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Split Equality Common Null Point Problem for Bregman Quasi-Nonexpansive Mappings

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Abstract. In this paper, we introduce a new algorithm for solving the split equality common null point problem and the equality fixed point problem for an infinite family of Bregman quasi-nonexpansive mappings in reflexive Banach spaces. We then apply this algorithm to the equality equilibrium problem and the split equality optimization problem. In this way, we improve and generalize the results of Takahashi and Yao [22], Byrne et al [9], Dong et al [11], and Sitthithakerngkiet et al [21].

1. Introduction

Let H_1, H_2 and H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets, and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. The *split equality problem* (SEP) which was first introduced by Moudafi [13] is to find

$$x \in C, \quad y \in Q \qquad \text{such that} \quad Ax = By. \tag{1}$$

The SEP (1) is actually an optimization problem with weak coupling in the constraint. The problem has numerous applications in the decomposition of domains for PDEs, game theory, and intensity-modulated radiation therapy. To see more applications of the SEP in optimal control theory, surface energy and potential games whose variational form can be seen as a SEP, we refer the reader to Attouch [2]. For solving the SEP (1), Moudafi [13] introduced the following alternating CQ algorithm:

$$x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)),$$
$$y_{n+1} = P_C(x_n - \beta_n B^*(Ax_n - By_n)),$$

where $\gamma_n, \beta_n \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A and λ_B are the spectral radii of A^*A and B^*B, respectively. If $B = I$ (the identity mapping) and $H_2 = H_3$, the problem (1) is equivalent to the well-known split feasibility problem (SFP).

In [8], Byrne et al considered the following problem: Let $A_i : H_1 \to 2^{H_1}$, $1 \leq i \leq m$, and $B_j : H_2 \to 2^{H_2}$, $1 \leq j \leq n$, be set-valued mappings, and $T_j : H_1 \to H_2$, $1 \leq j \leq n$, be bounded linear operators. The *split common null point problem* is to find a point $z \in H_1$ such that

$$z \in (\cap_{i=1}^{m} A_i^{-1}0) \cap (\cap_{j=1}^{n} T_j^{-1}(B_j^{-1}0)) \tag{2}$$

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where $A_i^{-1}0$ and $B_j^{-1}0$ are the null point sets of A_i and B_j, respectively.

Let $A : H \longrightarrow 2^H$ be a multivalued mapping with graph $G(A) = \{(x, y) : y \in Ax\}$, domain $D(A) = \{x \in H : Tx \neq \emptyset\}$ and range $R(A) = \cup\{Ax : x \in D(A)\}$. The mapping A is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $(x, x^*), (y, y^*) \in G(A)$. A monotone operator $A \subset H \times H$ is said to be maximal if its graph is not properly contained in the graph of any other monotone operator. One of the most important methods for solving (2) in a Hilbert space setting is to replace (2) with the fixed point problem for the operator $R_A : H \to 2^H$ defined by $R_A := (I + A)^{-1}$.

To tackle the problem in the Banach space setting, Teboulle [23] introduced a new type of resolvent. Let $f : E \to (-\infty, +\infty]$ be a proper, convex, lower semicontinuous and Gâteaux differentiable function on $int\, dom f$, and let A be a maximal monotone operator such that $int\, dom f \cap dom A \neq \emptyset$. Then the operator $Res_A^f : E \to 2^{E^*}$ where E^* is the dual of E, is defined by

$$Res_A^f := (\nabla f + A)^{-1} o \nabla f.$$

Note that the fixed points of Res_A^f are solutions of (2).

In 2015, Takahashi and Yao proposed the following iterative method to solve the problem (2): Let $x \in H$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_{\lambda_n}(x_n - \lambda_n T^* J_F(Tx_n - Q_{\mu_n} Tx_n)), \\ y_n = \alpha_n x_n + (1 - \alpha_n) z_n), \\ C_n = \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap D_n} x_1. \end{cases} \tag{3}$$

Observe that in the above algorithm, the determination of the step-size λ_n depends on the operator (matrix) norm $\|T\|$. This means that in order to implement the algorithm, first one has to compute the operator norm of T, which in general is not an easy task.

Here we consider the following *split equality common null point problem*:

$$\text{find} \quad x \in \cap_{i=1}^m h_i^{-1}0, \quad y \in \cap_{j=1}^n g_j^{-1}0 \quad \text{such that} \quad Ax = By, \tag{4}$$

where H_1, H_2, H_3 are real Hilbert spaces, $h_i : H_1 \to 2^{H_1}$ and $g_j : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are bounded linear operators.

We propose a new algorithm for solving the split equality common null point problem and the equality fixed point problem for an infinite family of Bregman quasi-nonexpansive mappings in reflexive Banach spaces. In this way, we extend the result of Takahashi and Yao [22]. At the same time, we present a useful method for estimating the step-size sequence (γ_n) which does not require any prior knowledge of the operator norms $\|A\|$ and $\|B\|$. As application, we consider the algorithm for the equality equilibrium problem and the split equality optimization problem. In this way, we improve and generalize the results of Takahashi and Yao [22], Byrne et al [9], Dong et al [11], and Sitthithakerngkiet et al [21].

2. Preliminaries

Let E be a real Banach space with the norm $\|.\|$ and the dual space E^*, and let $f : E \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. We denoted by $dom f$, the domain of f, that is the set $\{x \in E : f(x) < +\infty\}$. Let $x \in int\, dom f$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^*\rangle \leq f(y), \quad \forall y \in E\}.$$

The Fénchel conjugate of f is the convex function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

It is known that f satisfies the Young-Fénchel inequality

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*) \quad x \in E, x^* \in E^*,$$

moreover, the equality holds if $x^* \in \partial f(x)$.

Given $x \in int\, dom f$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^0(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{5}$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, the gradient of f at x is the linear function $\nabla f(x)$ defined by $\langle y, \nabla f(x)\rangle := f^0(x, y)$ for all $y \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at each $x \in int\ dom f$. When the limit as $t \to 0$ in (5) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is Fréchet differentiable at x. The function $f : E \to (-\infty, +\infty]$ is called Legendre if it satisfies the following two conditions:

(L_1) f is Gâteaux differentiable, $int\ dom f \neq \emptyset$ and $dom \nabla f = int\ dom f$,

(L_2) f^* is Gâteaux differentiable, $int\ dom f^* \neq \emptyset$ and $dom \nabla f^* = int\ dom f^*$.

Remark 2.1. *If E is a real reflexive Banach space, and f is a Legendre function, then we have*

(i) *f is a Legendre function if and only if f^* is a Legendre function,*

(ii) $(\partial f)^{-1} = \partial f^*$,

(iii) $\nabla f = (\nabla f^*)^{-1}$, $ran \nabla f = dom \nabla f^* = int(dom f^*)$, $ran \nabla f^* = dom \nabla f = int(dom f)$,

(iv) *f and f^* are strictly convex on the interior of their respective domains.*

Remark 2.2. *If $f : E \to \mathbb{R}$ is Gâteaux differentiable and convex, then*

$$\begin{aligned}\langle y, \nabla f(x)\rangle &= f^0(x, y) = \lim_{t\to 0} \frac{f(x + ty) - f(x)}{t} \\ &= \lim_{t\to 0} \frac{f((1 - t)x + t(x + y)) - f(x)}{t} \leq \lim_{t\to 0} \frac{(1 - t)f(x) + tf(x + y) - f(x)}{t} = f(x + y) - f(x).\end{aligned}$$

The Bregman distance with respect to f, (see [4]), is the bifunction $D_f : dom f \times int\, dom f \to [0, +\infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y)\rangle.$$

We mention in passing that D_f is not a distance in the usual sense; but it enjoys the following properties:

(i) $D_f(x, x) = 0$, but $D_f(x, y) = 0$ may not imply $x = y$,

(ii) D_f is not symmetric and does not satisfy the triangle inequality,

(iii) for $x \in dom f$ and $y, z \in int\, dom f$, we have

$$D_f(x, y) + D_f(y, z) - D_f(x, z) \leq \langle \nabla f(z) - \nabla f(y), x - y\rangle,$$

(iv) for each $z \in E$, we have $D_f(z, \nabla f^*(\sum_{i=1}^N t_i \nabla f(x_i) \leq \sum_{i=1}^N t_i D_f(z, x_i)$, where $\{x_i\}_{i=1}^N \subseteq E$ and $\{t_i\}_{i=1}^N \subseteq (0, 1)$ satisfies $\sum_{i=1}^N t_i = 1$.

Bregman distances have been studied by many researchers (see for instance [3, 5, 10]). We shall make use of the function $V_f : E \times E^* \to [0, +\infty]$ associated with f, which is defined by (see [7]):

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*.$$

Then $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality, we have

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*),$$

for all $x \in E$ and $x^*, y^* \in E^*$(see [14]).

The modulus of total convexity at x is the bifunction $\upsilon_f : int\ dom f \times [0, +\infty) \to [0, +\infty)$, defined by

$$\upsilon_f(x, t) := \inf\{D_f(y, x) : y \in dom f, \|y - x\| = t\}.$$

The function f is called totally convex at $x \in int\ dom f$ if $\upsilon_f(x, t)$ is positive for any $t > 0$. This notion was first introduced by Butnariu and Iusem in [7]. Let C be a nonempty subset of E. The modulus of total convexity of f on C is the bifunction $\upsilon_f : int\ dom f \times [0, +\infty) \to [0, +\infty)$, defined by

$$\upsilon_f(C, t) := \{\upsilon_f(x, t) : x \in C \cap int\ dom f\}.$$

The function f is called totally convex on bounded subsets if $\upsilon_f(C, t)$ is positive for any nonempty and bounded subset C and any $t > 0$.

Proposition 2.3. *[19] If $x \in int\ dom f$, then the following statements are equivalent:*

(i) *the function f is totally convex at x,*

(ii) *for any sequence $\{y_n\} \subset dom f$,*

$$\lim_{n\to\infty} D_f(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x\| = 0.$$

Recall that the function f is called sequentially consistent (see [6]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x_n\| = 0.$$

Proposition 2.4. *[7] If $dom f$ contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Proposition 2.5. *[17] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Definition 2.6. *Let C be a nonempty subset of $int\ dom f$. An operator $T : C \to int\ dom f$ is said to be:*

(i) *Bregman firmly nonexpansive (BFNE) if*

$$\langle Tx - Ty, \nabla f(Tx) - \nabla f(Ty) \rangle \leq \langle Tx - Ty, \nabla f(x) - \nabla f(y) \rangle,$$

for any $x, y \in C$, or equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

(ii) *Bregman quasi firmly nonexpansive (BQFNE) if $F(T) \neq \emptyset$, and*

$$\langle Tx - p, \nabla f(x) - \nabla f(Tx) \rangle \geq 0, \quad \forall x \in C, p \in F(T),$$

or equivalently,

$$D_f(p, Tx) + D_f(Tx, x) \leq D_f(p, x).$$

(iii) *Bregman quasi-nonexpansive (BQNE) if $F(T) \neq \emptyset$, and*

$$D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, p \in F(T).$$

Definition 2.7. *A point $u \in C$ is said to be an asymptotic fixed point of $T : C \to C$ if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup u$ and $\|x_n - Tx_n\| \to 0$. We denote the asymptotic fixed point set of T by $\hat{F}(T)$.*

The concept of an asymptotic fixed point was introduced by Reich in [15].

Proposition 2.8. *[16] Let $f : E \to \mathbb{R}$ be a Legendre function, and let C be a nonempty closed and convex subset of E. If $T : C \to E$ is a BQNE operator, then $F(T)$ is closed and convex.*

The gauge of uniform convexity of a function $f : E \to \mathbb{R}$ is defined by

$$\rho_r(t) = \inf \left\{ \frac{(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)}{\lambda(1-\lambda)} : \|x\|, \|y\| \leq r, \lambda \in (0,1), \|x - y\| = t \right\}.$$

A function f is said to be uniformly convex on bounded subsets if $\rho_r(t) > 0$ for all $r, t > 0$.

The gauge of uniform smoothness of f is defined by

$$\sigma_r(t) = \sup \left\{ \frac{(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y)}{\lambda(1-\lambda)} : \|x\|, \|y\| \leq r, \lambda \in (0,1), \|x - y\| = t \right\}.$$

Then the function f is said to be uniformly smooth on bounded subsets if $\lim_{t \to 0} \frac{\sigma_r(t)}{t} = 0$ for all $r > 0$.

Definition 2.9. *A function $f : E \to \mathbb{R}$ is said to be super coercive if*

$$\lim_{x \to \infty} \frac{f(x)}{\|x\|} = +\infty.$$

Definition 2.10. *Let C be a nonempty subset of a real Banach space E, and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into E such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the AKTT-condition (see [1]) if for each bounded subset K of C,*

$$\sum_{n=1}^{\infty} \sup \{\|T_{n+1}z - T_n z\| : z \in K\} < \infty.$$

Lemma 2.11. *[1] Let C be a nonempty subset of a real Banach space E, and let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings from C into E which satisfies the AKTT-condition. Then, for each $x \in C$, $\{T_n x\}_{n=1}^{\infty}$ is convergent. Furthermore, if we define a mapping $T : C \to E$ by*

$$Tx := \lim_{n \to \infty} T_n x, \quad \forall x \in C,$$

then, for each bounded subset K of C,

$$\lim_{n \to \infty} \sup\{\|T_n z - Tz\| : z \in K\} = 0.$$

In the sequel, we write $(\{T_n\}_{n=1}^{\infty}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and $F(T) = \cap_{n=1}^{\infty} F(T_n)$.

Proposition 2.12. *[18] If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E.*

Theorem 2.13. *[25] Let $f : E \to \mathbb{R}$ be a continuous convex function which is super coercive. Then the following are equivalent:*

(i) *f is bounded and uniformly smooth on bounded subsets of E,*

(ii) *f is Fréchet differentiable and ∇f is uniformly norm-to-norm continuous on bounded subsets of E,*

(iii) *$dom f^* = E^*$, f^* is super coercive and uniformly convex on bounded subsets of E^*.*

Theorem 2.14. *[25] Let $f : E \to \mathbb{R}$ be a continuous convex function which is bounded on bounded subsets of E. Then the following are equivalent:*

(i) *f is super coercive and uniformly convex on bounded subsets of E,*

(ii) *$dom f^* = E^*$, f^* is bounded and uniformly smooth on bounded subsets of E^*,*

(iii) *$dom f^* = E^*$, f^* is Fréchet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*.*

Theorem 2.15. *[10] Suppose that $f : E \to (-\infty, +\infty]$ is a Legendre function. The function f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets.*

Lemma 2.16. *[24] Let $\{\gamma_n\}$ be a sequence in $(0, 1)$ and $\{\delta_n\}$ be a sequence in $\mathbb{R}$ satisfying*

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) (2) $\limsup_{n\to\infty} \gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$.

If $\{a_n\}$ is a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n \delta_n,$$

for each $n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.17. *[12] Let $\{s_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that $s_{n_i} \leq s_{n_i+1}$ for all $i \geq 0$. For every $n \geq n_0$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{k \leq n : s_k < s_{k+1}\}.$$

Then $\tau(n) \to \infty$ and $\max\{s_{\tau(n)}, s_n\} \leq s_{\tau(n)+1}$.

3. The Main Result

We start this section by proving a strong convergence theorem for an infinite family of Bregman quasi-nonexpansive mappings.

Theorem 3.1. *Let E_1, E_2 and E_3 be reflexive Banach spaces, let $C \subseteq E_1$ and $Q \subseteq E_2$ be two nonempty closed convex sets, let $A : E_1 \longrightarrow E_3$ and $B : E_2 \longrightarrow E_3$ be two bounded linear operators and let $f_1 : E_1 \longrightarrow \mathbb{R}$ and $f_2 : E_2 \longrightarrow \mathbb{R}$ be super coercive Legendre functions which are bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E_1 and E_2, respectively, and $f_3 : E_3 \longrightarrow \mathbb{R}$ be convex, continuous and one-to-one on E_3 such that f_3^{-1} is continuous. Let, for $i = 1, 2, ..., N$, $h_i : E_1 \longrightarrow 2^{E_1^*}$ and $g_i : E_2 \longrightarrow 2^{E_2^*}$ be maximal monotone mappings with $dom(h_N) \subset C$ and $dom(g_N) \subset Q$. Assume that for $n \in \mathbb{N}$, $T_n : C \longrightarrow E_1$ and $S_n : Q \longrightarrow E_2$ are an infinite family of Bregman quasi-nonexpansive mappings such that $(\{T_n\}_{n=1}^{\infty}, T)$ and $(\{S_n\}_{n=1}^{\infty}, S)$ satisfy the AKTT-condition, and $F(T) = F(\hat{T})$ $F(S) = F(\hat{S})$. Put*

$$\Omega = \Big\{(x, y) : \quad x \in \cap_{n=1}^{\infty} F(T_n) \cap (\cap_{i=1}^{N} h_i^{-1} 0),\ y \in \cap_{n=1}^{\infty} F(S_n) \cap (\cap_{i=1}^{N} g_i^{-1} 0) \text{ such that } Ax = By\Big\} \neq \emptyset.$$

Let $\{x_n\}$ be the sequence generated by:

$$\begin{cases} z_n = Res^{f_1}_{\lambda_n^N h_N} o \cdots o Res^{f_1}_{\lambda_n^1 h_1} \nabla f_1^*((1-\gamma_n)\nabla f_1 x_n - \gamma_n A^* \nabla f_3(Ax_n - By_n)), \\ x_{n+1} = \nabla f_1^*(\beta_n \nabla f_1 z_n + (1-\beta_n)\nabla f_1 T_n z_n), \\ u_n = Res^{f_2}_{\lambda_n^N g_N} o \cdots o Res^{f_2}_{\lambda_n^1 g_1} \nabla f_2^*((1-\gamma_n)\nabla f_2 x_n + \gamma_n B^* \nabla f_3(Ax_n - By_n)), \\ y_{n+1} = \nabla f_2^*(\beta_n \nabla f_2 u_n + (1-\beta_n)\nabla f_2 S_n u_n), \end{cases} \tag{6}$$

where the step-size γ_n is chosen as follows:

$$\gamma_n = \sigma_n \min\left\{ \frac{|f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n)| + |f_1(x_n)|}, \frac{|f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n)| + |f_2(y_n)|} \right\},$$

where $\sigma_n \in (0,1)$ is defined in such a way that $\sum_{n=1}^{\infty} \gamma_n = \infty$. Assume that the sequences $\{\lambda_n^i\}, \{\beta_n\} \in (0,1)$ satisfy the following conditions:

(i) $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$,

(ii) $\liminf_{n\to\infty} \lambda_n^i > 0$.

Then the sequence $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Omega$.

Proof. It follows from Proposition 2.8 that Ω is closed and convex. Let

$$w_n = (1-\gamma_n)\nabla f_1 x_n - \gamma_n A^* \nabla f_3(Ax_n - By_n)$$

and $(\hat{x}, \hat{y}) \in \Omega$. Note that

$$\begin{aligned} D_{f_1}(\hat{x}, \nabla f_1^* w_n) &= D_{f_1}(\hat{x}, \nabla f_1^*((1-\gamma_n)\nabla f x_n - \gamma_n A^* \nabla f_3(Ax_n - By_n))) \\ &= f_1(\hat{x}) + f_1^*(w_n) - \langle \hat{x}, w_n \rangle \\ &\leq f_1(\hat{x}) + (1-\gamma_n) f_1^*(\nabla f_1 x_n) + \gamma_n f_1^*(-A^* \nabla f_3(Ax_n - By_n)) \\ &- \langle \hat{x}, (1-\gamma_n)\nabla f x_n - \gamma_n A^* \nabla f_3(Ax_n - By_n) \rangle \\ &\leq f_1(\hat{x}) + f_1^*(\nabla f_1 x_n) - \langle \hat{x}, \nabla f_1 x_n \rangle + \gamma_n [f_1^*(-A^* \nabla f_3(Ax_n - By_n)) \\ &+ \langle \hat{x}, \nabla f x_n + A^* \nabla f_3(Ax_n - By_n) \rangle] \\ &= D_{f_1}(\hat{x}, x_n) + \gamma_n [\sup_{x \in X} \{\langle -x, A^* \nabla f_3(Ax_n - By_n) \rangle - f_1(x)\} \\ &+ \langle \hat{x}, \nabla f x_n \rangle + \langle \hat{x}, A^* \nabla f_3(Ax_n - By_n) \rangle] \\ &\leq D_{f_1}(\hat{x}, x_n) + \gamma_n [\sup_{x \in X} \{\langle -x, A^* \nabla f_3(Ax_n - By_n) \rangle - f_1(x)\} \\ &+ f_1(x_n + \hat{x}) - f_1(x_n) + \langle \hat{x}, A^* \nabla f_3(Ax_n - By_n) \rangle] \\ &\leq D_{f_1}(\hat{x}, x_n) + \gamma_n [-\langle x_n + \hat{x}, A^* \nabla f_3(Ax_n - By_n) \rangle - f_1(x_n + \hat{x}) \\ &+ f_1(x_n + \hat{x}) - f_1(x_n) + \langle \hat{x}, A^* \nabla f_3(Ax_n - By_n) \rangle] \\ &= D_{f_1}(\hat{x}, x_n) + \gamma_n [-\langle Ax_n, \nabla f_3(Ax_n - By_n) \rangle - f_1(x_n)] \\ &\leq D_{f_1}(\hat{x}, x_n) + \gamma_n [f_3(-By_n) - f_3(Ax_n - By_n) - f_1(x_n)] \\ &= D_{f_1}(\hat{x}, x_n) - \gamma_n [f_3(Ax_n - By_n) + f_1(x_n) - f_3(-By_n)]. \end{aligned}$$

Therefore, we have

$$\begin{aligned} D_{f_1}(\hat{x}, z_n) = D_{f_1}(\hat{x}, Res^{f_1}_{\lambda_n^N h_N} o \cdots o Res^{f_1}_{\lambda_n^1 h_1} \nabla f_1^* w_n) &\leq D_{f_1}(\hat{x}, \nabla f_1^* w_n) \\ &\leq D_{f_1}(\hat{x}, x_n) - \gamma_n [f_3(Ax_n - By_n) + f_1(x_n) - f_3(-By_n)]. \end{aligned} \tag{7}$$

Following a similar argument as above, we obtain

$$D_{f_2}(\hat{y}, u_n) \leq D_{f_2}(\hat{y}, y_n) - \gamma_n [f_3(Ax_n - By_n) + f_2(y_n) - f_3(Ax_n)]. \tag{8}$$

From (7), (8) and the convexity of f_3, we obtain

$$\begin{aligned} D_{f_1}(\hat{x}, z_n) + D_{f_2}(\hat{y}, u_n) &\leq D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n) - \gamma_n[2f_3(Ax_n - By_n) \\ &\quad + f_1(x_n) + f_2(y_n) - f_3(Ax_n) - f_3(-By_n)] \\ &\leq D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n) - \gamma_n[f_3(Ax_n - By_n) + f_1(x_n) + f_2(y_n)]. \end{aligned} \tag{9}$$

Also from (6), by using (9), we obtain

$$\begin{aligned} D_{f_1}(\hat{x}, x_{n+1}) + D_{f_2}(\hat{y}, y_{n+1}) &\leq \beta_n[D_{f_1}(\hat{x}, z_n) + D_{f_2}(\hat{y}, u_n)] + (1 - \beta_n)[D_{f_1}(\hat{x}, Tz_n) + D_{f_2}(\hat{y}, Su_n)] \\ &\leq \beta_n[D_{f_1}(\hat{x}, z_n) + D_{f_2}(\hat{y}, u_n)] + (1 - \beta_n)D_{f_1}(\hat{x}, z_n) + D_{f_2}(\hat{y}, u_n)] \\ &\leq D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n) - \gamma_n(1 - \beta_n)[f_3(Ax_n - By_n) + f_1(x_n) + f_2(y_n)]. \end{aligned} \tag{10}$$

On the other hand, suppose that there is no x_n such that $|f_1(x_n)| \geq |f_2(y_n)|$ for all $n \geq n_0$. It follows that

$$\gamma_n = \sigma_n \frac{|f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n| + |f_1(x_n)|}$$

and

$$\begin{aligned} \gamma_n[f_3(Ax_n - By_n) + f_2(y_n) + f_1(x_n)] &\geq -\gamma_n|f_3(Ax_n - By_n) + f_2(y_n) + f_1(x_n)| \\ &\geq -\gamma_n[|f_2(y_n)| - |f_3(Ax_n - By_n)| - |f_1(x_n)|] \\ &= \gamma_n|[|f_3(Ax_n - By_n)| + |f_1(x_n)| - |f_2(y_n)|] \\ &= |f_3(Ax_n - By_n)|\sigma_n[1 - \frac{|f_2(y_n)|}{|f_3(Ax_n - By_n)| + |f_1(x_n)|}] \geq 0. \end{aligned} \tag{11}$$

Conversely, suppose there exists n_1 such that $|f_1(x_n)| \leq |f_2(y_n)|$ for all $n \geq n_1$. From (11) and $\beta_n \in (0, 1)$, we have

$$D_{f_1}(\hat{x}, x_{n+1}) + D_{f_2}(\hat{y}, y_{n+1}) \leq D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n).$$

Now we use induction to obtain

$$D_{f_1}(\hat{x}, x_{n+1}) + D_{f_2}(\hat{y}, y_{n+1}) \leq D_{f_1}(\hat{x}, x_1) + D_{f_2}(\hat{y}, y_1). \tag{12}$$

From Theorem 2.13, f_1^* and f_2^* are bounded on bounded subsets of E_1^* and E_2^*, respectively. Hence ∇f_1^* and ∇f_2^* are also bounded on bounded subsets of E_1^* and E_2^*, respectively. From (12) and Proposition 2.4, the sequences $\{(x_n, y_n)\}_{n=1}^\infty$, $\{(\nabla f_1^* z_n, \nabla f_2^* u_n)\}_{n=1}^\infty$ and $\{(T_n z_n, S_n u_n)\}_{n=1}^\infty$ are bounded. So by the boundedness of ∇f_1 and ∇f_2 on bounded subsets of E_1 and E_2, respectively, $\{(\nabla f_1 x_n, \nabla f_2 y_n)\}_{n=1}^\infty$, $\{(z_n, u_n)\}$ and $\{(\nabla f_1 T_n z_n, \nabla f_2 S_n u_n)\}_{n=1}^\infty$ are bounded. In view of Theorem 2.11 and Theorem 2.12, $dom\, f_1^* = E_1^*$, f_1^* is super coercive and uniformly convex on bounded subsets of E_1^*. Let

$$s \geq \sup\{\|z_n\|, \|\nabla f_1(T_n z_n)\|, \|\nabla f_1 z_n\| : n \in \mathbb{N}\}$$

be large enough and let $\rho_s^* : [0, \infty) \longrightarrow [0, \infty)$ be the gauge of uniform convexity of f_1^*. Now we have

$$\begin{aligned} D_{f_1}(\hat{x}, x_{n+1}) &= D_{f_1}(\hat{x}, \nabla f_1^*(\beta_n \nabla f_1 z_n + (1 - \beta_n)\nabla f_1 T_n z_n)) \\ &= f_1(\hat{x}) + f_1^*(\beta_n \nabla f_1 z_n + (1 - \beta_n)\nabla f_1 T_n z_n) - \langle \hat{x}, \beta_n \nabla f_1 z_n + (1 - \beta_n)\nabla f_1 T_n z_n \rangle \\ &\leq \beta_n f_1(\hat{x}) + (1 - \beta_n) f_1(\hat{x}) + \beta_n f_1^*(\nabla f_1 z_n) + (1 - \beta_n) f_1^*(\nabla f_1 T_n z_n) \\ &\quad - \beta_n(1 - \beta_n)\rho_s^*(\|\nabla f_1 z_n - \nabla f_1 T_n z_n\|) - \langle \hat{x}, \beta_n \nabla f_1 z_n + (1 - \beta_n)\nabla f_1 T_n z_n \rangle \\ &= \beta_n D_{f_1}(\hat{x}, z_n) + (1 - \beta_n) D_{f_1}(\hat{x}, T_n z_n) - \beta_n(1 - \beta_n)\rho_s^*(\|\nabla f_1 z_n - \nabla f_1 T_n z_n\|) \\ &\leq D_{f_1}(\hat{x}, z_n) - \beta_n(1 - \beta_n)\rho_s^*(\|\nabla f_1 z_n - \nabla f_1 T_n z_n\|) \\ &\leq D_{f_1}(\hat{x}, x_n) - \beta_n(1 - \beta_n)\rho_s^*(\|\nabla f_1 z_n - \nabla f_1 T_n z_n\|). \end{aligned}$$

It follows from the above inequality that

$$\beta_n(1-\beta_n)\rho_s^*(\|\nabla f_1 z_n - \nabla f_1 T_n z_n\|) \le D_{f_1}(\hat{x}, x_n) - D_{f_1}(\hat{x}, x_{n+1}). \tag{13}$$

Let $r \ge \sup_{n\in\mathbb{N}}\{\|u_n\|, \|\nabla f_2 S_n u_n\|, \|\nabla f_2 u_n\|\}$ be large enough, $\rho_r^* : [0,\infty) \longrightarrow [0,\infty)$ be the gauge of uniform convexity of f_2^*. We use a similar argument to obtain

$$\beta_n(1-\beta_n)\rho_r^*(\|\nabla f_1 u_n - \nabla f_1 S_n u_n\|) \le D_{f_1}(\hat{y}, y_n) - D_{f_1}(\hat{y}, y_{n+1}). \tag{14}$$

Also from (10), we obtain

$$\begin{aligned} \gamma_n(1-\beta_n)[f_3(Ax_n - By_n) + f_1(x_n) + f_2(y_n)] \le {} & [D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n)] \\ & - [D_{f_1}(\hat{x}, x_{n+1}) + D_{f_2}(\hat{y}, y_{n+1})]. \end{aligned} \tag{15}$$

For all $n \in \mathbb{N}$, we have

$$\begin{aligned} D_{f_1}(\hat{x}, x_{n+1}) + D_{f_2}(\hat{y}, y_{n+1}) &= D_{f_1}(\hat{x}, \nabla f_1^*(\beta_n \nabla f_1 z_n + (1-\beta_n)\nabla f_1 T_n z_n)) \\ &\quad + D_{f_2}(\hat{y}, \nabla f_2^*(\beta_n \nabla f_1 u_n + (1-\beta_n)\nabla f_2 S_n u_n)) \\ &\le D_{f_1}(\hat{x}, z_n) + D_{f_2}(\hat{y}, u_n) \\ &= V_{f_1}(\hat{x}, (1-\gamma_n)\nabla f_1 x_n + \gamma_n(-A^*\nabla f_3(Ax_n - By_n))) \\ &\quad + V_{f_2}(\hat{y}, (1-\gamma_n)\nabla f_2 y_n + \gamma_n(B^*\nabla f_3(Ax_n - By_n))) \\ &\le V_{f_1}(\hat{x}, (1-\gamma_n)\nabla f_1 x_n + \gamma_n(-A^*\nabla f_3(Ax_n - By_n)) + \gamma_n(A^*\nabla f_3(Ax_n - By_n))) \\ &\quad - \langle \nabla f_1^*((1-\gamma_n)\nabla f_1 x_n + \gamma_n(-A^*\nabla f_3(Ax_n - By_n))) - \hat{x}, \gamma_n(A^*\nabla f_3(Ax_n - By_n))\rangle \\ &\quad + V_{f_2}(\hat{y}, (1-\gamma_n)\nabla f_2 y_n + \gamma_n B^*\nabla f_3(Ax_n - By_n) - \gamma_n B^*\nabla f_3(Ax_n - By_n)) \\ &\quad - \langle \nabla f_2^*((1-\gamma_n)\nabla f_2 y_n + \gamma_n B^*\nabla f_3(Ax_n - By_n)) - \hat{y}, -\gamma_n B^*\nabla f_3(Ax_n - By_n)\rangle \\ &\le (1-\gamma_n)[D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n)] + \gamma_n[\langle \hat{x} - z_n, A^*\nabla f_3(Ax_n - By_n)\rangle \\ &\quad + \langle u_n - \hat{y}, B^*\nabla f_3(Ax_n - By_n)\rangle]. \end{aligned} \tag{16}$$

To prove that $\{x_n\}$ and $\{y_n\}$ converge in norm, we consider the following two cases.
Case1. Assume that the sequence $\{D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n)\}$ is a monotonically decreasing sequence. Then $\{D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n)\}$ is convergent. Clearly, we have

$$[\{D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n)\}] - [\{D_{f_1}(\hat{x}, x_{n+1}) + D_{f_2}(\hat{y}, y_{n+1})\}] \to 0.$$

Therefore, from (15) and $\beta_n \in (0,1)$, it follows that

$$\gamma_n(f_3(Ax_n - By_n) + f_1(x_n) + f_2(y_n)) \to 0, \quad n \to \infty.$$

Suppose that there exists n_0 such that $|f_1(x_n)| \ge |f_2(y_n)|$ for all $n \ge n_0$, which implies that

$$\gamma_n = \frac{\sigma_n |f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n)| + |f_1(x_n)|}.$$

Thus

$$\lim_{n\to\infty} \frac{\sigma_n |f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n)| + |f_1(x_n)|} |f_3(Ax_n - By_n) + f_1(x_n) + f_2(y_n)| = 0.$$

On the other hand, we consider

$$|f_2(y_n)| - |f_3(Ax_n - By_n)| - |f_1(x_n)| \le |f_3(Ax_n - By_n) + f_1(x_n) + f_2(y_n)|.$$

So, we have

$$\lim_{n\to\infty} \sigma_n |f_3(Ax_n - By_n)| \left(\frac{f_2(y_n)}{|f_3(Ax_n - By_n)| + |f_1(x_n)|} - 1\right) = 0.$$

This together with the condition on σ_n and $(\frac{f_2(y_n)}{|f_3(Ax_n-By_n)|+|f_1(x_n)|}-1)>0$ implies that

$$\lim_{n\to\infty} f_3(Ax_n - By_n) = 0. \tag{17}$$

Since f_3^{-1} is continuous, we have

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{18}$$

Conversely, suppose there exists n_1 such that $|f_1(x_n)| \le |f_2(y_n)|$ for all $n \ge n_1$. Following the above process, again we come to the same conclusion. Also, from (13), (14) and the condition (i), we obtain

$$\lim_{n\to\infty} \rho_s^*(\|\nabla f_1 z_n - \nabla f_1 T_n z_n\|) = 0$$

$$\lim_{n\to\infty} \rho_r^*(\|\nabla f_2 u_n - \nabla f_2 S_n u_n\|) = 0.$$

Next, we show that $\lim_{n\to\infty} \|\nabla f_1 z_n - \nabla f_1 T_n z_n\| = 0$. If not, there exists $\varepsilon_0 > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|\nabla f_1 z_{n_i} - \nabla f_1 T_n z_{n_i}\| \ge \varepsilon_0$ for all $i \in \mathbb{N}$. Since ρ_s^* is nondecreasing, we have $0 \ge \rho_s^*(\varepsilon_0)$. But this statement contradicts the uniform convexity of f_1^* on bounded sets. According to Theorems 2.13 and 2.14, ∇f_1^* is uniformly continuous on bounded subsets of E_1^*, hence we have

$$\lim_{n\to\infty} \|z_n - T_n z_n\| = 0. \tag{19}$$

By a similar argument, we have

$$\lim_{n\to\infty} \|u_n - S_n u_n\| = 0.$$

Since $(\{T_n\}_{n=1}^\infty, T)$ and $(\{S_n\}_{n=1}^\infty, S)$ satisfy the AKTT-condition, we conclude that

$$\begin{aligned} \|z_n - Tz_n\| &\le \|z_n - T_n z_n\| + \|T_n z_n - Tz_n\| \\ &\le \|z_n - T_n z_n\| + \sup\{\|T_n x - Tx\| : x \in k_1\} \end{aligned} \tag{20}$$

and

$$\begin{aligned} \|u_n - Su_n\| &\le \|u_n - S_n u_n\| + \|S_n u_n - Su_n\| \\ &\le \|u_n - S_n u_n\| + \sup\{\|S_n x - Sx\| : x \in k_2\} \end{aligned} \tag{21}$$

where $k_1 = sB = \{z \in E_1 : \|z\| \le s\}$ and $k_2 = rB = \{z \in E_2 : \|z\| \le s\}$. By using Lemma 2.10, (20) and (21), we get

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0 \tag{22}$$

$$\lim_{n\to\infty} \|u_n - Su_n\| = 0. \tag{23}$$

From (20), the boundedness of ∇f_1 and the uniform continuity of f_1 on bounded subsets of E_1, we have

$$D_{f_1}(T_n z_n, z_n) = f_1(T_n z_n) - f_1(z_n) - \langle T_n z_n - z_n, \nabla f_1 z_n\rangle \to 0, \quad n\to\infty. \tag{24}$$

This implies that

$$\begin{aligned} D_{f_1}(T_n z_n, x_{n+1}) &= D_{f_1}(T_n z_n, \nabla f_1^*(\beta_n \nabla f_1 z_n + (1-\beta_n)\nabla f_1 T_n z_n)) \\ &\le \beta_n D_{f_1}(T_n z_n, z_n) + (1-\beta_n) D_{f_1}(T_n z_n, T_n z_n) \to 0, \end{aligned} \tag{25}$$

as $n \to \infty$. From Proposition 2.3, (25) and (19), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0. \tag{26}$$

By the same argument as above, we have

$$\lim_{n\to\infty}\|y_{n+1}-u_n\|=0. \tag{27}$$

On the other hand, by the boundedness of ∇f_1 and ∇f_2 on bounded subsets of E_1 and E_2, respectively, we have

$$D_{f_1}(x_{n+1},x_n)=\langle x_n-\hat{x},\nabla f_1\hat{x}-\nabla f_1 x_{n+1}\rangle+D_{f_1}(x_n,\hat{x})-D_{f_1}(x_{n+1},\hat{x})\to 0$$
$$D_{f_2}(y_{n+1},y_n)=\langle y_n-\hat{y},\nabla f_2\hat{y}-\nabla f_2 y_{n+1}\rangle+D_{f_2}(y_n,\hat{y})-D_{f_2}(y_{n+1},\hat{y})\to 0,$$

as $n\to\infty$. From Proposition 2.3, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0,\qquad \lim_{n\to\infty}\|y_{n+1}-y_n\|=0.$$

So from (26) and (27), we obtain

$$\lim_{n\to\infty}\|z_n-x_n\|=0,\qquad \lim_{n\to\infty}\|u_n-y_n\|=0. \tag{28}$$

Since the sequence $\{(x_n,y_n)\}$ is bounded, there exists a subsequence $\{(x_{n_k},y_{n_k})\}$ of $\{(x_n,y_n)\}$ such that $x_{n_k}\rightharpoonup\bar{x}$ and $y_{n_k}\rightharpoonup\bar{x}$. Thus $z_{n_k}\rightharpoonup\bar{x}$ and $u_{n_k}\rightharpoonup\bar{x}$ and so by (22) and (23), $\bar{x}\in F(T)=\hat{F}(T)$ and $\bar{y}\in F(S)=\hat{F}(S)$. Now, we show that $\bar{x}\in\cap_{i=1}^N h_i^{-1}(0)$. Writing $\theta_n^0=I$ and $\theta_n^i=Res^{f_1}_{\lambda_n^i h_i}o\cdots oRes^{f_1}_{\lambda_n^1 h_1}$, we observe that

$$\begin{aligned}
D_{f_1}(\hat{x},z_n)&=D_{f_1}(\hat{x},\theta_n^N(\nabla f_1^*((1-\gamma_n)\nabla f_1x_n-\gamma_nA^*\nabla f_3(Ax_n-By_n)))\\
&=D_{f_1}(\hat{x},Res^{f_1}_{\lambda_n^N h_N}o\cdots oRes^{f_1}_{\lambda_n^1 h_1}(\nabla f_1^*((1-\gamma_n)\nabla f_1x_n-\gamma_nA^*\nabla f_3(Ax_n-By_n)))\\
&\le D_{f_1}(\hat{x},\nabla f_1^*((1-\gamma_n)\nabla f_1x_n-\gamma_nA^*\nabla f_3(Ax_n-By_n))\\
&\le D_f(\hat{x},x_n)-\gamma_n[f_3(Ax_n-By_n)+f_1(x_n)-f_3(-By_n)]\\
&\le D_{f_1}(\hat{x},x_n)\le D_{f_1}(\hat{x},x_{n-1}).
\end{aligned}$$

Since $\hat{x}\in h_N^{-1}(0)=F(Res^{f_1}_{\lambda_n^N h_N})$ and $Res^{f_1}_{\lambda_n^N h_n}$ is a BQFNE operator, it follows that for all $n\ge 1$, we have

$$\begin{aligned}
D_{f_1}(z_n,\theta_n^{N-1}\nabla f_1^*w_n)&\le D_{f_1}(\hat{x},\theta_n^{N-1}\nabla f_1^*w_n)-D_{f_1}(\hat{x},z_n)\\
&\le D_{f_1}(\hat{x},x_n)-D_{f_1}(\hat{x},x_{n+1})\to 0,\quad n\to\infty.
\end{aligned}$$

Therefore by Proposition 2.3, the uniform continuity of ∇f_1 on bounded subsets, and the boundedness of $\{\theta_n^{N-1}x_n\}$, we get

$$\lim_{n\to\infty}\|z_n-\theta_n^{N-1}\nabla f_1 w_n\|=0,\qquad \lim_{n\to\infty}\|\nabla f_1 z_n-\nabla f_1\theta_n^{N-1}\|=0. \tag{29}$$

Again since $\hat{x}\in h_{N-1}^{-1}(0)=F(Res^{f_1}_{\lambda_n^{N-1}h_{N-1}})$ and $Res^{f_1}_{\lambda_n^{N-1}h_{N-1}}$ is a BQFN operator for each $n\ge 1$, we have

$$\begin{aligned}
D_{f_1}(\theta_n^{N-1}\nabla f_1^*w_n,\theta_n^{N-2}\nabla f_1^*w_n)&\le D_{f_1}(\hat{x},\theta_n^{N-2}\nabla f_1^*w_n)-D_{f_1}(\hat{x},\theta_n^{N-1}\nabla f_1^*w_n)\\
&\le D_{f_1}(\hat{x},x_n)-D_{f_1}(\hat{x},\theta_n^{N-1}\nabla f_1^*w_n)\\
&\le D_{f_1}(\hat{x},x_n)-D_{f_1}(\hat{x},z_n)\\
&\le D_{f_1}(\hat{x},x_n)-D_{f_1}(\hat{x},x_{n+1}),\quad n\to\infty.
\end{aligned}$$

Hence $\lim_{n\to\infty}\|\theta_n^{N-1}\nabla f_1^*w_n-\theta_n^{N-2}\nabla f_1^*w_n\|=0$ and $\lim_{n\to\infty}\|\nabla f_1\theta_n^{N-1}\nabla f_1^*w_n-\nabla f_1\theta_n^{N-2}\nabla f_1^*w_n\|=0$. Similarly, we can verify that

$$\lim_{n\to\infty}\|\theta_n^{N-2}\nabla f_1^*w_n-\theta_n^{N-3}\nabla f_1^*w_n\|=\cdots=\lim_{n\to\infty}\|\theta_n^1\nabla f_1^*w_n-\nabla f_1^*w_n\|=0,$$

$$\lim_{n\to\infty} \|\nabla f_1 \theta_n^{N-2} \nabla f_1^* w_n - \nabla f_1 \theta_n^{N-3} \nabla f_1^* w_n\| = \cdots = \lim_{n\to\infty} \|\nabla f_1 \theta_n^1 \nabla f_1^* w_n - w_n\| = 0.$$

Therefore for any $i = 1, 2, ..., N$, we have

$$\lim_{n\to\infty} \|\theta_n^i \nabla f_1^* w_n - \theta_n^{i-1} \nabla f_1^* w_n\| = 0 \quad and \quad \lim_{n\to\infty} \|\nabla f_1 \theta_n^i \nabla f_1^* w_n - \nabla f_1 \theta_n^{i-1} \nabla f_1^* w_n\| = 0. \tag{30}$$

From (29) and (30), for $i = 1, 2, ..., N$ we have

$$\lim_{n\to\infty} \|z_n - \nabla f_1^* w_n\| = 0 \quad and \quad \lim_{n\to\infty} \|\theta_n^i \nabla f_1^* w_n - \nabla f_1^* w_n\| = 0. \tag{31}$$

From the definition of the f_1-resolvent, we have

$$\nabla f_1(\theta_n^{i-1} \nabla f_1^* w_n) \in (\nabla f_1 + \lambda_n^i h_i)(\theta_n^i \nabla f_1^* w_n).$$

Hence for any $i = 1, 2, ..., N$

$$\vartheta_n^i = \frac{1}{\lambda_n^i}(\nabla f_1(\theta_n^{i-1} \nabla f_1^* w_n) - \nabla f_1(\theta_n^i \nabla f_1^* w_n)) \in h_i(\theta_n^i \nabla f_1^* w_n). \tag{32}$$

It follows form (30), (32) and the condition (ii) that $\lim_{n\to\infty} \|\vartheta_n^i\| = 0$ for any $i = 1, 2, ..., N$. Since $x_{n_k} \rightharpoonup \bar{x}$, it follows from (29) that $z_{n_k} \rightharpoonup \bar{x}$. Also from (31), we obtain that $\theta_{n_k}^i(\nabla f_1^* w_n) \rightharpoonup \bar{x}$, for each $i = 1, 2, ..., N$. Note that from the monotonicity of h_i, we have

$$\langle \eta - \vartheta_n^i, z - \theta_{n_k}^i(\nabla f_1^* w_{n_k})\rangle \geq 0,$$

for all $(z, \eta) \in G(A_i)$ and for all $i = 1, 2, ..., N$. This implies that $\langle \eta, z - \bar{x}\rangle \geq 0$ for all $(z, \eta) \in G(h_i)$ and for any $i = 1, 2, ..., N$. Therefore by using the maximal monotonicity of A_i, we obtain $\bar{x} \in h_i^{-1}(0)$ for any $i = 1, 2, ..., N$. Thus $\bar{x} \in \cap_{i=1}^N h_i^{-1}(0)$. The same argument as above, reveals that $\hat{x} \in \cap_{i=1}^N g_i^{-1}(0)$. Furthermore, $Ax_n - By_n \rightharpoonup A\bar{x} - B\bar{y}$ and by using the lower semicontinuity of f_3, we have

$$f_3(A\bar{x} - B\bar{y}) \leq \liminf_{n\to\infty} f_3(Ax_n - By_n) = 0. \tag{33}$$

From (33) and the fact that f_3 is a one-to-one function, we have $A\bar{x} = B\bar{y}$. Hence $(\bar{x}, \bar{y}) \in \Omega$. Now we show that

$$\limsup_{n\to\infty}[\langle \hat{x} - z_n, A^*\nabla f_3(Ax_n - By_n)\rangle + \langle u_n - \hat{y}, B^*\nabla f_3(Ax_n - By_n)\rangle] \leq 0.$$

From (18) and the fact that ∇f_3 is uniformly continuous on bounded subset of E_3, we have

$$\begin{aligned} &\limsup_{n\to\infty}[\langle \hat{x} - z_n, A^*\nabla f_3(Ax_n - By_n)\rangle + \langle u_n - \hat{y}, B^*\nabla f_3(Ax_n - By_n)\rangle] \\ &= \lim_{m\to\infty}[\langle \hat{x} - z_{n_m}, A^*\nabla f_3(Ax_{n_m} - By_{n_m})\rangle + \langle u_{n_m} - \hat{y}, B^*\nabla f_3(Ax_{n_m} - By_{n_m})\rangle]. \end{aligned} \tag{34}$$

Since $\{(x_{n_m}, y_{n_m})\}$ is bounded, there exists a subsequence $\{(x_{n_{m_i}}, y_{n_{m_i}})\}$ of $\{(x_{n_m}, y_{n_m})\}$ such that $(x_{n_{m_i}}, y_{n_{m_i}}) \rightharpoonup (\bar{x}, \bar{y})$ and from (28), we have $(z_{n_{m_i}}, u_{n_{m_i}}) \rightharpoonup (\bar{x}, \bar{y})$ where $(\bar{x}, \bar{y}) \in \Omega$. It now follows that

$$\begin{aligned} &\lim_{m\to\infty}[\langle \hat{x} - z_{n_m}, A^*\nabla f_3(Ax_{n_m} - By_{n_m})\rangle + \langle u_{n_m} - \hat{y}, B^*\nabla f_3(Ax_{n_m} - By_{n_m})\rangle] \\ &= \lim_{i\to\infty}[\langle \hat{x} - z_{n_{m_i}}, A^*\nabla f_3(Ax_{n_{m_i}} - By_{n_{m_i}})\rangle + \langle u_{n_{m_i}} - \hat{y}, B^*\nabla f_3(Ax_{n_{m_i}} - By_{n_{m_i}})\rangle] = 0. \end{aligned} \tag{35}$$

Thus from (16), (35), $\sum_{n=1}^{\infty} \gamma_n = \infty$ and Lemma 2.15, we have $x_n \to \bar{x}$ and $y_n \to \bar{y}$.

Case2. Suppose $\{D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n)\}$ is not a monotone decreasing sequences. Then set $\Gamma_n = D_{f_1}(\hat{x}, x_n) + D_{f_2}(\hat{y}, y_n)$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for $n \geq N_0$, for some sufficiently large N_0, by

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then τ is a non-decreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n\tau)+1}$ for $n \geq N_0$. Using the condition $\beta \in (0,1)$ in (15), we obtain

$$\gamma_{\tau(n)}(f_3(Ax_{\tau(n)} - By_{\tau(n)}) + f_1(x_{\tau(n)}) + f_2(y_{\tau(n)})) \to 0, \quad n \to \infty.$$

Also, from (13), (14) and the condition (i), we obtain

$$\lim_{n\to\infty} \rho_s^*(\|\nabla f_1 z_{\tau(n)} - \nabla f_1 T_{\tau(n)} z_{\tau(n)}\|) = 0,$$

$$\lim_{n\to\infty} \rho_r^*(\|\nabla f_2 u_{\tau(n)} - \nabla f_2 S_{\tau(n)} u_{\tau(n)}\|) = 0.$$

Following the same argument as in Case 1, we have

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - z_{\tau(n)}\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|y_{\tau(n)+1} - u_{\tau(n)}\| = 0,$$

$$\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|y_{\tau(n)+1} - y_{\tau(n)}\| = 0.$$

As in the Case 1, we also obtain that $x_{\tau(n)} \rightharpoonup \bar{x}$ and $y_{\tau(n)} \to \bar{y}$ as $n \to \infty$, where $(\bar{x}, \bar{y}) \in \Omega$. Furthermore, for all $n \geq N_0$, we deduce from (16) that

$$\begin{aligned} D_{f_1}(\hat{x}, x_{\tau(n)+1}) + D_{f_2}(\hat{y}, y_{\tau(n)+1}) \leq\ & \gamma_{\tau(n)}[D_{f_1}(\hat{x}, x_{\tau(n)}) + D_{f_2}(\hat{y}, y_{\tau(n)})] \\ & + \gamma_{\tau(n)}[\langle y - z_{\tau(n)}, A^*\nabla f_3(Ax_{\tau(n)} - By_{\tau(n)})\rangle \\ & + \langle u_{\tau(n)} - y, B^*\nabla f_3(Ax_{\tau(n)} - By_{\tau(n)})\rangle]. \end{aligned} \tag{36}$$

It now follows from (36) that

$$\begin{aligned} D_{f_1}(\hat{x}, x_{\tau(n)}) + D_{f_2}(\hat{y}, y_{\tau(n)}) \leq\ & \langle y - z_{\tau(n)}, A^*\nabla f_3(Ax_{\tau(n)} - By_{\tau(n)})\rangle \\ & + \langle u_{\tau(n)} - y, B^*\nabla f_3(Ax_{\tau(n)} - By_{\tau(n)})\rangle \to 0, \quad n \to \infty. \end{aligned}$$

Thus

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1}.$$

Furthermore, for $n \geq N_0$, we have $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (i.e., $\tau(n) < n$), since $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. It then follows that for all $n \geq N_0$ we have

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

This implies that $\lim_{n\to\infty} \Gamma_{\tau(n)} = 0$, and hence $x_n \to \bar{x}$ and $y_n \to \bar{y}$ as $n \to \infty$, where $(\bar{x}, \bar{y}) \in \Omega$. □

In some special cases, our result reduces to the result already obtained by others.

Remark 3.2. *When for $n \in \mathbb{N}$, $T_n = S_n = 0$, Theorem 3.1 improves and extends the results of Sitthithakerngkiet et al [21] and Byrne et al [9].*

Remark 3.3. *When $h_i = \partial\delta_{C_i}$ and $g_i = \partial\delta_{Q_i}$ are the subdifferential of the indicator function of C_i and Q_i, respectively, and $T_n = S_n = 0$, Theorem 3.1 improves and extends the result of Dong et al [11].*

4. Application

In this section, we shall provide some applications of our main result to the split equality equilibrium problem, and to the split equality optimization problem.

4.1. Split equality equilibrium problem

Let C be a nonempty closed convex subset of a Banach space E and let $G : C \times C \to \mathbb{R}$ be a bifunction. For solving the equilibrium problem, let us assume that the bifunction G satisfies the following conditions:
(A_1) $G(x, x) = 0$ for all $x \in C$,
(A_2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0$ for any $x, y \in C$,
(A_3) G is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} G(tz + (1-t)x, y) \leq G(x, y),$$

(A_4) $G(x, 0)$ is convex and lower semicontinuous for each $x \in C$.

The equilibrium problem is to find $x^* \in C$ such that:

$$G(x^*, y) \geq 0 \quad \forall y \in C.$$

The set of solutions to this problem is denoted by $EP(G)$.

Lemma 4.1. *[17] Let $f : E \to (-\infty, +\infty]$ be a super coercive Legendre function and G be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A_1) - (A_4)$ and let $x \in E$. Define a mapping $S_G^f : E \longrightarrow C$ as follows:*

$$S_G^f(x) = \{z \in C : G(z, y) + \langle y - z, \nabla f z - \nabla f x\rangle \geq 0, \quad \forall y \in C\}.$$

Then

(i) *$dom S_G^f = E$,*

(ii) *S_G^f is single valued,*

(iii) *S_G^f is a BFNE operator,*

(iv) *the set of fixed points of S_G^f is the solution set of the corresponding equilibrium problem, i.e., $F(S_G^f) = EP(G)$,*

(v) *$EP(G)$ is closed and convex,*

(vi) *for all $x \in E$ and for all $u \in F(S_G^f)$, we have*

$$D_f(u, S_G^f(x)) + D_f(S_G^f(x), x) \leq D_f(u, x).$$

Proposition 4.2. *[20] Let $f : E \to (-\infty, +\infty]$ be a super coercive, Legendre, Fréchet differentiable and totally convex function. Let C be a closed and convex subset of E and assume that the bifunction $G : C \times C \to \mathbb{R}$ satisfies the conditions $(A_1) - (A_4)$. Let A_G be a set-valued mapping of E into 2^{E^*} defined by:*

$$A_G(x) = \begin{cases} \{z \in E^* : G(x, y) \geq \langle y - x, z\rangle \quad \forall y \in C\} & x \in C, \\ \emptyset \quad x \in E - C. \end{cases}$$

Then A_G is a maximal monotone operator, $EP(G) = A_G^{-1}(0)$ and $S_G^f = Res_{A_G}^f$.

Theorem 4.3. *Let E_1, E_2 and E_3 be reflexive Banach spaces, let $C \subseteq E_1$ and $Q \subseteq E_2$ be two nonempty closed convex sets, let $A : E_1 \longrightarrow E_3$ and $B : E_2 \longrightarrow E_3$ be two bounded linear operators and let $f_1 : E_1 \longrightarrow \mathbb{R}$ and $f_2 : E_2 \longrightarrow \mathbb{R}$ be super coercive Legendre functions which are bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of E_1 and E_2, respectively, and $f_3 : E_3 \longrightarrow \mathbb{R}$ be a convex, one-to-one and continuous function on E_3 with f_3^{-1} continuous, let for $i = 1, 2, ..., N, H_i : C \times C \longrightarrow \mathbb{R}$ and $G_i : Q \times Q \longrightarrow \mathbb{R}$ be bifunctions satisfying $(A_1) - (A_4)$. Let*

$$\Omega = \Big\{(x, y) : \quad x \in \cap_{i=1}^{N} EP(h_i),\ y \in \cap_{i=1}^{N} EP(g_i)\ \text{such that}\ Ax = By\Big\} \neq \emptyset,$$

Let $\{x_n\}$ be the sequence generated by

$$\begin{cases} z_n = S^{f_1}_{H_N} o \cdots o S^{f_1}_{H_1} \nabla f_1^*((1-\gamma_n)\nabla f_1 x_n - \gamma_n A^* \nabla f_3(Ax_n - By_n)), \\ u_n = S^{f_2}_{G_N} o \cdots o S^{f_2}_{G_1} \nabla f_2^*((1-\gamma_n)\nabla f_2 x_n + \gamma_n B^* \nabla f_3(Ax_n - By_n)), \end{cases} \tag{37}$$

where the step-size γ_n is chosen as follows:

$$\gamma_n = \sigma_n \min\left\{ \frac{|f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n)| + |f_1(x_n)|}, \frac{|f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n)| + |f_2(y_n)|} \right\},$$

where $\sigma_n \in (0, 1)$ is defined such that $\sum_{n=1}^{\infty} \gamma_n = \infty$. Then the sequence $\{(x_n, y_n)\}$ converges strongly to $(\bar{x}, \bar{y}) \in \Omega$.

Proof. For $1 \le i \le N$ and the bifunctions $H_i : C \times C \to \mathbb{R}$ and $G_i : Q \times Q \to \mathbb{R}$, we can define A_{H_i} and A_{G_i} as in Proposition 4.2. Putting $h_i = A_{H_i}$, $g_i = A_{G_i}$ and for $n \in \mathbb{N}$, $T_n = S_n = 0$ and $\beta_n = 0$ in Theorem 3.1, we obtain the desired result. □

4.2. Split equality optimization problem

Let E_1, E_2 and E_3 be Banach spaces, $D \subset E_1$ and $U \subset H_2$ be two nonempty closed convex subsets. Let $\{\tilde{h}_i\} : D \to \mathbb{R}$ and $\{\tilde{g}_i\} : U \to \mathbb{R}$ be two families of proper convex and lower semi-continuous functions. The so-called general split equality optimization problem with respect to $\{\tilde{h}_i\}, \{\tilde{g}_i\}$, D and U is to find $x^* \in D, y^* \in U$ such that

$$\tilde{h}_i(x^*) = \min_{x \in D} \tilde{h}_i(x), \quad \tilde{g}_i(y^*) = \min_{y \in U} \tilde{g}_i(y) \quad \text{and} \quad Ax^* = By^*, \quad \text{for each} \quad i \ge 1, \tag{38}$$

where $A : E_1 \to E_3$, $B : E_2 \to E_3$ are two bounded linear operators. We denote the solution set of the problem (38) by Γ

Theorem 4.4. *Let $E_1, E_2, E_3, C, Q, A, B, f_1, f_2$ and f_3 be the same as in Theorem 4.3. Let for $i = 1, 2, ..., N, \tilde{h}_i : C \longrightarrow \mathbb{R}$ and $\tilde{g}_i : Q \longrightarrow \mathbb{R}$ be two families of proper convex and lower semi-continuous functions. Let $\Gamma \neq \emptyset$ and the step-size γ_n is chosen as follows:*

$$\gamma_n = \sigma_n \min\left\{ \frac{|f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n)| + |f_1(x_n)|}, \frac{|f_3(Ax_n - By_n)|}{|f_3(Ax_n - By_n)| + |f_2(y_n)|} \right\},$$

where $\sigma_n \in (0, 1)$ is defined such that $\sum_{n=1}^{\infty} \gamma_n = \infty$. Then the sequence $\{(x_n, y_n)\}$ generated in Theorem 4.3 converges strongly to $(\bar{x}, \bar{y}) \in \Gamma$.

Proof. Put $H_i(x, y) = \tilde{h}_i(y) - \tilde{h}_i(x)$ and $G_i(x, y) = \tilde{g}_i(y) - \tilde{g}_i(x), i \ge 1$. It is easy to see that $\{H_i\} : C \times C \to \mathbb{R}$ and $\{G_i\} : Q \times Q \to \mathbb{R}$ are two families of equilibrium functions satisfying the conditions $(A_1) - (A_4)$. Thus the desired result follows from Theorem 4.3. □

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