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Hypergroups Constructed from Hypergraphs

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Abstract. The purpose of this paper is the study of hypergroups associated with hypergraphs. In this regard, we construct a hypergroupoid by defining a hyperoperation on the set of degrees of vertices of a hypergraph. We will see that the constructed hypergroupoid is always an H_v-group. We will investigate some conditions to have a hypergroup.

1. Introduction

The notion of a hypergraph appeared around 1960 and one of the initial concerns was to extend some classical results of graph theory. Hypergraphs are like simple graphs, except that instead of having edges that only connect two vertices, their edges are sets of any number of vertices. This happens to mean that all graphs are just a subset of hypergraphs. Hence some properties must be a generalization of graph properties. Moreover, hypergraph theory is a useful tool for discrete optimization problems. Hypergraphs is being widely and deeply investigated since last few decades as a successful tool to represent and model complex concepts and structures in various areas of computer science and discrete mathematics. A very good presentation of graph and hypergraph theory is in [1, 2], also see [5, 14, 20]. Hypergraphs have many other names. In computational geometry, a hypergraph may sometimes be called a range space and then the hyperedges are called ranges.

Algebraic hyperstructures represent a natural extension of classical algebraic structures and they were introduced by the French mathematician Marty [19]. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [3, 5, 7–9, 24]. In these books one can see the applications of hyperstructures in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs.

The connections between hyperstructure theory, binary relations and graph theory have been analyzed by many researchers (see for instance [4, 6, 10–17, 21]). In [4], Corsini present a commutative quasi-hypergroup H_Γ associated to a given hypergraph Γ. Necessary and sufficient conditions for H_Γ to be associative are found. For certain classes of hypergraphs that include finite hypergraphs, a sequence of hypergraphs is described such that the corresponding quasi-hypergroups form a join space. In [6], Corsini

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et al. considered the hypergroupoid $(H; \circ_m)$ associated with a finite path whereby the product $x \circ_m y$ is the set of vertices of the path at the graph distance at most m from x or y. They analyzed the derived hypergroupoids of $(H; \circ_m)$. In [17], Iranmanesh and Iradmusa defined a hyperoperation, they call it *PHO* hyperoperation, depending on two non-empty natural numbers. The most important result on the topic is a necessary and sufficient condition in order that a *PHO* hyperoperation defines a hypergroup in the sense of Marty. In [15], Farshi et al. constructed a ρ-hypergroup by means a given hypergraph by defining a special relation ρ, and then they investigated some related properties. Further, they introduced a special product of ρ-hypergroups. Also, They bridged between subhypergraphs and subhypergroups. Finally, the fundamental relation of a ρ-hypergroup is studied. In [15], Farshi et al. constructed a hypergroupoid by defining a hyperoperation on the set of degrees of vertices of a hypergraph and they call it a degree hypergroupoid. The constructed hypergroupoid is always an H_v-group. They presented some conditions on a degree hypergroupoid to have a hypergroup structure. Further, they studied the degree hypergroupoid associated with cartesian product of hypergraphs. In [21], Maryati and Davvaz investigated a general framework for the study of the relations between hypergraphs and hypergroups based on approximation operators.

2. Basic definitions

In this section, we gather all definitions we require of hyperstructures and hypergraphs. Let H be a nonempty set and $\rho^*(H)$ be the set of all nonempty subsets of H and $H \times H$ be the cartesian product of H. In general, a hyperoperation $\circ$ on H is a map from $H \times H$ to $\rho^*(H)$. More exactly, for all x, y of H, we have $x \circ y \subseteq H$. $x \circ y$ is called the hyperproduct of x and y. If A, B are non-empty subsets of H, then by $A \circ B$, we mean

$$A \circ B = \bigcup_{\substack{x \in A \\ y \in B}} x \circ y,$$

and for $x \in H$, $x \circ A = \{x\} \circ A$ and $A \circ x = A \circ \{x\}$. The hyperproduct of elements $x_1, ..., x_n$ of H is denoted by $\prod_{i=1}^{n} x_i$ and is equal to $x_1 \circ \prod_{i=2}^{n} x_i$. An algebraic system $(H, \circ)$ endowed with a hyperoperation is called a *hypergroupoid*. A hypergroupoid $(H, \circ)$ is called a:

- *semihypergroup* if for every $x, y, z \in H$, we have $x \circ (y \circ z) = (x \circ y) \circ z$;
- *quasihypergroup* if for every $x \in H$, $x \circ H = H = H \circ x$ (this condition is called the *reproduction axiom*);
- *hypergroup* if it is a semihypergroup and a quasihypergroup.

A hypergroup is called *commutative* if $x \circ y = y \circ x$ for all $x, y \in H$. A *join space* is a commutative hypergroup $(H, \circ)$ such that the following condition holds for all a, b, c, d in H:

$$a/b \cap c/d \neq \emptyset \Longrightarrow a \circ d \cap b \circ c \neq \emptyset,$$

where $a/b = \{x \in H \mid a \in x \circ b\}$. Join spaces have been introduced by Prenowitz [22] and used by him and Jantosciak [23] to rebuild several branches of geometry.

A *hypergraph* is a pair $\Gamma = (H, E)$, where H is a finite set of vertices and $E = \{E_1, ..., E_n\}$ is a set of hyperedges which are nonempty subsets of H such that $\bigcup_{i=1}^{m} E_i = H$.

Let $\Gamma = (H, E)$ be a hypergraph and $x, y \in H$. A hyperedge sequence $(E_1, ..., E_k)$ is called a *path of length k* from x to y if the following conditions are satisfied:

(1) $x \in E_1$ and $y \in E_k$,
(2) $E_i \neq E_j$ for $i \neq j$,
(3) $E_i \cap E_{i+1} \neq \emptyset$ for $1 \leq i \leq k - 1$.

In a hypergraph Γ, two vertices x and y are called *connected* if Γ contains a path from x to y. If two vertices are connected by a path of length 1, i.e., by a single hyperedge, the vertices are called *adjacent*. We use the notation $x - y$ to denote the adjacency of vertices x and y. A hypergraph is said to be *connected* if every pair of vertices in the hypergraph is connected. A *connected component* of a hypergraph is any maximal set of vertices which are pairwise connected by a path [14].

The length of shortest path between vertices x and y is denoted by $\mathrm{dist}(x, y)$ and the *diameter* of Γ is defined as follows:

$$d := \mathrm{diam}(\Gamma) = \begin{cases} \max\{\mathrm{dist}(x, y) \mid x, y \in H\} & \text{if } \Gamma \text{ is connected,} \\ \infty & \text{otherwise.} \end{cases}$$

Connections between hypergraphs and hypergroups are studied by many authors, for example, see [4, 6, 17, 18].

3. Special hyperoperation on hypergraphs

We define a hyperoperation ${}_n\circ_m$ for all $n, m \in \mathbb{N}$ on H as follows:

$$\forall (x, y) \in H^2,\ x \,{}_n\circ_m\, y = E^n(x) \cup E^m(y),$$

where $E^0(x) = x$, $E(x) = \bigcup\limits_{x \in E_i} E_i$, $E(A) = \bigcup\limits_{x \in A} E(x)$ for all non-empty subset A of H, and $E^n(x) = E^{n-1}(E(x))$. It is clear that for $n \geq m$, $x \,{}_n\circ_m\, x = E^n(x) \cup E^m(x) = E^n(x)$.

The hypergroupoid $H_\Gamma = (H, {}_n\circ_m)$ is called a *hypergraph hypergroupoid* or a *h.g. hypergroupoid*.

In the following, we consider three cases:

Case 1. $n \geq d$ or $m \geq d$.

Case 2. $n = m$ ($n < d$ and $m < d$).

Case 3. $n \neq m$ ($n < d$ and $m < d$).

For the case 1, we have the following:

Lemma 3.1. *Let Γ be a connected hypergraph of diameter d. If $n \geqslant d$, then for every $x \in H$, $E^n(x) = H$.*

Proof. By induction the result follows. So, we prove the claim for $n = d$, i.e., we show that $E^d(x) = H$.

By contradiction, suppose that $E^d(x) \neq H$. We know that $|E^d(x)| \geqslant d$. Since $E^d(x) \neq H$, it follows that there exists a vertex in Γ, say x_d, which is not in $E^d(x)$. Obviously, there is a vertex x_i in $E^d(x)$ such that $dist(x_i, x_d) > d$ and this is a contradiction. □

Remark 3.2. *If $n \geq d$ or $m \geq d$, then*

$$\forall (x, y) \in H^2,\ x \,{}_n\circ_m\, y = H.$$

For the case 2, we have (Theorems 3.3, 3.4, Proposition 3.5, Theorem 3.8, Corollary 3.9, Theorem 3.11).

In [4], Corsini defined the hyperoperation ${}_1\circ_1$ as follows:

$$\forall (x, y) \in H^2,\ x \,{}_1\circ_1\, y = E(x) \cup E(y).$$

The hyperoperation ${}_1\circ_1$ is commutative.

Theorem 3.3. *For each $(x, y) \in H^2$ and $n \in \mathbb{N}$, the hypergroupoid $H_\Gamma = (H, {}_n\circ_n)$ satisfies the following conditions:*

(1) $x \,{}_n\circ_n\, y = x \,{}_n\circ_n\, x \cup y \,{}_n\circ_n\, y$;

(2) $x \in x \,{}_n\circ_n\, x$;

(3) $y \in x \,{}_n\circ_n\, x \Leftrightarrow x \in y \,{}_n\circ_n\, y$.

Proof. We only prove (3). The proof of other conditions is straightforward.

Let $y \in x\ {}_n\circ_n\ x = E^n(x)$. Then $E(y) \cap E^{n-1}(x) \neq \emptyset$ and so $E^2(y) \cap E^{n-2}(x) \neq \emptyset$, ..., $E^{n-1}(y) \cap E(x) \neq \emptyset$. Hence $E^n(y) \cap x \neq \emptyset$. Therefor $x \in E^n(y)$. □

Theorem 3.4. *If the hypergroupoid $H_\Gamma = (H, {}_n\circ_n)$ satisfied the conditions (1), (2), (3) of the Theorem 3.3, then also satisfies the conditions:*

(4) $x\ {}_n\circ_n\ y \supset \{x, y\}$,

(5) $x\ {}_n\circ_n\ y = y\ {}_n\circ_n\ x$,

(6) $H\ {}_n\circ_n\ x = x\ {}_n\circ_n\ H = H$,

(7) $< H; \{x\ {}_n\circ_n\ x\}_{x\in H} >$ *is a hypergraph,*

(8) $(x\ {}_n\circ_n\ x)\ {}_n\circ_n\ x = \bigcup\limits_{x\in z\,{}_n\circ_n z} z\ {}_n\circ_n\ z$,

(9) $(x\ {}_n\circ_n\ x)\ {}_n\circ_n\ (x\ {}_n\circ_n\ x) = x\ {}_n\circ_n\ x\ {}_n\circ_n\ x$.

Proof. It is straightforward. □

Proposition 3.5. *The hypergroupoid $H_\Gamma = (H, {}_n\circ_m)$ is commutative, if one of the following conditions is satisfied:*

(1) $n = m$,

(2) $n \geq d$ *or* $m \geq d$.

Proof. (1) It is clear.

(2) It is straightforward, by Lemma 3.1. □

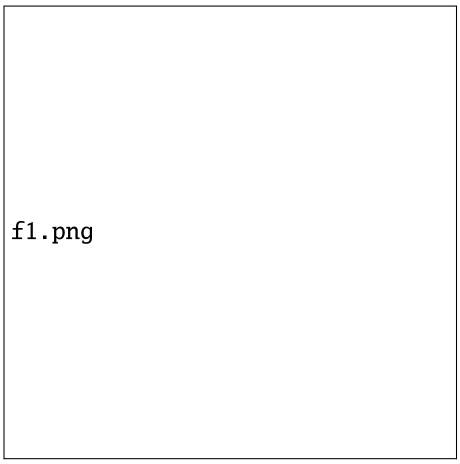

Figure 1: The hypergraph defineded in Example 3.6.

Example 3.6. *Diameter of the hypergraph presented in Figure 1 is* 3 ($d = 3$). *Suppose that* $n = 2$ *and* $m = 1$. *Then, we have*

${}_2\circ_1$	x	y	z	w	h
x	$\{x, y, z, h\}$	$\{x, y, z, h\}$	$\{x, y, z, h\}$	H	H
y	$\{x, y, z, h\}$	$\{x, y, z, h\}$	$\{x, y, z, h\}$	H	H
z	H	H	H	H	H
w	H	H	H	$\{w, h, z\}$	$\{w, h, z\}$
h	H	H	H	H	H

It is clear that $(H, {}_2\circ_1)$ *is not commutative.*

Remark 3.7. *By using the conditions (5) and (6) of Theorem 3.4, it is clear that $H_\Gamma = (H, {}_n\circ_n)$ is a commutative quasihypergroup.*

Theorem 3.8. *The hypergroupoid $(H, {}_n\circ_n)$ satisfying the conditions (1), (2) and (3) of Theorem 3.3 is a hypergroup if and only if the following condition is valid:*

$$\forall (a, c) \in H^2 \text{ and } n < \lceil \frac{d}{2} \rceil, \quad c \;{}_n\circ_n\; c \;{}_n\circ_n\; c \;-\; c \;{}_n\circ_n\; c \;\subset\; a \;{}_n\circ_n\; a \;{}_n\circ_n\; a.$$

Proof. By [4], the proof is clear. □

Corollary 3.9. *If the hypergroupoid satisfies the conditions (1), (2) and (3) of Theorem 3.3 and the condition:*

$$n < \lceil \frac{d}{2} \rceil,\ E^{2n}(x) = E^n(x), \text{ for all } x \in H,$$

then it is a hypergroup.

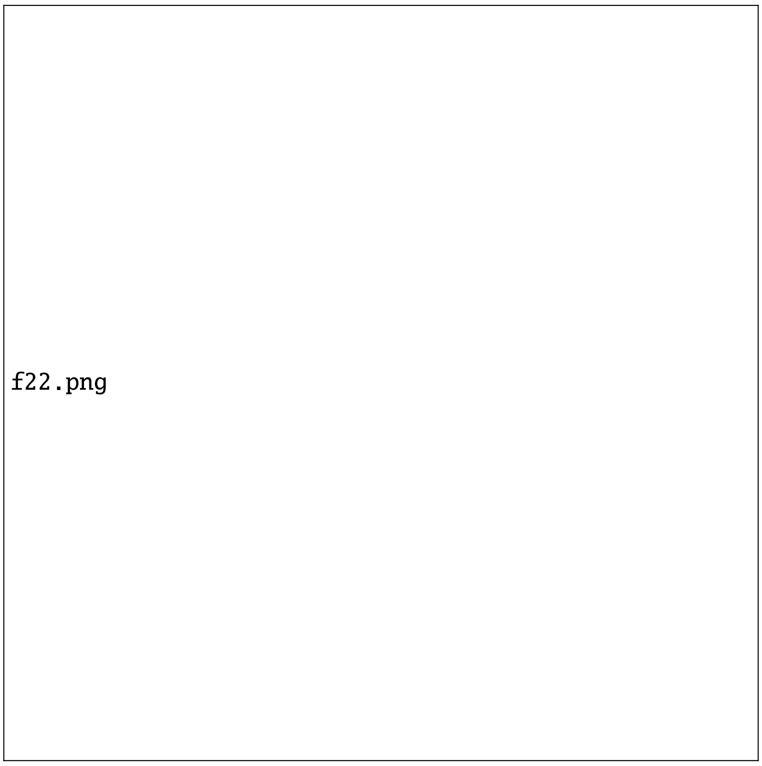

Figure 2: The hypergraph defineded in Example 3.10.

Example 3.10. *Suppose that $H = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$, $E = \{E_1, E_2, E_3, E_4, E_5, E_6, E_7\}$ and $d = 5$, where $E_1 = \{v_1, v_2, v_3\}$, $E_2 = \{v_3, v_4\}$, $E_3 = \{v_2, v_5\}$, $E_4 = \{v_4, v_6\}$, $E_5 = \{v_5, v_6\}$, $E_6 = \{v_6, v_7\}$ and $E_7 = \{v_7, v_8\}$, see Figure*

2. Then, we have

${}_2\circ_2$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_1	$\{v_1,...,v_5\}$	$\{v_1,...,v_6\}$	$\{v_1,...,v_6\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	H	H	H
v_2	$\{v_1,...,v_6\}$	$\{v_1,...,v_6\}$	$\{v_1,...,v_6\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	H	H	H
v_3	$\{v_1,...,v_6\}$	$\{v_1,...,v_6\}$	$\{v_1,...,v_6\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	H	H	H
v_4	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	H	H	H
v_5	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	$\{v_1,...,v_7\}$	H	H	H
v_6	H	H	H	H	H	$\{v_2,...,v_8\}$	$\{v_2,...,v_8\}$	$\{v_2,...,v_8\}$
v_7	H	H	H	H	H	$\{v_2,...,v_8\}$	$\{v_4,...,v_8\}$	$\{v_4,...,v_8\}$
v_8	H	H	H	H	H	$\{v_2,...,v_8\}$	$\{v_4,...,v_8\}$	$\{v_6,v_7,v_8\}$

It is clear that $(H, {}_2\circ_2)$ is a commutative hypergroup.

Theorem 3.11. *If the hypergroup $H_\Gamma = (H, {}_n\circ_n)$ satisfies the conditions (1), (2) and (3) of Theorem 3.3, then it is a join space.*

Proof. It is sufficient to prove that the following implication:

$$x/y \cap z/w \neq \emptyset \Rightarrow x \;{}_n\circ_n\; w \cap y \;{}_n\circ_n\; z \neq \emptyset,$$

where $x/y = \{z \mid x \in z \;{}_n\circ_n\; y\}$. We have:

$$u \in x/y \cap z/w \Longleftrightarrow [x \in u \;{}_n\circ_n\; y \text{ and } z \in u \;{}_n\circ_n\; w].$$

Moreover, $x \in u \;{}_n\circ_n\; y \Longleftrightarrow x \in u \;{}_n\circ_n\; u \cup y \;{}_n\circ_n\; y$ and $z \in u \;{}_n\circ_n\; w \Longleftrightarrow z \in u \;{}_n\circ_n\; u \cup w \;{}_n\circ_n\; w$. The following four cases are possible:

(1) If $x \in u \;{}_n\circ_n\; u$, $z \in u \;{}_n\circ_n\; u$, then $u \in x \;{}_n\circ_n\; x \cap z \;{}_n\circ_n\; z$ and so $u \in x \;{}_n\circ_n\; w \cap y \;{}_n\circ_n\; z$.
(2) If $x \in u \;{}_n\circ_n\; u$, $z \in w \;{}_n\circ_n\; w$, then $w \in z \;{}_n\circ_n\; z$. Hence, $w \in x \;{}_n\circ_n\; w \cap y \;{}_n\circ_n\; z$.
(3) If $x \in y \;{}_n\circ_n\; y$, $z \in u \;{}_n\circ_n\; u$, then $y \in x \;{}_n\circ_n\; x$. This implies that $y \in x \;{}_n\circ_n\; w \cap y \;{}_n\circ_n\; z$.
(4) If $x \in y \;{}_n\circ_n\; y$, $z \in w \;{}_n\circ_n\; w$, then $w \in z \;{}_n\circ_n\; z$. This implies that $w \in x \;{}_n\circ_n\; w \cap y \;{}_n\circ_n\; z$.
□

Remark 3.12. *If $n = m$ ($n < d$ and $m < d$), then*

$$\forall (x, y) \in H^2,\; x \;{}_n\circ_m\; y = E^n(x) \cup E^n(y).$$

For the case 3, we have (Theorems 3.13, 3.14, Lemma 3.17, Theorems 3.18):

Theorem 3.13. *For each $(x, y) \in H^2$, $n, m \in \mathbb{N}$ ($n \neq m, n < d$ and $m < d$), the hypergroupoid H_Γ satisfies the following conditions:*

(1) $x \;{}_n\circ_m\; y \subset x \;{}_n\circ_m\; x \cup y \;{}_n\circ_m\; y$. In the other words:

$$x \;{}_n\circ_m\; y = \begin{cases} x \;{}_n\circ_m\; x \cup E^m(y) & \text{if } n > m, \\ E^n(x) \cup y \;{}_n\circ_m\; y & \text{if } n < m. \end{cases}$$

(2) $x \in x \;{}_n\circ_m\; x$;
(3) $y \in x \;{}_n\circ_m\; x \Leftrightarrow x \in y \;{}_n\circ_m\; y$.

Proof. We only prove (3). The proof of other items is straightforward. Let $n > m$ and $y \in x \;{}_n\circ_m\; x = E^n(x)$. Then $E(y) \cap E^{n-1}(x) \neq \emptyset$ and so $E^2(y) \cap E^{n-2}(x) \neq \emptyset$, ..., $E^{n-1}(y) \cap E(x) \neq \emptyset$. Hence $E^n(y) \cap x \neq \emptyset$. Therefor $x \in E^n(y)$. □

Theorem 3.14. *If a hypergroupoid H_Γ satisfied the conditions (1), (2), (3) of Theorem 3.13, then also satisfies the following conditions:*

(4) $x\ {}_n\circ_m\ y \supset \{x, y\}$,
(5) $x\ {}_n\circ_m\ y \cap y\ {}_n\circ_m\ x \neq \emptyset$,
(6) $H\ {}_n\circ_m\ x = x\ {}_n\circ_m\ H = H$,
(7) $< H; \{x\ {}_n\circ_m\ x\}_{x\in H} >$ *is a hypergraph,*
(8) $(x\ {}_n\circ_m\ x)\ {}_n\circ_m\ x \subset \bigcup\limits_{x\in z_n\circ_m z} z\ {}_n\circ_m\ z$.

Proof. It is enough to prove (8). We have $(x\ {}_n\circ_m\ x)\ {}_n\circ_m\ x = \bigcup\limits_{z\in x_n\circ_m x} z\ {}_n\circ_m\ x$. By (1), we obtain

$$x\ {}_n\circ_m\ x\ {}_n\circ_m\ x \subset \bigcup_{z\in x_n\circ_m x} z\ {}_n\circ_m\ z \cup x\ {}_n\circ_m\ x.$$

Now, from (2) we have

$$x\ {}_n\circ_m\ x\ {}_n\circ_m\ x \subset \bigcup_{z\in x_n\circ_m x} z\ {}_n\circ_m\ z,$$

and finally from (3) we obtain (8). □

Remark 3.15. *From the condition (5) of Theorem 3.14, it is clear that hyperoperation ${}_n\circ_m$ is weak commutative and weak associative. In the other words:*

$$\forall\, x, y, z \in H, \quad x\ {}_n\circ_m\ (y\ {}_n\circ_m\ z) \cap (x\ {}_n\circ_m\ y)\ {}_n\circ_m\ z \neq \emptyset.$$

The H_v-structures are generalized algebraic hyperstructures where in the axioms of the classical hyperstructures the equality is replaced by the non-empty intersection. They were introduced by Vougiouklis [26], also see [24, 25]. A hypergroupoid $(H, \circ)$ is called an *H_v-group* if it is a quasihypergroup and for every $x, y, z \in H$, we have $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$.

Corollary 3.16. *From the condition (6) of Theorem 3.14 and Remark 3.15, it is clear that an h.g. hypergroupoid is an H_v-group.*

Lemma 3.17. *Let $(H, {}_n\circ_m)$ be a hypergroupoid. Then:*

$$\forall\, x, y \in H, \quad E^n(E^m(x) \cup E^p(y)) = E^{n+m}(x) \cup E^{n+p}(y).$$

Proof. Let $a \in E^n(E^m(x) \cup E^p(y))$. Then $a \in E^m(x)$ or $a \in E^p(y)$ or $a \in E^{n+m}(x)$ or $a \in E^{n+p}(y)$. But $E^m(x) \subseteq E^{n+m}(x)$ and $E^p(y) \subseteq E^{n+p}(y)$. So $a \in E^{n+m}(x)$ or $a \in E^{n+p}(y)$. Hence $E^n(E^m(x) \cup E^p(y)) \subseteq E^{n+m}(x) \cup E^{n+p}(y)$.

The converse is straightforward. □

Theorem 3.18. *The hypergroupoid $(H, {}_n\circ_m)$ is a hypergroup if and only if for every $a, b, c \in H$, $n < \lceil \frac{d}{2} \rceil$ and $m < \lceil \frac{d}{2} \rceil$ one of the following conditions is valid:*

(1) If $E^{2n}(a) \cup E^m(c) \subset E^n(a) \cup E^{2m}(c)$, then $(E^n(a) \cup E^{2m}(c)) - (E^{2n}(a) \cup E^m(c)) \subset E^{n+m}(b)$;
(2) If $E^n(a) \cup E^{2m}(c) \subset E^{2n}(a) \cup E^m(c)$, then $(E^{2n}(a) \cup E^m(c)) - (E^n(a) \cup E^{2m}(c)) \subset E^{n+m}(b)$;
(3) If $E^{2n}(a) \cup E^m(c) \nsubseteq E^n(a) \cup E^{2m}(c)$ and $E^n(a) \cup E^{2m}(c) \nsubseteq E^{2n}(a) \cup E^m(c)$, then $((E^n(a) \cup E^{2m}(c)) - (E^{2n}(a) \cup E^m(c))) \cup ((E^{2n}(a) \cup E^m(c)) - (E^n(a) \cup E^{2m}(c))) \subset E^{n+m}(b)$.

Proof. (1) Let $(H, {}_n\circ_m)$ is a hypergroup and suppose $a, b, c \in H$. Then $a\ {}_n\circ_m\ (b\ {}_n\circ_m\ c) = (a\ {}_n\circ_m\ b)\ {}_n\circ_m\ c$. So $E^n(a) \cup E^{n+m}(b) \cup E^{2m}(c) = E^{2n}(a) \cup E^{n+m}(b) \cup E^m(c)$ by Lemma 3.17. Since $E^{2n}(a) \cup E^m(c) \subset E^n(a) \cup E^{2m}(c)$, thus $(E^n(a) \cup E^{2m}(c)) - (E^{2n}(a) \cup E^m(c)) \subset E^{n+m}(b)$.

The converse is a routine verification.

(2) Follows directly from (1).

(3) Let $(H, {}_n\circ_m)$ is a hypergroup and suppose $a, b, c \in H$. Then $a\ {}_n\circ_m\ (b\ {}_n\circ_m\ c) = (a\ {}_n\circ_m\ b)\ {}_n\circ_m\ c$. So $E^n(a) \cup E^{n+m}(b) \cup E^{2m}(c) = E^{2n}(a) \cup E^{n+m}(b) \cup E^m(c)$ by Lemma 3.17. Since $E^{2n}(a) \cup E^m(c) \nsubseteq E^n(a) \cup E^{2m}(c)$ and $E^n(a) \cup E^{2m}(c) \nsubseteq E^{2n}(a) \cup E^m(c)$, thus $((E^n(a) \cup E^{2m}(c)) - (E^{2n}(a) \cup E^m(c))) \cup ((E^{2n}(a) \cup E^m(c)) - (E^n(a) \cup E^{2m}(c))) \subset E^{n+m}(b)$.

The converse is a routine verification. □

Corollary 3.19. *If the hypergroupoid* $(H, {}_n\circ_m)$ *satisfies the conditions:*

$$\forall a, \quad \begin{cases} E^{2n}(a) = E^n(a) & \text{if } n < \lceil \frac{d}{2} \rceil \\ E^{2m}(a) = E^m(a) & \text{if } m < \lceil \frac{d}{2} \rceil \end{cases}$$

then it is a hypergroup.

Remark 3.20. *If* $n \neq m$ $(n < d$ *and* $m < d)$, *then*

$$\forall (x, y) \in H^2,\ x\ {}_n\circ_m\ y = E^n(x) \cup E^m(y).$$

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