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Tauberian Theorems for Cesàro Summability of n^{th} Sequences

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Abstract. Tauberian theorem provides a criterion for the convergence of non convergent (summable) sequences. In this paper, we established a Tauberian theorem for n^{th} real sequences via Cesàro summability by using de la Vallée Poussin mean and slow oscillation. The discussion and findings are capable to unify several useful concepts in the literature, and should also provide nontrivial extension of several results. Some examples are incorporated in support of our definitions and results. The findings are further expected to be helpful in designing and study several other interesting problems in summability theory and applications.

1. Introduction and Definitions

Tauberian theorems for single sequences, that an Abel summable sequence is convergent under certain suitable conditions was introduced by Tauber [12]. A few researchers like Landau [7], Hardy and Littlewood [4], and Schmidt [10] obtained some classical Tauberian theorems for Cesàro and Abel summability methods of single sequences. Later on, Knopp [6] introduced some classical type of Tauberian theorems for Abel and $(C, 1, 1)$ summability methods of double sequences and obtained that Abel and $(C, 1, 1)$ summability methods are equivalent for the set of bounded sequences. Móricz [8], and Jena *et al.* [5] proved some Tauberian theorems for Cesàro $(C, 1, 1)$ summable double sequences. Very recently, Çanak and Totur [2] has extended some classical type of Tauberian theorems from double sequences to triple sequences and thereby established Tauberian theorems via $(C, 1, 1, 1)$ mean. In this paper, we aim at establishing classical Tauberian theorems via $(C, 1, 1, ..., 1)$ mean for n^{th} real sequences and that will generalize earlier existing results and unify several ideas. Aasma *et al.* [1] may be consulted for basic notions and ideas and Dutta and Rhoades [3] for some topics of current interest in summability theory and its applications.

Let $(u_{m_1,m_2,...,m_n})$ be a n^{th} real sequence. We have,

$$\Delta_{m_1}(u_{m_1,m_2,...,m_n}) = u_{m_1,m_2,...,m_n} - u_{m_1-1,m_2,...,m_n};$$

$$\Delta_{m_2}(u_{m_1,m_2,...,m_n}) = u_{m_1,m_2,...,m_n} - u_{m_1,m_2-1,...,m_n};$$

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... ...

$$\Delta_{m_n}(u_{m_1,m_2,...,m_n}) = u_{m_1,m_2,...,m_n} - u_{m_1,m_2,...,m_{n-1}}.$$

$$\Delta_{m_1,m_2}(u_{m_1,m_2,...,m_n}) = \Delta_{m_1}(\Delta_{m_2}u_{m_1,m_2,...,m_n}) = \Delta_{m_2}(\Delta_{m_1}u_{m_1,m_2,...,m_n});$$

... ...

$$\Delta_{m_{n-1},m_n}(u_{m_1,m_2,...,m_n}) = \Delta_{m_n}(\Delta_{m_{n-1}}u_{m_1,m_2,...,m_n}) = \Delta_{m_{n-1}}(\Delta_{m_n}u_{m_1,m_2,...,m_n}).$$

Similarly,

$$\Delta_{m_1,m_2,...,m_n}(u_{m_1,m_2,...,m_n}) = \Delta_{m_1}(\Delta_{m_2,...,m_n}(u_{m_1,m_2,...,m_n}))$$

A given sequence $(u_{m_1,m_2,...,m_n})$ is said to be convergent (in Pringsheims sense) to L (see [9]), if for each given $\varepsilon > 0$, there exists a positive integer N_0 such that $|(u_{m_1,m_2,...,m_n}) - L| < \epsilon$, for all nonnegative integers $m_1, m_2, ..., m_n > N_0$.

In this case, we write

$$\lim_{m_1,m_2,...,m_n\to\infty}(u_{m_1,m_2,...,m_n}) = L.$$

Note that, a n^{th} real sequence $(u_{m_1,m_2,...,m_n})$ is said to be bounded, if there exists a constant, $K > 0$ such that $|(u_{m_1,m_2,...,m_n})| < K$, for all nonnegative integers $m_1, m_2, ..., m_n$.

The $(C, 1, 1, ..., 1)$ mean of n^{th} sequence, denoted by $(\sigma_{m_1m_2...m_n}(u))$ is defined as

$$\sigma_{m_1m_2...m_n}(u) = \frac{1}{(m_1+1)(m_2+1)...(m_n+1)}\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}...\sum_{i_n=0}^{m_n}u_{i_1i_2...i_n}.$$

The sequence $(u_{m_1,m_2,...,m_n})$ is $(C, 1, 1, ..., 1)$ summable to L, if

$$\lim_{m_1,m_2,...,m_n\to\infty}\sigma_{m_1m_2...m_n}(u) = L. \tag{1}$$

Clearly, a bounded sequence $(u_{m_1,m_2,...,m_n})$ is P-convergent to L, if

$$\lim_{m_1,m_2,...,m_n\to\infty}u_{m_1m_2...m_n} = L. \tag{2}$$

Also, existence of (2) implies the existence of (1) but not conversely. To prove the converse part, we use some conditions such as slow oscillation and de la Vallée Poussin mean of n^{th} sequence. Such conditions are called Tauberian conditions and theorems with Tauberian conditions are called Tauberian theorems.

It is known that from [5], a double sequence can written as,

$$u_{m,n} - \sigma_{m,n} = v_{m,n}\Delta(u),$$

where

$$v_{m,n}\Delta(u) = \frac{1}{(m+1)(n+1)}\sum_{i_1=0}^{m}\sum_{i_2=0}^{n}i_1i_2\Delta_{i_1i_2}(u_{i_1i_2}).$$

Similarly, here for n^{th} sequence, we may write

$$u_{m_1,\dots,m_n} - \sigma_{m_1,\dots,m_n} = v_{m_1,\dots,m_n}\Delta(u),$$

where

$$v_{m_1,\dots,m_n}\Delta(u) = \frac{1}{(m_1+1)\dots(m_n+1)}\sum_{i_1=0}^{m_1}\dots\sum_{i_n=0}^{m_n} i_1\dots i_n\Delta_{i_1\dots i_n}(u_{i_1\dots i_n}).$$

Now, we present below an illustrative example to show that a sequence is $(C,1,1,\dots,1)$ summable but not P-convergent

Example 1.1. Let us consider a bounded sequence

$$u_{m_1m_2\dots m_n} = \begin{cases} 1 & (m_1 = \text{even}) \\ 0 & (\text{otherwise}). \end{cases}$$

Now

$$\sigma_{m_1m_2\dots m_n}(u) = \frac{1}{(m_1+1)(m_2+1)\dots(m_n+1)}\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\dots\sum_{i_n=0}^{m_n} u_{i_1i_2\dots i_n}$$

$$= \begin{cases} \frac{(m_1-1)m_2\dots m_n}{2\{(m_1+1)(m_2+1)\dots(m_n+1)\}}, & if\ m_1 = \text{odd}, \\ \frac{m_1m_2\dots m_n}{2\{(m_1+1)(m_2+1)\dots(m_n+1)\}}, & if\ m_1 = \text{even}. \end{cases}$$

Clearly, the sequence is $(C,1,1,\dots,1)$ summable to $\frac{1}{2}$. But it is not P-convergent.

Next, it will be interesting to see that unlike single sequences, every P- convergent n^{th} sequence need not be bounded and further every P- convergent n^{th} sequence does not have to be $(C,1,1,\dots,1)$ summable. It can be illustrated in the following example.

Example 1.2. Let us consider a sequence

$$u_{m_1m_2\dots.m_n} = \begin{cases} m_1, & (m_2 = \dots = m_n = 0,\ m_1 = 0,1,2,\dots,) \\ m_2, & (m_1 = \dots = m_n = 0,\ m_2 = 0,1,2,\dots,) \\ 0, & (\text{otherwise}). \end{cases}$$

Obviously, the sequence is unbounded but P-convergent to 0.

Furthermore,

$$\sigma_{m_1m_2\dots m_n}(u) = \lim_{m_1,m_2,\dots,m_n\to\infty}\frac{1}{(m_1+1)(m_2+1)\dots(m_n+1)}\sum_{i_1=0}^{m_1}\sum_{i_2=0}^{m_2}\dots\sum_{i_n=0}^{m_n} u_{i_1i_2\dots i_n}$$

$$= \lim_{m_1,m_2,\dots,m_n\to\infty}\frac{(m_1^2+m_1)(m_2^2+m_2)\dots(m_n^2+m_n)}{(n+1)\{(m_1+1)(m_2+1),\dots,(m_n+1)\}}$$

does not tend to a finite limit. Therefore, it is not $(C, 1, 1, ..., 1)$ summable.

Now, the definition of slow oscillation for n^{th} sequences is introduced in the sense of Stanojević [11] as follows:

A n^{th} sequence $(u_{m_1,...,m_n})$ is said to be slowly oscillating in sense $(0, 0, 0, ...1, ..., 0)$ with 1 is in the k^{th} place if,

$$\lim_{\lambda\to 1^+} \lim_{m_1 m_2 ... m_n\to\infty} \sup \max_{m_k+1\le i\le \lambda_{m_k}} \left| \sum_{r=m_k+1}^{i} \Delta_r u_{r m_2 m_3 ... m_n} \right| = 0. \tag{3}$$

Next, the de la Vallée Poussin mean $\tau^{>}_{m_1 m_2 ... m_n}(u)$ of the n^{th} sequence $(u_{m_1,...,m_n})$ is defined for $\lambda > 1$ as,

$$\frac{1}{(\lambda_{m_1} - m_1)(\lambda_{m_2} - m_2)...(\lambda_{m_n} - m_n)} \sum_{i_1=m_1+1}^{\lambda_{m_1}} \sum_{i_2=m_2+1}^{\lambda_{m_2}} ... \sum_{i_n=m_n+1}^{\lambda_{m_n}} (u_{i_1,i_2,...,i_n}) \tag{4}$$

and for $0 < \lambda < 1$ as,

$$\frac{1}{(m_1 - \lambda_{m_1})(m_2 - \lambda_{m_2})...(m_n - \lambda_{m_n})} \sum_{i_1=\lambda_{m_1}+1}^{m_1} \sum_{i_2=\lambda_{m_2}+1}^{m_2} ... \sum_{i_n=\lambda_{m_n}+1}^{m_n} (u_{i_1,i_2,...,i_n}). \tag{5}$$

Moreover, now we define the Cesàro mean for each sequence of non-negative integers $(k_1, k_2, ..., k_n)$,

$$\sigma^{k_1,k_2,...,k_n} = \begin{cases} \frac{1}{(m_1+1)...(m_n+1)} \sum_{i_1=0}^{m_1} ... \sum_{i_n=0}^{m_n} \sigma^{k_1-1,...,k_n-1}, & \text{for } k_1, ..., k_n \ge 1 \\ u_{m_1 m_2 ... m_n}, & \text{for } k_1, ..., k_n = 0. \end{cases}$$

A sequence $(u_{m_1,...,m_n})$ is said to be $(C, k_1, ..., k_n)$ summable to L, if $\lim_{m_1,m_2,...,m_n\to\infty} \sigma^{k_1,...,k_n}(u) = L$.

Remark 1.1 If $k_1 = ... = k_n = 1$ then $(C, k_1, ..., k_n)$ summability reduces to $(C, 1, ..., 1)$ summability.

The following is a list of some known theorems.

Theorem 1.1 *(see [13]) If $(u_{m,n})$ is Cesàro $(C, 1, 1)$ summable to s and $(u_{m,n})$ is slowly oscillating, then* $\lim_{n\to\infty}(u_{m,n}) = s$.

Theorem 1.2 *(see [5]) If $(u_{m,n})$ is Cesàro (C, k, r) summable to s and $(u_{m,n})$ is slowly oscillating, then* $\lim_{n\to\infty}(u_{m,n}) = s$.

Theorem 1.3 *(see [2]) If $(u_{m,n,s})$ is Cesàro $(C, 1, 1, 1)$ summable to L and $(u_{m,n,s})$ is slowly oscillating, then* $\lim_{n\to\infty}(u_{m,n,s}) = L$.

2. Main Theorems

The objective of the present paper is to prove the generalized Littlewood-Tauberian theorem [2] for $(C, 1, ..., 1)$ summability of a n^{th} sequence by using slow oscillations and de la Vallée mean.

Theorem 2.1 *Let $(u_{m_1 m_2 ... m_n})$ be $(C, 1, 1, ..., 1)$ summable to L. If $(u_{m_1 m_2 ... m_n})$ is slowly oscillating in sense $(1, 0, 0, 0, ..., 0)$, $(0, 1, 0, 0, ..., 0)$,..., $(0, 0, 0, 0, ..., 1)$; then $(u_{m_1 m_2 ... m_n})$ is P-convergent to L.*

The following lemmas give two representations of the difference $u_{m_1m_2...m_n} - \sigma_{m_1m_2...m_n}(u)$ and are required for the proof of our main Theorem 2.1.

Lemma 2.1 *(see [2], p. 4, Lemma 4.1) Let $\lambda_{m_1}, \lambda_{m_2}, ..., \lambda_{m_n}$ denote the integral part of $\lambda m_1, \lambda m_2, ..., \lambda m_n$ respectively.*

(i) *If $\lambda > 1$, then*

$$u_{m_1m_2...m_n} - \sigma_{m_1m_2...m_n}(u)$$

$$= \frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} \times$$

$$[\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_n}}(u) - \sigma_{m_1\lambda_{m_2}...\lambda_{m_n}}(u) - ... - \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u)$$

$$+\sigma_{\lambda_{m_1}m_2m_3...\lambda_{m_n}}(u) + \sigma_{m_1\lambda_{m_2}m_3...\lambda_{m_n}}(u) + ... + \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-2}}m_{n-1}m_n}(u)$$

$$... - \sigma_{m_1m_2...m_n}(u)]$$

$$-\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_{n-1}}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_{n-1}}-m_{n-1})} \times [-\sigma_{\lambda_{m_1}\lambda_{m_2}\lambda_{m_3}...\lambda_{m_{n-1}}m_n}(u)$$

$$+\sigma_{m_1\lambda_{m_2}\lambda_{m_3}...\lambda_{m_{n-1}}m_n}(u) + ... + \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-2}}\lambda_{m_{n-1}}m_n}(u)... - \sigma_{m_1m_2...m_n}(u)]$$

$$-... - \frac{(\lambda_{m_2}+1)(\lambda_{m_3}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_2}-m_2)(\lambda_{m_3}-m_3)...(\lambda_{m_n}-m_n)} \times [-\sigma_{m_1\lambda_{m_2}m_3...\lambda_{m_n}}(u)$$

$$+\sigma_{m_1\lambda_{m_2}\lambda_{m_3}...m_n}(u) + ... + \sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_{n-1}}\lambda_{m_n}}(u) + ... - \sigma_{m_1m_2m_3...m_n}(u)]$$

$$+\frac{(\lambda_{m_1}+1)}{\lambda_{m_1-m_1}}(\sigma_{\lambda_{m_1}m_2...m_n}(u) - \sigma_{m_1m_2m_3...m_n}(u)) + ... + \frac{(\lambda_{m_1}+1)}{\lambda_{m_n-m_n}} \times$$

$$[\sigma_{m_1m_2...\lambda_{m_n}}(u) - \sigma_{m_1m_2...m_n}(u)]$$

$$-\frac{1}{(\lambda_{m_1}-m_1)...(\lambda_{m_n}-m_n)} \sum_{i_1=m_1+1}^{\lambda_{m_1}} \sum_{i_2=m_2+1}^{\lambda_{m_2}} ... \sum_{i_n=m_n+1}^{\lambda_{m_n}} (u_{i_1i_2...i_n} - u_{m_1m_2...m_n}).$$

(ii) *If $0 < \lambda < 1$, then*

$$u_{m_1m_2...m_n} - \sigma_{m_1m_2...m_n}(u)$$

$$= \frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(m_1-\lambda_{m_1})(m_2-\lambda_{m_2})...(\lambda_{m_n}-m_n)} \times$$

$$[\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_n}}(u) - \sigma_{m_1\lambda_{m_2}...\lambda_{m_n}}(u) - ... - \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u)$$

$$+\sigma_{\lambda_{m_1} m_2 m_3 \ldots \lambda_{m_n}}(u) + \sigma_{m_1 \lambda_{m_2} m_3 \ldots \lambda_{m_n}}(u) + \ldots + \sigma_{\lambda_{m_1} \lambda_{m_2} \ldots \lambda_{m_{n-2}} m_{n-1} m_n}(u)$$

$$\ldots - \sigma_{m_1 m_2 \ldots m_n}(u)]$$

$$-\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)\ldots(\lambda_{m_{n-1}}+1)}{(m_1-\lambda_{m_1})(m_2-\lambda_{m_2})\ldots(m_{n-1}-\lambda_{m_{n-1}})} \times [-\sigma_{\lambda_{m_1}\lambda_{m_2}\lambda_{m_3}\ldots\lambda_{m_{n-1}} m_n}(u)$$

$$+\sigma_{m_1 \lambda_{m_2} \lambda_{m_3} \ldots \lambda_{m_{n-1}} m_n}(u) + \ldots + \sigma_{\lambda_{m_1} \lambda_{m_2} \ldots \lambda_{m_{n-2}} \lambda_{m_{n-1}} m_n}(u) - \ldots - \sigma_{m_1 m_2 \ldots m_n}(u)]$$

$$-\ldots - \frac{(\lambda_{m_2}+1)(\lambda_{m_3}+1)\ldots(\lambda_{m_n}+1)}{(m_2-\lambda_{m_2})(m_3-\lambda_{m_3})\ldots(m_n-\lambda_{m_n})} \times [-\sigma_{m_1 \lambda_{m_2} m_3 \ldots \lambda_{m_n}}(u)$$

$$+\sigma_{m_1 \lambda_{m_2} \lambda_{m_3} \ldots m_n}(u) + \ldots + \sigma_{m_1 m_2 \lambda_{m_3} \ldots \lambda_{m_{n-1}} \lambda_{m_n}}(u) + \ldots . - \sigma_{m_1 m_2 m_3 \ldots m_n}(u)]$$

$$+\frac{(\lambda_{m_1}+1)}{m_1-\lambda_{m_1}}(\sigma_{\lambda_{m_1} m_2 \ldots m_n}(u) - \sigma_{m_1 m_2 m_3 \ldots m_n}(u)) + \ldots + \frac{(\lambda_{m_1}+1)}{m_n-\lambda_{m_n}}$$

$$\times[\sigma_{m_1 m_2 \ldots \lambda_{m_n}}(u) - \sigma_{m_1 m_2 \ldots m_n}(u)]$$

$$-\frac{1}{(m_1-\lambda_{m_1})\ldots(m_n-\lambda_{m_n})} \sum_{i_1=m_1+1}^{\lambda_{m_1}} \sum_{i_2=m_2+1}^{\lambda_{m_2}} \ldots \sum_{i_n=m_n+1}^{\lambda_{m_n}} (u_{m_1 m_2 \ldots . m_n} - u_{i_1 i_2 \ldots .. i_n}).$$

Proof (i) For $\lambda > 1$, by the definition of de la Vallée Poussin means of $(u_{m_1 m_2 \ldots m_n})$, we have

$$\tau_{m_1 m_2 \ldots m_n}(u) = \frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)\ldots(\lambda_{m_n}-m_n)} \sum_{i_1=m_1+1}^{\lambda_{m_1}} \ldots \sum_{i_n=m_n+1}^{\lambda_{m_n}} (u_{i_1 i_2 \ldots i_n})$$

$$= \frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)\ldots(\lambda_{m_n}-m_n)} \left[\left(\sum_{i_1=0}^{\lambda_{m_1}} - \sum_{i_1=0}^{m_1}\right) \ldots \left(\sum_{i_n=0}^{\lambda_{m_n}} - \sum_{i_n=0}^{m_n}\right)\right] u_{i_1 i_2 \ldots i_n}$$

$$= \frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)\ldots(\lambda_{m_n}-m_n)} \times$$

$$\left[\left(\sum_{i_1=0}^{\lambda_{m_1}} \sum_{i_2=0}^{\lambda_{m_2}} \ldots \sum_{i_n=0}^{\lambda_{m_n}}\right) - \left(\sum_{i_1=0}^{m_1} \sum_{i_2=0}^{\lambda_{m_2}} \ldots \sum_{i_n=0}^{\lambda_{m_n}}\right) - \ldots - \left(\sum_{i_1=0}^{\lambda_{m_1}} \sum_{i_2=0}^{\lambda_{m_2}} \ldots \sum_{i_{n-1}=0}^{\lambda_{m_{n-1}}} \sum_{i_n=0}^{m_n}\right)\right] u_{i_1 i_2 \ldots i_n}$$

$$+\frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)\ldots(\lambda_{m_n}-m_n)} \times$$

$$\left[\left(\sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \sum_{i_3=0}^{\lambda_{m_3}} \ldots \sum_{i_n=0}^{\lambda_{m_n}}\right) + \ldots + \left(\sum_{i_1=0}^{\lambda_{m_1}} \sum_{i_2=0}^{\lambda_{m_2}} \ldots \sum_{i_{n-1}=0}^{m_{n-1}} \sum_{i_n=0}^{m_n}\right)\right] u_{i_1 i_2 \ldots i_n}$$

$$+\ldots + \frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)\ldots(\lambda_{m_n}-m_n)} \times$$

$$\left[\left(\sum_{i_1=0}^{\lambda_{m_1}} \sum_{i_2=0}^{m_2} \ldots \sum_{i_n=0}^{m_n}\right) - \ldots - \left(\sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \ldots \sum_{i_{n-1}=0}^{m_n-1} \sum_{i_n=0}^{\lambda_{m_n}}\right) + \left(\sum_{i_1=0}^{m_1} \sum_{i_2=0}^{m_2} \ldots \sum_{i_n=0}^{m_n}\right)\right] (u_{i_1 i_2 \ldots i_n}).$$

Hence,

$$\tau_{m_1m_2...m_n}(u) = \frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} \times [((\lambda_{m_1}+1)(\lambda_{m_2}+1)...$$

$$(\lambda_{m_n}-m_n))\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_n}}(u) - ((m_1+1)(\lambda_{m_2}+1)...(\lambda_{m_n}-m_n))\,\sigma_{m_1\lambda_{m_2}...\lambda_{m_n}}$$

$$-((\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_{n-1}}+1)(m_n+1))\,\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u)$$

$$-(((m_1+1)(\lambda_{m_2}+1)...(\lambda_{m_{n-1}}+1)(\lambda_{m_n}+1))\,\sigma_{m_1\lambda_{m_2}...\lambda_{m_{n-1}}\lambda_{m_n}}(u)$$

$$-...-((\lambda_{m_1}+1)(\lambda_{m_2}+1)...(m_{n-1}+1)(\lambda_{m_n}+1))\,\sigma_{m_1\lambda_{m_2}...m_{n-1}\lambda_{m_n}}(u)$$

$$+\Big((m_1+1)(m_2+1)(\lambda_{m_3}+1)\sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_n}}\Big)(u) + ... + (\lambda_{m_1}+1)(\lambda_{m_2}+1)...$$

$$(\lambda_{m_{n-2}}+1)(m_{n-1}+1)(m_n+1)\sigma_{m_1...\lambda_{m_{n-2}}m_{n-1}m_n} - ... -$$

$$(\lambda_{m_1}+1)...(m_n+1)\sigma_{\lambda_{m_1}m_2...m_n}(u) - ... - (m_1+1)...(\lambda_{m_n}+1)\sigma_{m_1...\lambda_{m_n}}$$

$$-(m_1+1)(m_2+1)...(m_n+1)\sigma_{m_1m_2...m_n}(u)]$$

and

$$\tau_{m_1m_2...m_n}(u) = \frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)}\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_n}}(u)$$

$$-\left[\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} - \frac{(\lambda_{m_2}+1)(\lambda_{m_3}+1)....(\lambda_{m_n}+1)}{(\lambda_{m_2}-m_2)(\lambda_{m_3}-m_3)...(\lambda_{m_n}-m_n)}\right]\sigma_{m_1\lambda_{m_2}...\lambda_{m_n}}$$

$$-...-\left[\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} - \frac{(\lambda_{m_1}+1)...(\lambda_{m_{n-1}}+1)}{(\lambda_{m_1}-m_1)...(\lambda_{m_{n-1}}-m_{n-1})}\right]\sigma_{\lambda_{m_1}...\lambda_{m_{n-1}}m_n}$$

$$+\left[\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} - \frac{(\lambda_{m_1}+1)(\lambda_{m_3}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_3}-m_3)...(\lambda_{m_n}-m_n)}\right]\sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_n}}$$

$$-...-\left[\frac{(\lambda_{m_2}+1)(\lambda_{m_3}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_2}-m_2)(\lambda_{m_3}-m_3)...(\lambda_{m_n}-m_n)} + \frac{(\lambda_{m_n}+1)}{(\lambda_{m_n}-m_n)}\right]\sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_n}}$$

$$+\left[\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} - \frac{(\lambda_{m_1}+1)(\lambda_{m_3}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_3}-m_3)...(\lambda_{m_n}-m_n)}\right]\sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_n}}$$

$$-...-\left[\frac{(\lambda_{m_2}+1)(\lambda_{m_3}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_2}-m_2)(\lambda_{m_3}-m_3)...(\lambda_{m_n}-m_n)} + \frac{(\lambda_{m_1}+1)}{(\lambda_{m_1}-m_1)} + ... + \frac{(\lambda_{m_n}+1)}{(\lambda_{m_n}-m_n)}\right]\sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_n}}$$

$$+\sigma_{m_1m_2...m_n}(u).$$

The difference $\tau_{m_1m_2...m_n}(u) - \sigma_{m_1m_2...m_n}$ can be written as

$$\tau_{m_1m_2...m_n}(u) - \sigma_{m_1m_2...m_n}(u)$$

$$= \frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)}[\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_n}}(u) - \sigma_{m_1\lambda_{m_2}\lambda_{m_3}...\lambda_{m_n}}(u)$$

$$-... - \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u) + \sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_n}} + ... + \sigma_{\lambda_{m_1}\lambda_{m_2}...m_{n-1}m_n}$$

$$+\sigma_{\lambda_{m_1}m_2m_3...m_n} + ... + \sigma_{m_1m_2...\lambda_{m_n}}(u) - \sigma_{m_1m_2m_3...\lambda_{m_n}}(u)]$$

$$-\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_{n-1}}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_{n-1}}-m_n)}[-\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u) + \sigma_{\lambda_{m_1}\lambda_{m_2}...m_{n-1}m_n}(u)$$

$$+\sigma_{\lambda_{m_1}m_2...m_n} - \sigma_{m_1m_2...m_n}(u)]$$

$$-... - \frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)}[-\sigma_{m_1\lambda_{m_2}\lambda_{m_3}...\lambda_{m_n}}(u) + \sigma_{m_1\lambda_{m_2}\lambda_{m_3}...\lambda_{m_n}}(u)$$

$$+\sigma_{m_1\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u) - ... - \sigma_{m_1\lambda_{m_2}m_3...\lambda_{m_n}}(u) - \sigma_{m_1m_2...m_n}(u)]$$

$$+\frac{\lambda_{m_1}+1}{\lambda_{m_1}-m}[\sigma_{\lambda_{m_1}...m_n}(u) - \sigma_{m_1...m_n}(u)] + ... + \frac{\lambda_{m_n}+1}{\lambda_{m_n}-m_n}[\sigma_{m_1...m_{n-1}m_n}(u) - \sigma_{m_1...m_n}(u)].$$

It follows from the previous equation that,

$$u_{m_1m_2...m_n} - \sigma_{m_1m_2...m_n}(u)$$

$$= \frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)}[\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_n}}(u) - \sigma_{m_1\lambda_{m_2}...\lambda_{m_n}}(u)$$

$$-... - \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u) + \sigma_{\lambda_{m_1}m_2m_3...\lambda_{m_n}}(u) + \sigma_{m_1\lambda_{m_2}m_3...\lambda_{m_n}}(u)$$

$$+... + \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-2}}m_{n-1}m_n}(u)... - \sigma_{m_1m_2...m_n}(u)]$$

$$-\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_{n-1}}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_{n-1}}-m_{n-1})}[-\sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u)$$

$$+\sigma_{m_1\lambda_{m_2}\lambda_{m_3}...\lambda_{m_{n-1}}m_n}(u) + ... + \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-2}}\lambda_{m_{n-1}}m_n}(u)... - \sigma_{m_1m_2...m_n}(u)]$$

$$-\frac{(\lambda_{m_2}+1)(\lambda_{m_3}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_2}-m_2)(\lambda_{m_3}-m_3)...(\lambda_{m_n}-m_n)}[-\sigma_{m_1\lambda_{m_2}m_3...\lambda_{m_n}}(u) + \sigma_{m_1\lambda_{m_2}\lambda_{m_3}...m_n}(u)$$

$$+... + \sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_{n-1}}\lambda_{m_n}}(u) + ... - \sigma_{m_1m_2m_3...m_n}(u)]$$

$$+\frac{(\lambda_{m_1}+1)}{\lambda_{m_1-m_1}}[\sigma_{\lambda_{m_1}...m_n}(u) - \sigma_{m_1...m_n}(u)] + ... + \frac{(\lambda_{m_1}+1)}{\lambda_{m_n-m_n}}[\sigma_{m_1...\lambda_{m_n}}(u) - \sigma_{m_1...m_n}(u)]$$

$$-\frac{1}{(\lambda_{m_1}-m_1)...(\lambda_{m_n}-m_n)}\sum_{i_1=m_1+1}^{\lambda_{m_1}}\sum_{i_2=m_2+1}^{\lambda_{m_2}}...\sum_{i_n=m_n+1}^{\lambda_{m_n}}(u_{i_1i_2...i_n} - u_{m_1m_2...m_n}).$$

This completes the proof of Lemma (2.1)(i). □

(ii) The proof for $0 < \lambda < 1$ is similar to that of first part of Lemma 2.1(i).

Lemma 2.2 *(see [5], Lemma 6. p. 2-3) A sequence $(u_{m_1m_2...m_n})$ is slowly oscillating if and only if $v_{u_{m_1m_2...m_n}}\Delta(u)$ is slowly oscillating and bounded.*

Proof of Theorem 2.1 By Lemma 2.1(i) we obtain

$$|u_{m_1m_2...m_n} - \sigma_{m_1m_2...m_n}(u)| \leq |\tau_{m_1m_2...m_n}(u) - \sigma_{m_1m_2...m_n}(u)| + \left| -\frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} \sum_{i_1=m_1+1}^{\lambda_{m_1}} \sum_{i_2=m_2+1}^{\lambda_{m_2}} ... \sum_{i_n=m_n+1}^{\lambda_{m_n}} (u_{i_1i_2...i_n} - u_{m_1m_2...m_n}) \right|. \tag{6}$$

For the second term on the right hand side of the inequality (6), we have

$$\left| \frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} \sum_{i_1=m_1+1}^{\lambda_{m_1}} \sum_{i_2=m_2+1}^{\lambda_{m_2}} ... \sum_{i_n=m_n+1}^{\lambda_{m_n}} (u_{i_1i_2...i_n} - u_{m_1m_2...m_n}) \right|$$
$$\leq \frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} \sum_{i_1=m_1+1}^{\lambda_{m_1}} \sum_{i_2=m_2+1}^{\lambda_{m_2}} ... \sum_{i_n=m_n+1}^{\lambda_{m_n}} \left(\left| \sum_{r_1=m_1+1}^{\lambda_{m_1}} \Delta_{r_1} u_{r_1i_2...i_n} \right| + \left| \sum_{r_2=m_2+1}^{\lambda_{m_2}} \Delta_{r_2} u_{m_1r_2i_3...i_n} \right| + ... + \left| \sum_{r_n=m_n+1}^{\lambda_{m_n}} \Delta_{r_n} u_{m_1m_2...m_{n-1}r_n} \right| \right)$$

and then

$$\left| -\frac{1}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)} \sum_{i_1=m_1+1}^{\lambda_{m_1}} \sum_{i_2=m_2+1}^{\lambda_{m_2}} ... \sum_{i_n=m_n+1}^{\lambda_{m_n}} (u_{i_1i_2...i_n} - u_{m_1m_2...m_n}) \right|$$
$$\leq \max_{m_1+1\leq i\leq \lambda_{m_1}} \left| \sum_{r_1=m_1+1}^{i_1} \Delta_{r_1} u_{r_1i_2i_3...i_n} \right| + \max_{m_2+1\leq i_2\leq \lambda_{m_2}} \left| \sum_{r_2=m_2+1}^{i_2} \Delta_{r_2} u_{m_1r_2i_3...i_n} \right| + ... + \max_{m_n+1\leq i_n\leq \lambda_{m_n}} \left| \sum_{r_n=m_n+1}^{i_n} \Delta_{r_n} u_{m_1m_2m_3...m_{n-1}r_n} \right|. \tag{7}$$

By taking the lim sup on both sides of the inequality (7) as $m_1...m_n \to \infty$, and the first term on the right hand side of the inequality (7) vanishes by Lemma 2.1(i). We have,

$$\lim_{m_1,m_2,...,m_n\to\infty} \sup \left| (u_{m_1,m_2,...,m_n} - \sigma_{m_1,m_2,...,m_n}) \right| \leq \lim_{m_1,m_2,...,m_n\to\infty} \sup \max_{m_1+1\leq i_1\leq \lambda_{m_1}} \left| \sum_{r_1=m_1+1}^{i_1} \Delta_{r_1} u_{r_1i_2i_3...i_n} \right|$$
$$+ \lim_{m_1,m_2,...,m_n\to\infty} \sup \max_{m_2+1\leq i_2\leq \lambda_{m_2}} \left| \sum_{r_2=m_2+1}^{i_2} \Delta_{r_2} u_{m_1r_2i_3...i_n} \right|$$
$$... \quad ... \quad ... \quad ... \quad ...$$
$$+ \lim_{m_1,m_2,....,m_n\to\infty} \sup \max_{m_n+1\leq i_n\leq \lambda_{m_n}} \left| \sum_{r_n=m_n+1}^{i_n} \Delta_{r_n} u_{m_1m_2...m_{n-1}r_n} \right|.$$

Taking limit to both sides of the last inequality as $\lambda \to 1^+$, again since $(u_{m_1m_2...m_n})$ is slowly oscillating in senses $(1,0,0,...,0)$, $(0,1,0,...,0)$,...,$(0,0,0,...,1)$; we have

$$\lim_{m_1,m_2,...,m_n\to\infty} \sup |u_{m_1m_2...m_n} - \sigma_{m_1,m_2,...,m_n}(u)| \leq 0.$$

Therefore, $(u_{m_1,m_2,...,m_n})$ is P-convergent to L.

This completes the proof of of Theorem 2.1 □

Corollary 2.1 *Let $(u_{m_1m_2...m_n})$ be $(C,k_1,k_2,...,k_n)$ summable to L. If $(u_{m_1m_2...m_n})$ is slowly oscillating in sense $(1,0,0,0,...,0)$, $(0,1,0,0,...,0)$,..., $(0,0,0,0,...,1)$; then $(u_{m_1m_2...m_n})$ is P-convergent to L.*

Proof. Let $(u_{m_1m_2...m_n})$ be slowly oscillating, then $(\sigma^{k_1...k_n})$ is slowly oscillating (by Lemma 2.2). Further, since $u = (u_{m_1m_2...m_n})$ is $(C,k_1,k_2,...,k_r)$ summable to L; so by Theorem 2.1,

$$\lim_{m_1m_2...m_n\to\infty} (\sigma^{k_1...k_n})(u) = L. \tag{8}$$

Next from the definition,

$$(\sigma^{k_1...k_n})(u) = \sigma^{1...1}(u)(\sigma^{k_1-1...k_n-1}(u)) = L. \tag{9}$$

Clearly, (8) and (9) imply that $u = (u_{m_1m_2...m_n})$ is $(C,k_1-1,k_2-1,...,k_r-1)$ summable to L. Again, $(\sigma^{k_1-1...k_n-1}(u))$ is also slowly oscillating (by Lemma 2.2); Thus, by Theorem 2.1, we have

$$\lim_{m_1m_2...m_n\to\infty} (\sigma^{k_1-1...k_n-1}(u)) = L.$$

Continuing in this way, we get $(u_{m_1m_2....m_n})$ is P-convergent to L.

This completes the proof of of Corollary 2.1 □

Theorem 2.2 *Let $u_{m_1m_2...m_n}$ be $(C,1,1,1,...,1)$ summable to L. If $m_1\Delta_{m_1}u_{m_1m_2...m_n} \geq -K$, $m_2\Delta_{m_2}u_{m_1m_2...m_n} \geq -K$,..., $m_n\Delta_{m_n}u_{m_1m_2...m_n} \geq -K$; then $(u_{m_1m_2...m_n})$ is P-convergent to L.*

Proof. Taking lim sup on both sides of the identity in Lemma 2.1(i) as $m_1m_2...m_n \to \infty$ for $\lambda > 1$, we have

$$\limsup_{m_1...m_n\to\infty} u_{m_1m_2...m_n} - \sigma_{m_1m_2...m_n}(u)$$

$$\leq \limsup_{m_1...m_n\to\infty} \Big(\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_n}-m_n)}[-\sigma_{m_1\lambda_{m_2}...\lambda_{m_n}}(u)$$

$$-... - \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-1}}m_n}(u) + \sigma_{\lambda_{m_1}m_2m_3...\lambda_{m_n}}(u) + \sigma_{m_1\lambda_{m_2}m_3...\lambda_{m_n}}(u)$$

$$+... + \sigma_{\lambda_{m_1}\lambda_{m_2}...\lambda_{m_{n-2}}m_{n-1}m_n}(u) - ... - \sigma_{m_1m_2...m_n}(u)]$$

$$-\frac{(\lambda_{m_1}+1)(\lambda_{m_2}+1)...(\lambda_{m_{n-1}}+1)}{(\lambda_{m_1}-m_1)(\lambda_{m_2}-m_2)...(\lambda_{m_{n-1}}-m_{n-1})}[-\sigma_{\lambda_{m_1}\lambda_{m_2}\lambda_{m_3}...\lambda_{m_{n-1}}m_n}(u)$$

$$+\sigma_{m_1...\lambda_{m_{n-1}}m_n}(u) + ... + \sigma_{\lambda_{m_1}...\lambda_{m_{n-2}}\lambda_{m_{n-1}}m_n}(u) - ... - \sigma_{m_1m_2...m_n}(u)]$$

$$-... - \frac{(\lambda_{m_2}+1)(\lambda_{m_3}+1)...(\lambda_{m_n}+1)}{(\lambda_{m_2}-m_2)(\lambda_{m_3}-m_3)...(\lambda_{m_n}-m_n)}[-\sigma_{m_1\lambda_{m_2}m_3...\lambda_{m_n}}(u) + \sigma_{m_1\lambda_{m_2}\lambda_{m_3}...m_n}(u)$$

$$+... + \sigma_{m_1m_2\lambda_{m_3}...\lambda_{m_{n-1}}\lambda_{m_n}}(u) + ... - \sigma_{m_1m_2m_3...m_n}(u)]$$

$$+\frac{(\lambda_{m_1}+1)}{\lambda_{m_1-m_1}}[\sigma_{\lambda_{m_1}\dots m_n}(u)-\sigma_{m_1\dots m_n}(u)]+\dots+\frac{(\lambda_{m_1}+1)}{\lambda_{m_n-m_n}}[\sigma_{m_1\dots\lambda_{m_n}}(u)-\sigma_{m_1\dots m_n}(u)]$$

$$+\limsup_{m_1\dots m_n\to\infty}-\frac{1}{(\lambda_{m_1}-m_1)\dots(\lambda_{m_n}-m_n)}\sum_{i_1=m_1+1}^{\lambda_{m_1}}\sum_{i_2=m_2+1}^{\lambda_{m_2}}\dots\sum_{i_n=m_n+1}^{\lambda_{m_n}}(u_{i_1\dots i_n}-u_{m_1\dots m_n}).$$

Since the first term on the right-hand side of the last inequality vanishes by Lemma 2.1(i), we have

$$\limsup_{m_1\dots m_n\to\infty} u_{m_1m_2\dots m_n}-\sigma_{m_1m_2\dots m_n}(u)$$

$$\leq \limsup_{m_1\dots m_n\to\infty}(-\frac{1}{(\lambda_{m_1}-m_1)\dots(\lambda_{m_n}-m_n)}\sum_{i_1=m_1+1}^{\lambda_{m_1}}\sum_{i_2=m_2+1}^{\lambda_{m_2}}\dots\sum_{i_n=m_n+1}^{\lambda_{m_n}}\times$$

$$\left(\sum_{r_1=m_1+1}^{\lambda_{m_1}}\Delta_{r_1}u_{r_1i_2\dots.i_n}+\sum_{r_2=m_2+1}^{\lambda_{m_2}}\Delta_{r_2}u_{m_1r_2i_3\dots i_n}+\dots+\sum_{r_n=m_n+1}^{\lambda_{m_n}}\Delta_{r_n}u_{m_1m_2\dots m_{n-1}r_n})\right).$$

From the given conditions of Theorem 2.2, we obtain

$$\limsup_{m_1\dots m_n\to\infty} u_{m_1\dots m_n}-\sigma_{m_1m_2\dots m_n}(u)$$

$$\leq \limsup_{m_1\dots m_n\to\infty}(-\frac{1}{(\lambda_{m_1}-m_1)\dots(\lambda_{m_n}-m_n)}\sum_{i_1=m_1+1}^{\lambda_{m_1}}\sum_{i_2=m_2+1}^{\lambda_{m_2}}\dots\sum_{i_n=m_n+1}^{\lambda_{m_n}}\times$$

$$\left(\sum_{r_1=m_1+1}^{\lambda_{m_1}}-\frac{K}{r_1}+\sum_{r_2=m_2+1}^{\lambda_{m_2}}-\frac{K}{r_2}+\dots+\sum_{r_n=m_n+1}^{\lambda_{m_n}}-\frac{K}{r_n})\right)$$

for some $K\geq 0$. Hence we get,

$$\limsup_{m_1m_2\dots m_n}(u_{m_1m_2\dots m_n}-\sigma_{m_1m_2\dots m_n}(u))\leq\limsup_{m_1m_2\dots m_n}\left(K_1\log(\frac{\lambda_{m_1}}{m_1})+\dots+K_n\log(\frac{\lambda_{m_n}}{m_n})\right)$$

for some $K_1,K_2,\dots,K_n\geq 0$. Therefore, we have

$$\limsup_{m_1m_2\dots m_n\to\infty}\ (u_{m_1m_2\dots m_n}-\sigma_{m_1m_2\dots m_n}(u))\leq K_{n+1}\log\lambda,\ \text{for some } K_{n+1}\geq 0.$$

Taking the limit to both sides as $\lambda\to 1^+$, we obtain

$$\limsup_{m_1m_2\dots m_n\to\infty}\ (u_{m_1m_2\dots m_n}-\sigma_{m_1m_2\dots m_n}(u))\leq 0. \qquad (10)$$

Similarly, for $0<\lambda<1$ Lemma 2.1(ii) implies,

$$\liminf_{m_1m_2\dots m_n\to\infty}(u_{m_1m_2\dots m_n}-\sigma_{m_1m_2\dots m_n}(u))\geq 0. \qquad (11)$$

Clearly, by the inequalities (10) and (11); $(u_{m_1m_2\dots m_n})$ is P-convergent to L.

This completes the proof of of Theorem 2.2 □

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