the basic vector field corresponding to $\nabla^N_{X_*} Y_*$.*

The fundamental tensors of a submersion were defined by O'Neill. They are $(1, 2)$-tensors on M, given by the formula:

$$\mathcal{T}(E, F) = \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{12}$$

$$\mathcal{A}(E, F) = \mathcal{A}_E F = \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F, \tag{13}$$

for any vector field E and F on M, where ∇ denotes the Levi-Civita connection of (M, g_M). It is easy to see that a Riemannian submersion $F : M \to N$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew-symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. In the other words,

$$g(\mathcal{T}_D E, G) = -g(E, \mathcal{T}_D G), \tag{14}$$

$$g(\mathcal{A}_D E, G) = -g(E, \mathcal{A}_D G), \tag{15}$$

for any $D, E, G \in \Gamma(TM)$. For any U, V vertical vector fields and X, Y horizontal vector fields, $\mathcal{T}$ and $\mathcal{A}$ satisfy:

$$\mathcal{T}_U V = \mathcal{T}_V U, \tag{16}$$

$$\mathcal{A}_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X, Y]. \tag{17}$$

Moreover, from (12) and (13), we have

$$\nabla_V W = \mathcal{T}_V W + \hat{\nabla}_V W, \tag{18}$$

$$\nabla_V X = \mathcal{H}\nabla_V X + \mathcal{T}_V X, \tag{19}$$

$$\nabla_X V = \mathcal{A}_X V + \mathcal{V}\nabla_X V, \tag{20}$$

$$\nabla_X Y = \mathcal{H}\nabla_X Y + \mathcal{A}_X Y, \tag{21}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V, W \in \Gamma(\ker F_*)$, where $\hat{\nabla}_V W = \mathcal{V}\nabla_V W$.

2.3. Foliations on manifold and decomposition theorem

A foliation $\mathcal{D}$ on a manifold M is an integrable distribution. A foliation $\mathcal{D}$ on a semi-Riemannian manifold M is called totally umbilical, if every leaf of $\mathcal{D}$ is a totally umbilical semi-Riemannian submanifold of M. If, in addition, the mean curvature vector of every leaf is parallel in the normal bundle, then $\mathcal{D}$ is called a sphenic foliation, because in this case each leaf of $\mathcal{D}$ is an extrinsic sphere of M. If every leaf of $\mathcal{D}$ is a totally geodesic submanifold of $\mathcal{D}$, then $\mathcal{D}$ is called a totally geodesic foliation [4]. The following results were proved in [21].

Let (M, g) be a simply-connected semi-Riemannian manifold which admits two complementary foliations $\mathcal{D}_1$ and $\mathcal{D}_2$ whose leaves intersect perpendicularly.
1. If $\mathcal{D}_1$ is totally geodesic and $\mathcal{D}_2$ is totally umbilical, then (M, g) is isometric to a twisted product $M_1 \times_f M_2$.
2. If $\mathcal{D}_1$ is totally geodesic and $\mathcal{D}_2$ is spherical, then (M, g) is isometric to a warped product $M_1 \times_f M_2$.
3. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are totally geodesic, then (M, g) is isometric to a direct product $M_1 \times M_2$, where M_1 and M_2 are integral manifolds of distributions $\mathcal{D}_1$ and $\mathcal{D}_2$.

2.4. Harmonic maps

We now recall the notion of harmonic maps between semi-Riemannian manifolds. Let (M, g_M) and (N, g_N) be semi-Riemannian manifolds and suppose that $\varphi : M \to N$ is a smooth mapping between them. Then the differential φ_* of φ can be viewed a section of the bundle $Hom(TM, \varphi^{-1}TN) \to M$, where $\varphi^{-1}TN$ is the pullback bundle which has fibers $\varphi^{-1}(TN_p) = T_{\varphi(p)}N, p \in M$. $Hom(TM, \varphi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of φ is given by

$$(\nabla\varphi_*)(X, Y) = \nabla^{\varphi}_X \varphi_*(Y) - \varphi_*(\nabla^M_X Y), \tag{22}$$

for $X, Y \in \Gamma(TM)$, where ∇^{φ} is the pullback connection. It is known that the second fundamental form is symmetric. For a semi-Riemannian submersion F, one can easily obtain

$$(\nabla F_*)(X, Y) = 0, \tag{23}$$

for $X, Y \in \Gamma((\ker F_*)^{\perp})$. A smooth map $\varphi : M \to N$ is said to be harmonic if $\operatorname{trace}(\nabla\varphi_*) = 0$. On the other hand, the tension field of φ is the section $\tau(\varphi)$ of $\Gamma(\varphi^{-1}TN)$ defined by

$$\tau(\varphi) = \operatorname{div} \varphi_* = \sum_{i=1}^{m} \epsilon_i (\nabla\varphi_*)(e_i, e_i), \tag{24}$$

where $\{e_1, \ldots, e_m\}$ is the orthonormal frame on M and $\epsilon_i = g_M(e_i, e_i)$. Then it follows that φ is harmonic if and only if $\tau(\varphi) = 0$, for details, see [8].

3. Anti-invariant semi-Riemannian submersions

In this section, we study a semi-Riemannian submersion from a Lorentzian almost (para) contact manifold $M(\phi, \xi, \eta, g_M)$ to a semi-Riemannian manifold (N, g_N) and give necessary and sufficient conditions for the characteristic vector field to be vertical or horizontal.

Definition 3.1. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para) contact manifold and (N, g_N) be a semi-Riemannian manifold. A semi-Riemannian submersion $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ is said to be anti-invariant if $\ker F_*$ is anti-invariant with respect to ϕ, $\phi(\ker F_*) \subseteq (\ker F_*)^{\perp}$.* We denote the complementary orthogonal distribution to $\phi(\ker F_*)$ in $(\ker F_*)^{\perp}$ by μ. Then, we have

$$(\ker F_*)^{\perp} = \phi(\ker F_*) \oplus \mu. \tag{25}$$

3.1. Examples

We now give some examples of anti-invariant semi-Riemannian submersion.

Example 3.2. *Let N be $\mathbb{R}^5 = \{(y_1, y_2, y_3, y_4, z) | y_1, y_2, y_3, z \in \mathbb{R}\}$ and $\mathbb{R}^7$ be a Lorentzian Sasakian manifold as in Example 2.1. The semi-Riemannian metric tensor field g_N is given by*

$$g_N = \frac{1}{4}\begin{pmatrix} \frac{1}{2} - y_1^2 & -y_1y_2 & -y_1y_3 & 0 & y_1 \\ -y_1y_2 & \frac{1}{2} - y_2^2 & -y_2y_3 & 0 & y_2 \\ -y_1y_3 & -y_2y_3 & \frac{1}{2} - y_3^2 & 0 & y_3 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ y_1 & y_2 & y_3 & 0 & -1 \end{pmatrix}$$

on N. Let $F : \mathbb{R}^7 \to N$ be a map defined by

$$F(x_1, x_2, x_3, y_1, y_2, y_3, z) = \left(x_1 + y_1, x_2 + y_2, x_3 + y_3, x_3 - y_3, \frac{y_1^2}{2} + \frac{y_2^2}{2} + \frac{y_3^2}{2} + z\right).$$

After some calculations, we have $\ker F_* = \mathrm{span}\{V_1 = E_1 - E_4, V_2 = E_2 - E_5\}$ *and*

$$\ker F_*^\perp = \mathrm{span}\{H_1 = E_1 + E_4, H_2 = E_2 + E_5, H_3 = E_3, H_4 = E_6, H_5 = E_7\}.$$

It is easy to see that F is a semi-Riemannian submersion and $\phi_{-1}(V_1) = H_1, \phi_{-1}(V_2) = H_2$ imply that $\phi_{-1}(\ker F_*) \subset (\ker F_*)^\perp = \phi_{-1}(\ker F_*) \oplus \mathrm{span}\{H_3, H_4, H_5\}$. Thus, F is an anti-invariant semi-Riemannian submersion such that ξ is a horizontal vector field and $\mu = \mathrm{span}\{H_3, H_4, H_5\}$. Moreover, $\phi_{-1}(\ker F_*)$ is Riemannian Distribution.

It is clear that $F : (\mathbb{R}^7, \phi_1, \eta_1, \xi, g) \to N$ is an anti-invariant semi-Riemannian submersion from Lorentzian para Sasakian manifold to semi-Riemannian manifold.

Example 3.3. *$\mathbb{R}^5$ has a Lorentzian Sasakian structure as in Example 2.1. The Riemannian metric tensor field $g_{\mathbb{R}^2}$ is defined by $g_{\mathbb{R}^2} = \frac{1}{8}(du \otimes du + dv \otimes dv)$ on $\mathbb{R}^2 = \{(u, v)|u, v \in \mathbb{R}\}$. Let $F : \mathbb{R}^5 \to \mathbb{R}^2$ be a map defined by $F(x_1, x_2, y_1, y_2, z) = (x_1 + y_1, x_2 + y_2)$. By direct calculations* $\ker F_* = \mathrm{span}\{V_1 = E_1 - E_3, V_2 = E_2 - E_4, V_3 = E_5 = \xi\}$ *and* $(\ker F_*)^\perp = \mathrm{span}\{H_1 = E_1 + E_3, H_2 = E_2 + E_4\}$*. Therefore, it is easy to see that F is a semi-Riemannian submersion. However, $\phi_{-1}(V_1) = H_1, \phi_{-1}(V_2) = H_2$. That is, F is an anti-invariant semi-Riemannian submersion from Lorentzian para Sasakian manifold $(\mathbb{R}^5, \phi_1, \eta_1, \xi, g)$ to Riemannian manifold $(\mathbb{R}^2, g_{\mathbb{R}})$ and $\phi(\ker F_*) = (\ker F_*)^\perp$.*

Example 3.4. *Let N be $\mathbb{R}^3 = \{(y_1, y_2, z)|y_1, y_2, z \in \mathbb{R}\}$ and $\mathbb{R}^5$ be a Lorentzian Sasakian manifold as in Example 2.1. The Lorentzian metric tensor field g_N is given by*

$$g_N = \frac{1}{4}\begin{bmatrix} \frac{1}{2} - y_1^2 & -y_1 y_2 & y_1 \\ -y_1 y_2 & \frac{1}{2} - y_2^2 & y_2 \\ y_1 & y_2 & -1 \end{bmatrix}$$

on N. Let $F : \mathbb{R}^5 \to N$ be a map defined by

$$F(x_1, x_2, y_1, y_2, z) = \left(x_1 + y_1, x_2 + y_2, \frac{y_1^2}{2} + \frac{y_2^2}{2} + z\right).$$

After some calculations, we have $\ker F_* = \mathrm{span}\{V_1 = E_3 - E_1, V_2 = E_4 - E_2\}$ *and* $(\ker F_*)^\perp = \mathrm{span}\{H_1 = E_1 + E_3, H_2 = E_2 + E_4, H_3 = E_5\}$*. Then, it is easy to see that F is an anti-invariant semi-Riemannian submersion and* $(\ker F_*)^\perp = \phi_{-1}(\ker F_*) \oplus \mathrm{span}\{\xi\}$.

In the following results, we find necessary and sufficient conditions for the characteristic vector field to be vertical or horizontal.

Theorem 3.5. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para)contact manifold of dimension $2m + 1$ and (N, g_N) be a semi-Riemannian manifold of dimension n and $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be a semi-Riemannian submersion. Then the following statements hold:*

1. *The characteristic vector field ξ is vertical vector field if and only if N is a Riemannian manifold.*
2. *The characteristic vector field ξ is a horizontal vector field if and only if N is a Lorentzian manifold.*

Proof. Let F be a semi-Riemannian submersion. Then F_* is an isometry from $(\ker F_*)_p^\perp$ to $T_{F(p)}N$ for every point p of M. So, they have the same dimension and index. ξ is a (horizontal) vertical vector field if and only if (horizontal) vertical distribution is Lorentzian distribution and (vertical) horizontal distribution is Riemannian distribution. □

Theorem 3.6. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para)contact manifold of dimension $2m + 1$ and (N, g_N) be a semi-Riemannian manifold of dimension n. Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant semi-Riemannian submersion. Then the following statements hold:*

(a) If the characteristic vector field ξ is a vertical vector field then $m \leqslant n \leqslant 2m$.
(b) If $m = n$ then the characteristic vector field ξ is a vertical vector field.
(c) If the characteristic vector field ξ is a horizontal vector field then $m + 1 \leqslant n$.

Proof. Assume that the characteristic vector field ξ is a vertical vector field. We have $0 \leqslant \dim \phi(\ker F_*) = 2m - n \leqslant n$, then $m \leqslant n \leqslant 2m$. So the proof of (a) ends.
Assume that $m = n$ and $k = \dim\{X \in \ker F_* | \phi(X) = 0\}$. If ξ is not a vertical vector field, then $k = 0$. Therefore, $\dim \phi(\ker F_*) = n + 1 \leqslant n$, it is a contradiction, which proves (b).
If the characteristic vector field ξ is a horizontal vector field, then $\dim \phi(\ker F_*) = 2m + 1 - n \leqslant n$. Therefore, $1 \leqslant 2(n - m)$, we have $1 \leqslant n - m$. So the proof of (c) ends. □

Theorem 3.7. *Let F be a semi-Riemannian submersion from a K-(para)contact manifold $M(\phi, \xi, \eta, g_M)$ of dimension $2m + 1$ onto a semi-Riemannian manifold (N, g_N) of dimension n. If ξ is a horizontal vector field, then F is an anti-invariant submersion and $m + 1 \leqslant n$.*

Proof. From (6), (14) and (16), we have

$$g_M(\phi U, V) = g_M(\varepsilon \nabla_U \xi, V) = \varepsilon g_M(\mathcal{T}_U \xi, V) = -\varepsilon g_M(\xi, \mathcal{T}_U V)$$

for any $U, V \in \Gamma(\ker F_*)$. Since ϕ is skew-symmetric and $\mathcal{T}$ is symmetric, that is, (19), we have $g_M(\phi U, V) = 0$. Thus F is an anti-invariant submersion. From part (c) of Theorem 3.6 we have $m + 1 \leqslant n$. □

Corollary 3.8. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para)contact manifold of dimension $2m + 1$ and (N, g_N) is a semi-Riemannian manifold of dimension n and $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant semi-Riemannian submersion. If $m = n$, then $\phi(\ker F_*) = (\ker F_*)^\perp$. Moreover, N is a Riemannian manifold.*

Proposition 3.9. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para)contact manifold of dimension $2m+1$ and (N, g_N) is a semi-Riemannian manifold of dimension n and $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant semi-Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^\perp$. Then the characteristic vector field ξ is a vertical vector field and $m = n$. Moreover, N is a Riemannian manifold.*

Proof. If ξ is not a vertical vector field, then $\dim \phi(\ker F_*) = 2m + 1 - n = n$. Therefore, $2(n - m) = 1$, it is a contradiction. So $\xi \in \ker F_*$. That is, ξ is a vertical vector field. Now, since ξ is a vertical vector field. We have $\dim \phi(\ker F_*) = 2m - n = n$. Thus, $m = n$ and by Theorem 3.5, N is a Riemannian manifold. □

Proposition 3.10. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para) contact manifold of dimension $2m + 1$ and (N, g_N) be a semi-Riemannian manifold of dimension n and $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant semi-Riemannian submersion such that $\phi(\ker F_*) = \{0\}$. Then the characteristic vector field ξ is a vertical vector field, $2m = n$ and $\ker F_* = \mathrm{span}\{\xi\}$. Moreover, N is a Riemannian manifold.*

Proof. If ξ is not a vertical vector field, then $\dim \phi(\ker F_*) = 2m + 1 - n = 0$. Therefore, $\dim \ker F_* = 0$, it is contradiction. So ξ is a vertical vector field. In this case $\dim \phi(\ker F_*) = 2m - n = 0$ and $\dim \ker F_* = 1$, Thus $2m = n, \ker F_* = \mathrm{span}\{\xi\}$ and by Theorem 3.5, N is a Riemannian manifold. □

Proposition 3.11. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para) contact manifold of dimension $2m + 1$ and (N, g_N) be a semi-Riemannian manifold of dimension n and $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant semi-Riemannian submersion. If $2m = n$, then ξ is a vertical vector field, $\ker F_* = \mathrm{span}\{\xi\}, \phi(\ker F_*) = \{0\}$ and N is a Riemannian manifold or ξ is a horizontal vector field and N is a Lorentzian manifold*

Proof. If ξ is not a vertical vector field, then $\dim \phi(\ker F_*) = 2m + 1 - n = 0$. Therefore, $\dim \ker F_* = 0$, it is contradiction. So ξ is a vertical vector field. In this case $\dim \phi(\ker F_*) = 2m - n = 0$ and $\dim \ker F_* = 1$, Thus $2m = n, \ker F_* = \mathrm{span}\{\xi\}$ and by Theorem 3.5, N is a Riemannian manifold. □

Proposition 3.12. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para)contact manifold of dimension $2m+1$ and (N, g_N) is a Lorentzian manifold of dimension n. Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant semi-Riemannian submersion. $(\ker F_*)^\perp = \phi(\ker F_*) \oplus \mathrm{span}\{\xi\}$ if and only if $m + 1 = n$.*

Proof. Obviously, ξ is a horizontal vector field, if $(\ker F_*)^\perp = \phi(\ker F_*) \oplus \mathrm{span}\{\xi\}$ then $\dim \phi(\ker F_*) = 2m + 1 - n = n - 1$, so $m + 1 = n$. Conversely, by using (25), we have $2m + 1 - n + \dim \mu = n$. So $\dim \mu = 1$ then $\mu = \mathrm{span}\{\xi\}$. □

Remark 3.13. *We note that Example 3.4 satisfies Proposition 3.12.*

4. Anti-invariant submersions admitting vertical structure vector field

In this section, we will study anti-invariant submersions from a Lorentzian (para) Sasakian manifold onto a Riemannian manifold such that the characteristic vector field ξ is a vertical vector field. It is easy to see that μ is an invariant distribution of $(\ker F_*)^\perp$, under the endomorphism ϕ. Thus, for $X \in \Gamma((\ker F_*)^\perp)$ we have

$$\phi X = BX + CX. \tag{26}$$

where $BX \in \Gamma(\ker F_*), CX \in \Gamma(\mu)$. On the other hand, since $F_*((\ker F_*)^\perp) = TN$ and F is a semi-Riemannian submersion, using (26) we derive $g_N(F_*\phi V, F_*CX) = 0$, for every $X \in \Gamma((\ker F_*)^\perp), V \in \Gamma(\ker F_*)$ which implies that

$$TN = F_*(\phi(\ker F_*)^\perp) \oplus F_*(\mu). \tag{27}$$

Theorem 4.1. *Let $M(\phi, \xi, \eta, g_M)$ be a Lorentzian almost (para) contact manifold of dimension $2m + 1$ and (N, g_N) be a Riemannian manifold of dimension n. Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant semi-Riemannian submersion and ξ is a vertical vector field. Then the fibers are not totally umbilical.*

Proof. From (18), we have that, for $U \in \Gamma(\ker F_*)$, $\nabla_U \xi = \mathcal{T}_U \xi + \mathcal{V}\nabla_U \xi$. And from (6), we have $\nabla_U \xi = \varepsilon\phi U$. So, we have

$$\varepsilon\phi U = \mathcal{T}_U \xi. \tag{28}$$

If the fibers are totally umbilical, then we have $\mathcal{T}_U V = g_M(U, V)H$ for any vertical vector fields U, V, where H is the mean curvature vector field of any fibers. Since $\mathcal{T}_\xi \xi = 0$, we have $H = 0$, which shows that fibres are minimal. Hence the fibers are totally geodesic, which is a contradiction to the fact that $\mathcal{T}_U \xi = \varepsilon\phi U \neq 0$. □

Lemma 4.2. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$BCX = 0, C^2X + \phi BX = \epsilon X, \tag{29}$$

$$\nabla_X Y = g(X, \phi Y)\xi + \varepsilon\phi\nabla_X \phi Y, \tag{30}$$

where $X, Y \in \Gamma((\ker F_*)^\perp)$.

Proof. First, by using (1) and (26) for $X \in \Gamma(\ker F_*)$, we obtain $\epsilon X = BCX + C^2X + \phi BX$. This proves (29). Next, (30) is obtained from (1), (6) and (7). □

Lemma 4.3. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N). Then we have*

$$CX = \varepsilon\mathcal{A}_X \xi, \tag{31}$$

$$g_M(\mathcal{A}_X \xi, \phi U) = 0, \tag{32}$$

$$g_M(\nabla_Y \mathcal{A}_X \xi, \phi U) = -g_M(\mathcal{A}_X \xi, \phi\mathcal{A}_Y U) - \varepsilon\eta(U) g_M(\mathcal{A}_X \xi, Y), \tag{33}$$

$$g_M(X, \mathcal{A}_Y \xi) = \varepsilon g_M(Y, \mathcal{A}_X \xi), \tag{34}$$

where $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Proof. By using (20) and (6) for $X \in \Gamma((\ker F_*)^\perp)$ and $V = \xi$, the equality (31) is obvious. Next, from (2), (26) and (31), the equality (32) is obtained. Now from (32), for $X, Y \in \Gamma((\ker F_*)^\perp)$, we get $g_M(\nabla_Y \mathcal{A}_X\xi, \phi U) + g_M(\mathcal{A}_X\xi, \nabla_Y \phi U) = 0$ and $g_M(\mathcal{A}_X\xi, \nabla_Y\phi U) = g_M\big(\mathcal{A}_X\xi, (\nabla_Y\phi)U\big) + g_M\big(\mathcal{A}_X\xi, \phi(\nabla_Y U)\big)$. By using (7) and (20), we obtain

$$\begin{aligned} g_M(\mathcal{A}_X\xi, \nabla_Y\phi U) =& \varepsilon g_M\big(\mathcal{A}_X\xi, \eta(U)Y\big) + g_M\big(\mathcal{A}_X\xi, \phi\mathcal{A}_Y U\big) \\ &+ g_M\big(\mathcal{A}_X\xi, \phi(\mathcal{V}\nabla_Y U)\big). \end{aligned}$$

Finally, by using (31), (33) is obtained. From (5), (6) and (31), we have (34). □

Theorem 4.4. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Riemannian manifold (N, g_N), then for all $U, V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$, the following assertions are equivalent to each other:*

(*i*) $(\ker F_*)^\perp$ *is integrable.*

(*ii*) $g_N\big((\nabla F_*)(Y, BX), F_*\phi V\big) = g_N\big((\nabla F_*)(X, BY), F_*\phi V\big) + \varepsilon g_M(A_X\xi, \phi A_Y V) - \varepsilon g_M(A_Y\xi, \phi A_X V)$.

(*iii*) $g_M(A_X BY - A_Y BX, \phi V) = \varepsilon g_M(A_X\xi, \phi A_Y U) - \varepsilon g_M(A_Y\xi, \phi A_X V)$.

Proof. (*i*) $\Longleftrightarrow$ (*ii*). Assume that $U, V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$. From (30) and (5), we obtain.

$$\begin{aligned} g_M([X, Y], V) =& g_M(\nabla_X Y, V) - g_M(\nabla_Y X, V) \\ =& g_M(\varepsilon\phi\nabla_X\phi Y, V) + g_M\big(g_M(Y, \phi X)\xi, V\big) \\ &- g_M\big(\varepsilon\phi\nabla_Y\phi X, V) - g_M\big(g_M(X, \phi Y)\xi, V\big) \\ =& g_M(\nabla_X\phi Y, \phi V) - g_M\big(\nabla_Y\phi X, \phi V\big) + (1-\varepsilon)\varepsilon g_M(\phi X, Y)\eta(V). \end{aligned}$$

Now from (26), (31) and since F is an anti-invariant submersion, we have

$$\begin{aligned} g_M([X, Y], V) =& g_N(F_*\nabla_X BY, F_*\phi V) + \varepsilon g_M(\nabla_X\mathcal{A}_Y\xi, \phi V) - g_N(F_*\nabla_Y BX, F_*\phi V) \\ &- \varepsilon g_M(\nabla_Y\mathcal{A}_X\xi, \phi V) + (1-\varepsilon) g_M(\mathcal{A}_X\xi, Y)\eta(V). \end{aligned}$$

On the other hand, according to (22), (33) and (34), we get

$$\begin{aligned} g_M([X, Y], V) =& - g_N(\nabla F_*(X, BY), F_*\phi V) + \varepsilon g_M(\mathcal{A}_Y\xi, \phi\mathcal{A}_Y V) \\ &+ g_N(\nabla F_*(Y, BX), F_*\phi V) - \varepsilon g_M(\mathcal{A}_X\xi, \phi\mathcal{A}_Y V). \end{aligned} \tag{35}$$

(*ii*) $\Longleftrightarrow$ (*iii*). By using (20) and (22), we have

$$g_N(F_*\nabla_Y BX - \nabla_X BY, F_*\phi V) = g_M(\mathcal{A}_Y BX, \phi V) - g_M(\mathcal{A}_X BY, \phi V).$$

Thus according to part (*ii*), we have

$$g_M(\mathcal{A}_Y BX - \mathcal{A}_X BY, \phi V) = -\varepsilon g_M(\mathcal{A}_X\xi, \phi\mathcal{A}_Y V) + \varepsilon g_M(\mathcal{A}_Y\xi, \phi\mathcal{A}_X V). \tag{36}$$

□

Remark 4.5. *If $\phi(\ker F_*) = (\ker F_*)^\perp$, then we get $\varepsilon\mathcal{A}_X\xi = CX = 0$ and $BX = \phi X$.*

Hence we have the following corollary.

Corollary 4.6. *Let $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ be an anti-invariant semi-Riemannian submersion such that $\phi(\ker F_*) = (\ker F_*)^\perp$, where $M(\phi, \xi, \eta, g_M)$ is a Lorentzian (para) Sasakian manifold and (N, g_N) is a Riemannian manifold. Then for every $X, Y \in \Gamma(\ker F_*)^\perp$, the following assertions are equivalent to each other;*

(i) $(\ker F_*)^\perp$ *is integrable.*
(ii) $(\nabla F_*)(Y, \phi X) = (\nabla F_*)(X, \phi Y)$.
(iii) $A_X \phi Y = A_Y \phi X$.

Theorem 4.7. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant semi-Riemannian submersion, where* $M(\phi, \xi, \eta, g_M)$ *is a Lorentzian (para) Sasakian manifold and* (N, g_N) *is a Riemannian manifold. Then the following assertions are equivalent to each other;*

(i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on M.*
(ii) $g_M(A_X BY, \phi V) = \varepsilon g_M(A_Y \xi, \phi A_X V)$.
(iii) $g_N\Big((\nabla F_*)(X, \phi Y), F_* \phi V\Big) = -\varepsilon g_M(A_Y \xi, \phi A_X V)$.

for every $X, Y \in \Gamma((\ker F_*)^\perp)$ *and* $V \in \Gamma(\ker F_*)$.

Proof. (i) $\Longleftrightarrow$ (ii). Assume that $V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^\perp)$. By using (30), we have

$$g_M(\nabla_X Y, V) = g_M(\nabla_X \phi Y, \phi V) + \varepsilon \eta(V) g_M(X, \phi Y), \tag{37}$$

and from (20) and (26), we have

$$g_M(\nabla_X \phi Y, \phi V) = g_M(\mathcal{A}_X BY, \phi V) + \varepsilon g_M(\nabla_X \mathcal{A}_Y \xi, \phi V), \tag{38}$$

and from (33), we have

$$g_M(\nabla_X \phi Y, \phi V) = g_M(\mathcal{A}_X BY, \phi V) - \varepsilon g_M(\mathcal{A}_Y \xi, \phi \mathcal{A}_X V) - \eta(V) g_M(\mathcal{A}_Y \xi, X). \tag{39}$$

Now, from (26), (31), (37), (38) and (39), $(\ker F_*)^\perp$ is a totally geodesic foliation on M if and only if

$$g_M(\mathcal{A}_X BY, \phi V) = \varepsilon g_M(\mathcal{A}_Y \xi, \phi \mathcal{A}_X V). \tag{40}$$

Finally, by using (22), (23), (26), (27) and (39), we have (ii) $\Longleftrightarrow$ (iii). □

Corollary 4.8. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant semi-Riemannian submersion such that* $\phi(\ker F_*) = (\ker F_*)^\perp$*, where* $M(\phi, \xi, \eta, g_M)$ *is a Lorentzian (para) Sasakian manifold and* (N, g_N) *is a Riemannian manifold. Then, for every* $X, Y \in \Gamma((\ker F_*)^\perp)$*, the following assertions are equivalent to each other;*

(i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on M.*
(ii) $A_X \phi Y = 0$.
(iii) $(\nabla F_*)(X, \phi Y) = 0$.

We note that a differentiable map F between two semi-Riemannian manifolds is called totally geodesic if $\nabla F_* = 0$. Using Theorem 4.1 one can easily prove that the fibers are not totally geodesic. Hence, we have the following theorem.

Theorem 4.9. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant semi-Riemannian submersion such that* $\phi(\ker F_*) = (\ker F_*)^\perp$*, where* $M(\phi, \xi, \eta, g_M)$ *is a Lorentzian (para) Sasakian manifold and* (N, g_N) *is a Riemannian manifold. Then F is not totally geodesic map.*

Finally, we give a necessary and sufficient condition for an anti-invariant Riemannian submersion to be harmonic.

Theorem 4.10. *Let* $F : M(\phi, \xi, \eta, g_M) \to (N, g_N)$ *be an anti-invariant semi-Riemannian submersion such that* $m = n$*, where* $M(\phi, \xi, \eta, g_M)$ *is a Lorentzian (para) Sasakian manifold of dimension* $2m + 1$ *and* (N, g_N) *is a Riemannian manifold of dimension n. Then F is harmonic if and only if* $\operatorname{trace} \phi(\mathcal{T}_V) = -n\eta(V)$*, where* $V \in \Gamma(\ker F_*)$.

Proof. We know that F is harmonic if and only if F has minimal fibres [5]. Thus, F is harmonic if and only if $\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i = 0$, where $\{e_1, \ldots, e_{k-1}, e_k = \xi\}$ is the orthonormal basis for $\ker F_*$ and $k = 2m + 1 - n = n + 1$ is dimension of $\ker F_*$.
On the other hand, from (18), (19) and (7), we get

$$g_M(\mathcal{T}_V \phi W, U) = \varepsilon g_M(\phi V, \phi W)\eta(U) + \eta(W) g_M(\phi^2 V, U) + \varepsilon g_M(\mathcal{T}_V W, \phi U). \tag{41}$$

By using (41) and (14), we get

$$-\varepsilon \sum_{i=1}^{k} g_M(e_i, \phi \mathcal{T}_{e_i} U) = \varepsilon\Big((k-1)\eta(U) + g_M(\sum_{i=1}^{k} \mathcal{T}_{e_i} e_i, \phi U)\Big). \tag{42}$$

Since F is a Harmonic mapping, $\sum_{i=1}^{k} (\mathcal{T} e_i e_i, \phi U) = 0$. Then we have

$$\operatorname{trace} \phi(\mathcal{T}_U) = \sum_{i=1}^{k} g_M(e_i, \phi \mathcal{T}_{e_i} U) = -n\eta(U). \tag{43}$$

□

5. Anti-invariant submersions admitting horizontal structure vector field

In this section, we will study anti-invariant submersions from a Lorentzian (para) Sasakian manifold onto a Lorentzian manifold such that the characteristic vector field ξ is a horizontal vector field. From (25), it is easy to see that $\phi(\mu) \subset \mu$ and $\xi \in \mu$. Thus, for $X \in \Gamma((\ker F_*)^\perp)$ we have

$$\phi X = BX + CX, \tag{44}$$

where $BX \in \Gamma(\ker F_*), CX \in \Gamma(\mu)$. On the other hand, since $F_*((\ker F_*)^\perp) = TN$ and F is a semi-Riemannian submersion, using (44) we derive $g_N(F_* \phi V, F_* CX) = 0$, for every $X \in \Gamma((\ker F_*)^\perp), V \in \Gamma(\ker F_*)$, which implies that

$$TN = F_*(\phi(\ker F_*)) \oplus F_*(\mu). \tag{45}$$

Lemma 5.1. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Lorentzian manifold (N, g_N). Then we have*

$$BX = \varepsilon A_X \xi, \tag{46}$$
$$\mathcal{T}_U \xi = 0, \tag{47}$$
$$g_M(\nabla_X CY, \phi U) = -g_M(CY, \phi \mathcal{A}_X U), \tag{48}$$

where $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Proof. Assume that $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$. By using (21) and (6), we have

$$BX = \varepsilon \mathcal{A}_X \xi, \tag{49}$$

and also from (19) and (6), we get

$$\mathcal{T}_U \xi = 0. \tag{50}$$

From (7) and (20), we obtain (48). □

Theorem 5.2. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold $M(\phi, \xi, \eta, g_M)$ onto a Lorentzian manifold (N, g_N). Then for all $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$, the following assertions are equivalent.*

(*i*) $(\ker F_*)^\perp$ *is integrable.*

(*ii*)

$$g_N\Big((\nabla F_*)(Y,BX),F_*\phi V\Big)=g_N\Big((\nabla F_*)(X,BY),F_*\phi V\Big)-g_M(CX,\phi\mathcal{A}_YV)$$
$$+g_M(CY,\phi\mathcal{A}_XV)+\varepsilon g_M(X,\phi V)\eta(Y)-\varepsilon g_M(Y,\phi V)\eta(X).$$

(*iii*)

$$g_M(\mathcal{A}_X\mathcal{A}_Y\xi-\mathcal{A}_Y\mathcal{A}_X\xi,\phi V)=-\,g_M(CX,\phi\mathcal{A}_YV)+g_M(CY,\phi\mathcal{A}_XV)$$
$$+\,\varepsilon g_M(X,\phi V)\eta(Y)-\varepsilon g_M(Y,\phi V)\eta(X).$$

Proof. Assume that $X,Y\in\Gamma((\ker F_*)^\perp)$ and $V\in\Gamma(\ker F_*)$. From (2), (7) and (5), we obtain.

$$\begin{aligned}g_M([X,Y],V)=&g_M(\nabla_XY,V)-g_M(\nabla_YX,V)\\=&g_M(\nabla_X\phi Y,\phi V)-\varepsilon\eta(Y)g_M(X,\phi V)\\&-g_M(\nabla_Y\phi X,\phi V)+\varepsilon\eta(X)g_M(Y,\phi V)\\=&g_M(\nabla_XBY,\phi V)+g_M(\nabla_XCY,\phi V)-\varepsilon\eta(Y)g_M(X,\phi V)\\&-g_M(\nabla_YBX,\phi V)-g_M(\nabla_YCX,\phi V)+\varepsilon\eta(X)g_M(Y,\phi V).\end{aligned}$$

Since F is an anti-invariant submersion, we have

$$\begin{aligned}g_M([X,Y],V)=&g_N(F_*\nabla_XBY,F_*\phi V)+g_M(\nabla_XCY,\phi V)-\varepsilon\eta(Y)g_M(X,\phi V)\\&-g_N(F_*\nabla_YBX,F_*\phi V)-g_M(\nabla_YCX,\phi V)+\varepsilon\eta(X)g_M(Y,\phi V).\end{aligned}$$

On the other hand, according to (22), (48) and (34), we get

$$\begin{aligned}g_M([X,Y],V)=&-g_N(\nabla F_*(X,BY),F_*\phi V)-g_M(CY,\phi\mathcal{A}_XV)-\varepsilon\eta(Y)g_M(X,\phi V)\\&+g_N(\nabla F_*(Y,BX),F_*\phi V)+g_M(CX,\phi\mathcal{A}_YV)+\varepsilon\eta(X)g_M(Y,\phi V)\end{aligned}\tag{51}$$

which proves (*i*) $\Longleftrightarrow$ (*ii*). By using (20), (22), we have

$$g_N(F_*\nabla_YBX-\nabla_XBY,F_*\phi V)=-(g_M(\mathcal{A}_YBX,\phi V)-g_M(\mathcal{A}_XBY,\phi V))$$

Thus according to part (*ii*), we have (*ii*) $\Longleftrightarrow$ (*iii*). □

Corollary 5.3. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold* $M(\phi,\xi,\eta,g_M)$ *onto a Lorentzian manifold* (N,g_N) *with* $(\ker F_*)^\perp=\phi(\ker F_*)\oplus\mathrm{span}\{\xi\}$. *Then for all* $X,Y\in\Gamma((\ker F_*)^\perp)$, *the following assertions are equivalent.*

(*i*) $(\ker F_*)^\perp$ *is integrable.*

(*ii*) $(\nabla F_*)(Y,BX)=(\nabla F_*)(X,BY)+\varepsilon\eta(Y)F_*X-\varepsilon\eta(X)F_*Y$.

(*iii*) $\mathcal{A}_X\mathcal{A}_Y\xi-\mathcal{A}_Y\mathcal{A}_X\xi=\varepsilon\eta(Y)X-\varepsilon\eta(X)Y$.

Theorem 5.4. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold* (M,g_M,ϕ,ξ,η) *onto a Lorentzian manifold* (N,g_N). *Then for all* $X,Y\in\Gamma((\ker F_*)^\perp)$ *and* $V\in\Gamma(\ker F_*)$, *the following three statements are equivalent.*

(i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on M.*

(ii) $g_M(\mathcal{A}_XBY,\phi V)=g_M(CY,\phi\mathcal{A}_XV)+\varepsilon\eta(Y)g(X,\phi V)$.

(iii) $g_N\Big((\nabla F_*)(Y,\phi X),F_*(\phi V)\Big)=g_M(CY,\phi\mathcal{A}_XV)+\varepsilon\eta(Y)g(X,\phi V)$.

Proof. For $X,Y\in\Gamma\Big((\ker F_*)^\perp\Big)$ and $V\in\Gamma(\ker F_*)$, from (2), (7), and (48), we obtain

$$g_M(\nabla_XY,V)=g_M(\mathcal{A}_XBY,\phi V)-g_M(CY,\phi\mathcal{A}_XV)-\varepsilon\eta(Y)g(X,\phi V),$$

which shows (*i*) $\Longleftrightarrow$ (*ii*). From (20) and (22), we have (*ii*) $\Longleftrightarrow$ (*iii*). □

Corollary 5.5. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold $(M, g_M, \phi, \xi, \eta)$ onto a Lorentzian manifold (N, g_N) with $(\ker F_*)^\perp = \phi(\ker F_*) \oplus \mathrm{span}\{\xi\}$. Then for all $X, Y \in \Gamma((\ker F_*)^\perp)$, the following three statements are equivalent.*

(i) $(\ker F_*)^\perp$ *defines a totally geodesic foliation on M.*
(ii) $\mathcal{A}_X BY = \varepsilon\eta(Y)X$.
(iii) $(\nabla F_*)(Y, \phi X) = \varepsilon\eta(Y)F_*X$.

Theorem 5.6. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold $(M, g_M, \phi, \xi, \eta)$ onto a Lorentzian manifold (N, g_N). Then for $X \in \Gamma\Big((\ker F_*)^\perp\Big)$ and $V, W \in \Gamma(\ker F_*)$, the following three statements are equivalent.*

(a) $\ker F_*$ *defines a totally geodesic foliation on M.*
(b) $g_N\Big((\nabla F_*)(V, \phi X), F_*\phi W\Big) = 0$.
(c) $\mathcal{T}_V BX + \mathcal{A}_{CX} V \in \Gamma(\mu)$.

Proof. Assume that $X \in \Gamma\Big((\ker F_*)^\perp\Big)$ and $V, W \in \Gamma(\ker F_*)$. From (6) and $g_M(W, \xi) = 0$, we obtain $g_M(\nabla_V W, \xi) = \varepsilon g_M(W, \nabla_V \xi) = g(W, \phi V) = 0$. Thus, we have

$$\begin{aligned} g_M(\nabla_V W, X) =& g_M(\phi\nabla_V W, \phi X) - \eta(\nabla_V W)\eta(X) \\ =& g_M(\phi\nabla_V W, \phi X) \\ =& g_M(\nabla_V \phi W, \phi X) - g_M\Big((\nabla_V \phi)W, \phi X\Big) \\ =& - g_M(\phi W, \nabla_V \phi X). \end{aligned}$$

Since F is a semi-Riemannian submersion, we have

$$g_M(\nabla_V W, X) = -g_N(F_*\phi W, F_*\nabla_V \phi X) = g_N\Big(F_*\phi W, (\nabla F_*)(V, \phi X)\Big),$$

which proves $(a) \Leftrightarrow (b)$.
By direct calculation, we derive

$$\begin{aligned} g_N\Big(F_*\phi W, (\nabla F_*)(V, \phi X)\Big) =& - g_M(\phi W, \nabla_V \phi X) \\ =& - g_M(\phi W, \nabla_V BX + \nabla_V CX) \\ =& - g_M(\phi W, \nabla_V BX + [V, CX] + \nabla_{CX} V) \end{aligned}$$

Since $[V, CX] \in \Gamma(\ker F_*)$, from (18) and (20), we obtain

$$g_N\Big(F_*\phi W, (\nabla F_*)(V, \phi X)\Big) = -g_M(\phi W, \mathcal{T}_V BX + \mathcal{A}_{CX} V),$$

which proves $(b) \Longleftrightarrow (c)$. □

Corollary 5.7. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold $(M, g_M, \phi, \xi, \eta)$ onto a Lorentzian manifold (N, g_N) with $(\ker F_*)^\perp = \phi(\ker F_*) \oplus \mathrm{span}\{\xi\}$. Then the following three statements are equivalent.*

(a) $\ker F_*$ *defines a totally geodesic foliation on M.*
(b) $(\nabla F_*)(V, \phi X) = 0$.
(c) $\mathcal{T}_V \phi W = 0$, *for* $X \in \Gamma\Big((\ker F_*)^\perp\Big)$ *and* $V, W \in \Gamma(\ker F_*)$.

The proof of the following two theorems are exactly the same with Theorem 3.10 and Theorem 3.11 in [15] for Riemannian case. Therefore, we omit them here.

Theorem 5.8. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Lorentzian manifold* (N, g_N) *with* $(\ker F_*)^\perp = \phi(\ker F_*) \oplus \mathrm{span}\{\xi\}$. *Then F is a totally geodesic map if and only if*

$$\mathcal{T}_V \phi W = 0, \quad V, W \in \Gamma(\ker F_*) \tag{52}$$

and

$$\mathcal{A}_X \phi W = 0, \quad X \in \Gamma\Big((\ker F_*)^\perp\Big). \tag{53}$$

Theorem 5.9. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Lorentzian manifold* (N, g_N) *with* $(\ker F_*)^\perp = \phi(\ker F_*) \oplus \mathrm{span}\{\xi\}$. *Then F is a harmonic map if and only if* $\mathrm{trace}(\phi \mathcal{T}_V) = 0$ *for* $V \in \Gamma(\ker F_*)$.

In the following, we obtain decomposition theorems for an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold onto a Lorentzian manifold. By using results in subsection 2.3 and Theorems 5.2, 5.4 and 5.6, we have the following theorem.

Theorem 5.10. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Lorentzian manifold* (N, g_N). *Then M is a locally product manifold if and only if*

$$g_N\Big((\nabla F_*)(Y, BX), F_* \phi V\Big) = g_M(CY, \phi \mathcal{A}_X V) + \varepsilon \eta(Y) g_M(X, \phi V)$$

and

$$g_N\Big((\nabla F_*)(V, \phi X), F_* \phi W\Big) = 0$$

for $X, Y \in \Gamma\Big((\ker F_*)^\perp\Big)$ *and* $V, W \in \Gamma(\ker F_*)$.

Theorem 5.11. *Let F be an anti-invariant semi-Riemannian submersion from a Lorentzian (para) Sasakian manifold* $(M, g_M, \phi, \xi, \eta)$ *onto a Lorentzian manifold* (N, g_N) *with* $(\ker F_*)^\perp = \phi(\ker F_*) \oplus \mathrm{span}\{\xi\}$. *Then M is a locally twisted product manifold of the form* $M_{(\ker F_*)^\perp} \times_f M_{\ker F_*}$ *if and only if*

$$\mathcal{T}_V \phi X = -g_M(X, \mathcal{T}_V V) \|V\|^{-2} \phi V$$

and

$$\mathcal{A}_X \phi Y = \eta(Y) X$$

for $X, Y \in \Gamma\Big((\ker F_*)^\perp\Big)$ *and* $V, W \in \Gamma(\ker F_*)$, *where* $M_{(\ker F_*)^\perp}$ *and* $M_{\ker F_*}$ *are integral manifolds of the distributions* $(\ker F_*)^\perp$ *and* $\ker F_*$.

Theorem 5.12. *Let* $(M, g_M, \phi, \xi, \eta)$ *be a Lorentzian (para) Sasakian manifold and* (N, g_N) *be a Lorentzian manifold. Then it does not exist an anti-invariant semi-Riemannian submersion from M to N with* $(\ker F_*)^\perp = \phi(\ker F_*) \oplus \mathrm{span}\{\xi\}$ *such that M is a locally proper twisted product manifold of the form* $M_{(\ker F_*)^\perp} \times_f M_{\ker F_*}$.

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