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Einstein Statistical Warped Product Manifolds

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Abstract. We consider Einstein statistical warped product manifolds $I \times_f N$, $M \times_f N$ and $M \times_f I$, where I, M and N are 1, m and n dimensional statistical manifolds, respectively.

1. Introduction and Preliminaries

A Riemannian manifold (M, g), $(n \geq 2)$, is said to be an *Einstein manifold* if its Ricci tensor S satisfies the condition

$$S = \lambda g, \tag{1}$$

where $\lambda = \frac{\tau}{n}$ and τ denotes the *scalar curvature* of M. It is well-known that if $n > 2$, then λ is a constant.

Let ∇ be an affine connection on a Riemannian manifold (M, g). An affine connection ∇^* is said to be *dual or conjugate* of ∇ with respect to the metric g if

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g\left(Y, \nabla^*_X Z\right). \tag{2}$$

Given an affine connection ∇ on a Riemannian manifold (M, g), there exists a unique affine connection dual of ∇, denoted by ∇^*. So a pair of (∇, ∇^*) is called a *dualistic structure* on M (see [1], [11]).

If ∇ is a torsion-free affine connection and for all $X, Y, Z \in TM$

$$\nabla_X g(Y, Z) = \nabla_Y g(X, Z)$$

then, (M, g, ∇) is called a *statistical manifold*, in this case a pair of (∇, g) is called a *statistical structure* on M [1].

Denote by R and R^* the curvature tensor fields of ∇ and ∇^*, respectively.

A statistical structure (∇, g) is said to be of constant curvature $c \in \mathbb{R}$ (see [2], [7]) if

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}. \tag{3}$$

The curvature tensor fields R and R^* satisfy

$$g(R^*(X, Y)Z, W) = -g(Z, R(X, Y)W), \tag{4}$$

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(see [4]).

Let ∇^0 be the Levi-Civita connection of g. Certainly, a pair $\left(\nabla^0, g\right)$ is a statistical structure, which is called *Riemannian statistical structure* or a trivial statistical structure (also see [4]).

An n-dimensional, $(n > 2)$, statistical manifold (M, g, ∇) is called an *Einstein statistical manifold* if the scalar curvature τ is a constant and the equation (1) is fulfilled on M ([6]).

Example 1.1. *[8] Let $\left(\mathbb{R}^3, g\right)$ be a statistical manifold with Riemannian metric $g = \sum_{i=1}^{3} de_i de_i$ and ∇ an affine connection defined by*

$$\nabla_{e_1} e_1 = b e_1,\ \nabla_{e_2} e_2 = \frac{b}{2} e_1,\ \nabla_{e_3} e_3 = \frac{b}{2} e_1,$$

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \frac{b}{2} e_2,\ \nabla_{e_1} e_3 = \nabla_{e_3} e_1 = \frac{b}{2} e_3,\ \nabla_{e_3} e_2 = \nabla_{e_2} e_3 = 0,$$

where $\{e_1, e_2, e_3\}$ is an orthonormal frame field and b is a constant. Then, $\left(\mathbb{R}^3, g\right)$ is a statistical manifold of constant curvature $c = \frac{b^2}{4} > 0$ and it is an Einstein statistical manifold with $\lambda = \frac{b^2}{2}$.

In [10], Todjihounde defined dualistic structures on warped product manifolds. It is known that (M, ∇, g_M) and $\left(N, \widetilde{\nabla}, g_N\right)$ are statistical manifolds if and only if $\left(B = M \times_f N, D, g\right)$ is a statistical manifold (see [10] and [3]). In [5], A. Gebarowski studied Einstein warped product manifolds. He considered Einstein warped products $I \times_f F$, $\dim I = 1$, $\dim F = n - 1$ $(n \geq 3)$, $B \times_f F$ of a complete connected r-dimensional $(1 < r < n)$ Riemannian manifold B and $(n - r)$-dimensional Riemannian manifold F and $B \times_f I$ of a complete connected $(n - 1)$-dimensional Riemannian manifold B and one-dimensional Riemannian manifold I. Motivated by the studies [5] and [10], in the present study, we consider Einstein statistical warped product manifolds.

2. Dualistic Structures on Warped Product Manifolds

Let (M, g_M) and (N, g_N) be two Riemannian manifolds of dimension m and n, respectively and $f \in C^\infty(M)$ be a positive function on M. The warped product of (M, g_M) and (N, g_N) (see [9]) with warping function f is the $(m \times n)$-dimensional manifold $M \times N$ endowed with the metric g given by:

$$g =: \pi^* g_M + (f \circ \pi)^2 \sigma^* g_N, \tag{5}$$

where π^* and σ^* are the pull-backs of the projections π and σ of $M \times N$ on M and N, respectively. The tangent space $T_{(p,q)}(M \times N)$ at a point $(p, q) \in M \times N$ is isomorphic to the direct sum $T_p M \oplus T_q N$. Let $L_H M$ (resp. $L_V N$) be the set of all vector fields on $M \times N$, each of which is the horizontal lift (resp. the vertical lift) of a vector field on M (resp. on N). We have:

$$T(M \times N) = L_H M \oplus L_V N;$$

and thus a vector field A on $M \times N$ can be written as

$$A = X + U, \quad \text{with } X \in L_H M \text{ and } U \in L_V N.$$

Obviously,

$$\pi_*(L_H M) = TM \text{ and } \sigma_*(L_V N) = TN.$$

For any vector field $X \in L_H M$, we denote $\pi_*(X)$ by $\overline{X}$, and for any vector field $U \in L_V N$, we denote by $\sigma_*(U)$ by $\widetilde{U}$ [9].

Let (∇,∇^*), $\left(\widetilde{\nabla},\widetilde{\nabla}^*\right)$ and (D,D^*) be dualistic structures on M, N, and $M\times N$, respectively. For any $X,Y\in L_HM$ and $U,V\in L_VN$ we put [10]

$$\pi_*\left(D_XY\right)=\nabla_{\overline{X}}\overline{Y},\text{ and }\pi_*\left(D^*_XY\right)=\nabla^*_{\overline{X}}\overline{Y},$$

and

$$\sigma_*\left(D_UV\right)=\nabla_{\widetilde{U}}\widetilde{V},\text{ and }\sigma_*\left(D^*_UV\right)=\widetilde{\nabla}^*_{\widetilde{U}}\widetilde{V}.$$

Given fields $X,Y\in L_HM$ and $U,V\in L_VN$ then:

1. $D_XY=\nabla_{\overline{X}}\overline{Y}$,
2. $D_XU=D_UX=\frac{Xf}{f}U$,
3. $D_UV=-\frac{g(U,V)}{f}grad\ f+\widetilde{\nabla}_{\widetilde{U}}\widetilde{V}$,
4. $D^*_XY=\nabla^*_{\overline{X}}\overline{Y}$,
5. $D^*_XU=D^*_UX=\frac{Xf}{f}U$,
6. $D^*_UV=-\frac{g(U,V)}{f}grad\ f+\widetilde{\nabla}^*_{\widetilde{U}}\widetilde{V}$,

where we use the notation by writing f for $f\circ\pi$ and $grad\ f$ for $grad\ (f\circ\pi)$ and denote by g the inner product with respect to $M\times N$. Obviously, D and D^* define dual affine connections on $T(M\times N)$ [10].

The Hessian function H^f_D of f with respect to connection D is a $(0,2)$-tensor field such that

$$H^f_D(X,Y)=XY(f)-(D_XY)f. \tag{6}$$

Let M be an n-dimensional Riemannian manifold, D an affine connection, $\{e_1,e_2,...,e_n\}$ an orthonormal frame field. Then the Laplacian Δ^Df of a function f with respect to connection D is defined by

$$\Delta^Df=div(grad\ f)=\sum_{i=1}^{n}g\left(D_{e_i}grad\ f,e_i\right). \tag{7}$$

Let ${}^MR,{}^NR$ and R be the Riemannian curvature operators w.r.t. $\nabla,\widetilde{\nabla}$ and D respectively. Then Todjihounde [10] gave the following lemma:

Lemma 2.1. *Let (g_M,∇,∇^*) and $\left(g_N,\widetilde{\nabla},\widetilde{\nabla}^*\right)$ be dualistic structures on M and N, respectively, $B=M\times_fN$ a warped product with curvature tensor R. For $X,Y,Z\in L_HM$ and $U,V,W\in L_VN$,*

(i) $R(X,Y)Z=({}^MR\left(\overline{X},\overline{Y}\right)\overline{Z})$,

(ii) $R(V,Y)Z=-\frac{1}{f}H^f_D(Y,Z)V$,

(iii) $R(X,Y)V=R(V,W)X=0$,

(iv) $R(X,V)W=-\frac{1}{f}g(V,W)D_X(grad\ f)$,

(v) $R(V,W)U=({}^NR(\widetilde{V},\widetilde{W})\widetilde{U})+\frac{1}{f^2}\left\|grad\ f\right\|^2(g(V,U)W-g(W,U)V)$.

For the calculations of the Ricci tensors of the warped product $B=M\times_fN$, by a similar way of [9], we can state the following lemma:

Lemma 2.2. *Let (g_M,∇,∇^*) and $\left(g_N,\widetilde{\nabla},\widetilde{\nabla}^*\right)$ be dualistic structures on M and N, respectively, $B=M\times_fN$ a warped product with Ricci tensor ${}^BS$. Given fields $X,Y\in L_HM$ and $U,V\in L_VN$, then*

(i) ${}^BS(X,Y)={}^MS(X,Y)-\frac{d}{f}H^f_D(X,Y)$, *where* $d=\dim N$,

(ii) ${}^BS(X,V)=0$,

(iii) ${}^BS(U,V)={}^NS(U,V)-g(U,V)\left[\frac{\Delta^Df}{f}+\frac{\left\|grad\ f\right\|^2}{f^2}(d-1)\right]$.

3. Einstein Warped Products in Statistical Manifolds

In this section, we consider Einstein statistical warped product manifolds and prove some results concerning these type manifolds.

Now, let (g, D, D^*) be a dualistic structure on $M \times_f N$. So we can state the following theorems:

Theorem 3.1. *Let $(B = I \times_f N, D, D^*, g)$ be a statistical warped product with a 1-dimensional statistical manifold I with trivial statistical structure and an $(n-1)$-dimensional statistical manifold N.*

i) If (B, g) is an Einstein statistical manifold, then N is an Einstein statistical manifold with scalar curvature $\tau^N = -(n-1)(n-2)a^2$, $f(t) = \cosh(at+b)$ and a, b are real constants.

ii) Conversely, if N is an Einstein statistical manifold with scalar curvature $\tau^N = -(n-1)(n-2)a^2$, $f(t) = \cosh(at+b)$ and a, b are real constants, then B is an Einstein statistical manifold with scalar curvature $\tau^B = -n(n-1)a^2$.

Proof. Denote by $(dt)^2$, the metric on I. Making use of Lemma 2.2, we can write

$$^B S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{n-1}{f}\left[f'' - f' g\left(D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)\right].$$

Since I is a 1-dimensional statistical manifold with trivial statistical structure, we have

$$g\left(D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = 0. \tag{8}$$

So the above equation reduces to

$$^B S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{n-1}{f} f''. \tag{9}$$

On the other hand, for $U, V \in L_V N$

$$^B S(U, V) = {}^N S(U, V) - \left(\frac{f'' + f' g\left(D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right)}{f} + (n-2)\frac{f'^2}{f^2}\right) g(U, V).$$

Then using (8) and the definition of warped product metric (5), we get

$$^B S(U, V) = {}^N S(U, V) - \left[f'' f + (n-2) f'^2\right] g_N(U, V). \tag{10}$$

Since B is an Einstein statistical manifold, from (1), we have

$$^B S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \lambda g_I\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) \tag{11}$$

and

$$^B S(U, V) = \lambda f^2 g_N(U, V). \tag{12}$$

If we consider (11) and (9) together, then we find

$$\lambda = -\frac{n-1}{f} f''. \tag{13}$$

Hence from (1), λ is a constant.

Using (12) and (13) in (10) we obtain

$$^{N}S\,(U,V)=(n-2)\left[-f''f+f'^{2}\right]g_{N}\,(U,V)\,.$$

If $-f''f+f'^{2}$ is a constant, then N is an Einstein statistical manifold. Since λ is a constant, $\frac{f''}{f}$ is also a constant. Since $f>0$, we get $f(t)=\cosh(at+b)$, where a and b are real constants. In this case, N is an Einstein statistical manifold with scalar curvature $\tau^{N}=-\,(n-1)\,(n-2)a^{2}$.

Conversely, assume that N is an Einstein statistical manifold with scalar curvature $\tau^{N}=-\,(n-1)\,(n-2)a^{2}$, $f(t)=\cosh(at+b)$ and a,b are real constants. Then

$$^{N}S=-(n-2)a^{2}g_{N}.$$

From Lemma 2.2 (iii), (i) and the definition of warped product metric (5), we have

$$^{B}S\,(U,V)=-(n-1)a^{2}g(U,V) \tag{14}$$

and

$$^{B}S\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=-(n-1)a^{2}g(\frac{\partial}{\partial t},\frac{\partial}{\partial t}). \tag{15}$$

So B is an Einstein statistical manifold with scalar curvature $\tau^{B}=-n(n-1)a^{2}$.

Hence we get the result as required. □

From Lemma 2.1, it can be easily seen that if $(M,\nabla,\nabla^{*},g_{M})$ and $\left(N,\widetilde{\nabla},\widetilde{\nabla}^{*},g_{N}\right)$ are statistical manifolds of constant curvatures c and $\widetilde{c}$, respectively,

$$H_{D}^{f}(X,Y)=-cfg(X,Y),\quad D_{X}\,(grad\ f)=-cfX$$

and $\frac{1}{f}\left\|grad\ f\right\|$ is a constant, then $\left(B=M\times_{f}N,D,D^{*},g\right)$ is also a statistical manifold of constant curvature c, where $c=\widetilde{c}-\frac{1}{f^{2}}\left\|grad\ f\right\|^{2}$.

Theorem 3.2. *Let $\left(B=M\times_{f}N,D,D^{*},g\right)$ be a statistical warped product of an r-dimensional $(1<r<n)$ statistical manifold $(M,\nabla,\nabla^{*},g_{M})$ and $(n-r)$-dimensional statistical manifold $\left(N,\widetilde{\nabla},\widetilde{\nabla}^{*},g_{N}\right)$. Assume that (B,g) is a statistical manifold of constant curvature c. Then*

i) N is Einstein if $cf^{2}+\left\|grad\ f\right\|^{2}$ is a constant.

ii) M is Einstein if $\lambda g_{M}\,(X,Y)=H_{D}^{f}\,(X,Y)\,,$ where λ is a differentiable function on M and $\frac{\lambda}{f}$ is a constant.

Proof. Assume that B is a statistical manifold of constant curvature c. So B is an Einstein statistical manifold with scalar curvature $\tau^{B}=n\,(n-1)\,c$. From (3), we can write

$$\begin{aligned} g\,(R\,(X,U)\,V,Y) &= c\,\{g\,(U,V)\,g\,(X,Y)-g\,(X,V)\,g\,(U,Y)\} \\ &= cg\,(U,V)\,g\,(X,Y)\,, \end{aligned} \tag{16}$$

where $X,Y\in L_{H}M$, $U,V\in L_{V}N$.

Since $M\times_{f}N$ is a warped product, then from Lemma 2.2 (iv), we have

$$g\,(R\,(X,U)\,V,Y)=-\frac{1}{f}g(U,V)g\,(D_{X}grad\ f,Y) \tag{17}$$

for $X,Y\in L_{H}M$, $U,V\in L_{V}N$. If we choose a local orthonormal frame $e_{1},...,e_{n}$ such that $e_{1},...,e_{r}$ are tangent to M and $e_{r+1},...,e_{n}$ are tangent to N, in view of (16) and (17), then

$$\sum_{1\leq j\leq r,\ r+1\leq s\leq n}g\left(R\left(e_{j},e_{s}\right)e_{s},e_{j}\right)=\sum_{1\leq j\leq r,\ r+1\leq s\leq n}cg\,(e_{s},e_{s})\,g\left(e_{j},e_{j}\right)=-\frac{1}{f}\sum_{1\leq j\leq r,\ r+1\leq s\leq n}g(e_{s},e_{s})g\left(D_{e_{j}}grad\ f,e_{j}\right).$$

So we find

$$-\frac{\Delta^D f}{f} = cr. \tag{18}$$

From Lemma 2.2 (iii), using (18), we get

$${}^{N}S(U,V) = (n-r-1)\left(cf^2 + \left\|grad\ f\right\|^2\right) g_N(U,V),$$

which means that N is Einstein if $cf^2 + \left\|grad f\right\|^2$ is a constant.

Now assume that the Hessian of the affine connection D is proportional to the metric tensor g_M, then we can write

$$\lambda g_M(X,Y) = H_D^f(X,Y), \tag{19}$$

where λ is a differentiable function on M. On the other hand, from Lemma 2.2 (i) and (19), we get

$${}^{M}S(X,Y) = \left((n-1)c + \lambda\frac{n-r}{f}\right) g_M(X,Y).$$

So M is Einstein if $\frac{\lambda}{f}$ is a constant.

This proves the theorem. □

Theorem 3.3. *Let $\left(B = M \times_f I, D, D^*, g\right)$ be a statistical warped product of an $(n-1)$-dimensional statistical manifold $(M, \nabla, \nabla^*, g_M)$ and 1-dimensional statistical manifold I. Assume that $\lambda g_M(X,Y) = H_D^f(X,Y)$, where λ is a differentiable function on M.*

i) If (B,g) is an Einstein statistical manifold, then (M,g_M) is an Einstein statistical manifold with scalar curvature $\tau^M = (n-1)\left(\frac{\lambda}{f} - \frac{\Delta^D f}{f}\right)$, when $\frac{\lambda}{f}$ is a constant.

ii) Conversely, if (M,g_M) is an Einstein statistical manifold when $\frac{\lambda}{f}$ is a constant, then (B,g) is an Einstein statistical manifold with scalar curvature $\tau^B = -n\frac{\Delta^D f}{f}$, when $\frac{\Delta^D f}{f}$ is a constant.

Proof. Since (B,g) is an Einstein statistical manifold, from Lemma 2.2 (i) and (iii), we have

$${}^{M}S(X,Y) = \frac{\tau^B}{n} g(X,Y) + \frac{1}{f} H_D^f(X,Y) \tag{20}$$

and

$${}^{B}S\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = -\frac{\Delta^D f}{f} g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right), \tag{21}$$

respectively. Since the Hessian of the affine connection D is proportional to the metric tensor g_M, then using (20) and (19), we have

$${}^{M}S(X,Y) = \left(\frac{\tau^B}{n} + \frac{\lambda}{f}\right) g_M(X,Y). \tag{22}$$

Since (B,g) is an Einstein statistical manifold, from (1), we get

$$\frac{\tau^B}{n} = -\frac{\Delta^D f}{f} = \text{ constant},$$

where the scalar curvature τ^B is a constant. Substituting the last equation in (22) we obtain

$$^{M}S\left(X,Y\right)=\left(\frac{\lambda}{f}-\frac{\Delta^{D}f}{f}\right)g_{M}\left(X,Y\right).$$

Since $\frac{\Delta^D f}{f}$ is a constant, (M, g_M) is an Einstein statistical manifold, if $\frac{\lambda}{f}$ is also a constant.

Conversely, if (M, g_M) is an Einstein statistical manifold with scalar curvature $\tau^M = (n-1)\left(\frac{\lambda}{f} - \frac{\Delta^D f}{f}\right)$, when $\frac{\lambda}{f}$ is a constant, then

$$^{M}S\left(X,Y\right)=\left(\frac{\lambda}{f}-\frac{\Delta^{D}f}{f}\right)g_{M}\left(X,Y\right).$$

So using Lemma 2.2 (i) and (iii), we have

$$^{B}S\left(X,Y\right)=-\frac{\Delta^{D}f}{f}g\left(X,Y\right)$$

and

$$^{B}S\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)=-\frac{\Delta^{D}f}{f}g\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right).$$

Hence (B, g) is an Einstein statistical manifold with scalar curvature $\tau^B = -n\frac{\Delta^D f}{f}$, if $\frac{\Delta^D f}{f}$ is a constant.

This proves the theorem. □

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References

[1] S. Amari, Differential-Geometrical Methods in Statistics, Springer-Verlag, 1985.
[2] M. E. Aydın, A. Mihai, I. Mihai, Some inequalities on submanifolds in statistical manifolds of constant curvature, Filomat 29 (2015) 465–477.
[3] D. Djebbouri, S. Ouakkas, Product of statistical manifolds with doubly warped product, Gen. Math. Notes, 31 (2015) 16–28.
[4] H. Furuhata, Hypersurfaces in statistical manifolds, Differential Geom. Appl. 27 (2009) 420–429.
[5] A. Gebarowski, On Einstein warped products, Tensor (N.S.) 52 (1993) 204–207.
[6] I. Hasegawa, K. Yamauchi, Conformally-projectively flat statistical structures on tangent bundles over statistical manifolds, Differential Geometry and its Applications, Proc. Conf., in Honour of Leonhard Euler, Olomouc, 2008 World Scientific Publishing Company, 239–251.
[7] A. Mihai, I. Mihai, Curvature invariants for statistical submanifolds of Hessian manifolds of constant Hessian curvature, Mathematics, (2018) 6 44.
[8] C. R. Min, S. O. Choe, Y. H. An, Statistical immersions between statistical manifolds of constant curvature, Glob. J. Adv. Res. Class. Mod. Geom. 3 (2014) 66–75.
[9] B. O'Neill, Semi-Riemannian Geometry with Applications to Relativity, Pure and Applied Mathematics, 103. Academic Press, Inc., New York, (1983).
[10] L. Todjihounde, Dualistic structures on warped product manifolds, Differential Geometry-Dynamical Systems 8 (2006) 278-284.
[11] P. W. Vos, Fundamental equations for statistical submanifolds with applications to the Bartlett correction, Ann. Inst. Statist. Math. 41 (1989) 429-450.