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Basic Properties of Finite Sum of Weighted Composition Operators

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Abstract. In this paper,we continue the study of finite sum of weighted composition operators between different L^p-spaces that was investigated by Jabbarzadeh and Estaremi in 2012. Indeed, we first obtain some necessary and sufficient conditions for boundedness of the finite sums of weighted composition operators between distinct L^p-spaces. In the sequel, we investigate the compactness of finite sum of weighted composition operators. By using theorems of boundedness and compactness, we estimate the essential norms of these operators. Finally, some examples to illustrate the main results are given.

1. Introduction

In recent years, considerable attention has been given to the delineation of weighted composition operators with regard to basic properties like boundedness, compactness, essential norm and some others. There are many great papers on the investigation of weighted composition operators acting on the spaces of measurable functions. For instance, one can see [4–8, 10, 11, 14, 18, 20]. Also, some basic properties of weighted composition operators on L^p-spaces were studied by Parrott [16], Nordgern [15], Singh and Manhas [17], Takagi [19] and some other mathematicians. As far as we know finite sum of weighted composition operators were studied by Jabbarzadeh and Estaremi in [9] for the first time.

Let (X, Σ, μ) be a σ-finite measure space. We denote the linear space of all complex-valued Σ-measurable functions on X by $L^0(\Sigma)$. For any σ-finite subalgebra $\mathcal{A} \subseteq \Sigma$ such that $(X, \mathcal{A}, \mu_{\mathcal{A}})$ is also σ-finite , the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative f as well as for all $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, where $E^{\mathcal{A}} f$ is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad A \in \mathcal{A}.$$

As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. For more details on the properties of $E^{\mathcal{A}}$ see [12] and [13]. For a measurable function $u : X \to \mathbb{C}$ and non-singular measurable transformation $\varphi : X \to X$, i.e, the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to μ, we can define an operator $uC_\varphi : L^p(\Sigma) \to L^0(\Sigma)$ with $uC_\varphi(f) = u.f \circ \varphi$ and it is called a weighted composition operator. For non-singular measurable transformations $\{\varphi_i\}_{i=1}^n$, we put $W = \sum_{i=1}^n u_i C_{\varphi_i}$. In this paper, we investigate some

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basic properties of the operator W between different L^p-spaces. In section 2, we provide some necessary and sufficient condition for W to be a bounded operator from $L^p(\Sigma)$ into $L^q(\Sigma)$ in the cases that $1 \le p < q \le \infty$ and $1 \le q < p \le \infty$. In the section 3, we discuss about compactness of W as an operator between different L^p-spaces. In section 4 we provide some bounds for the essential norm of W.

2. Boundedness

In this section, we study the boundedness of W between two distinct L^p-spaces. First, we give some necessary and sufficient conditions for W to be bounded as an operator from $L^p(\mu)$ into $L^q(\mu)$ in case $1 < q < p < \infty$.

Theorem 2.1. *Let $1 < q < p < \infty$. Then the following assertions hold.*

(a) If W is a bounded operator from $L^p(\mu)$ into $L^q(\mu)$ and u_i's are non-negative, then we have $\sum_{i=1}^n h_i E^i(|u_i|^q) \circ \varphi_i^{-1} \in L^{\frac{p}{p-q}}(\mu)$.

(b) If $\sum_{i=1}^n h_i E^i(|u_i|^q) \circ \varphi_i^{-1} \in L^{\frac{p}{p-q}}(\mu)$, then W is a bounded operator from $L^p(\mu)$ into $L^q(\mu)$.

(c) If u_i's are non-negative, then W is a bounded if and only if $\sum_{i=1}^n h_i E^i(|u_i|^q) \circ \varphi_i^{-1} \in L^{\frac{p}{p-q}}(\mu)$, in this case we have $\| \sum_{i=1}^n h_i E^i(u_i^q) \circ \varphi_i^{-1} \|_{\frac{p}{p-q}}^{\frac{1}{q}} \le \|W\| \le n^{\frac{q-1}{q}} \| \sum_{i=1}^n h_i E^i(|u_i|^q) \circ \varphi_i^{-1} \|_{\frac{p}{p-q}}^{\frac{1}{q}}$.

Proof. (a) Let u_i's be non-negative and define a linear functional Φ on $L^{\frac{p}{q}}(\mu)$ by

$$\Phi(f) := \int_X \sum_{i=1}^n h_i E^i(u_i^q) \circ \varphi_i^{-1} f d\mu, \qquad \text{for any } f \in L^{\frac{p}{q}}(\mu).$$

We show that Φ is bounded. For every $f \in L^{\frac{p}{q}}(\mu)$ we have $\int_X (|f|^{\frac{1}{q}})^p d\mu = \int_X |f|^{\frac{p}{q}} d\mu < \infty$ and so $|f|^{\frac{1}{q}} \in L^p(\mu)$. Hence

$$\begin{aligned}
|\Phi(f)| &\le \int_X \sum_{i=1}^n h_i E^i(u_i^q) \circ \varphi_i^{-1} |f| d\mu \\
&\le \int_X (\sum_{i=1}^n u_i |f|^{\frac{1}{q}} \circ \varphi_i)^q d\mu = \|W(|f|^{\frac{1}{q}})\|_q^q \\
&\le \|W\|^q \|f\|_{\frac{p}{q}}.
\end{aligned}$$

This implies that Φ is a bounded linear functional on $L^{\frac{p}{q}}(\mu)$ and $\|\Phi\| \le \|W\|^q$. By Riesz Representation Theorem, there exists unique function $g \in L^{\frac{p}{p-q}}(\mu)$ such that

$$\Phi(f) = \int_X g f d\mu, \qquad \text{for any} \quad f \in L^{\frac{p}{q}}(\mu).$$

Therefore, we have

$$\int_X g f d\mu = \int_X \sum_{i=1}^n h_i E^i(u_i^q) \circ \varphi_i^{-1} d\mu \quad \forall \ f \in L^{\frac{p}{q}}(\mu).$$

Hence, we get that

$$g = \sum_{i=1}^n h_i E^i(u_i^q) \circ \varphi_i^{-1} \qquad \mu - a.e.$$

This means that $\sum_{i=1}^n h_i E^i(u_i^q) \circ \varphi_i^{-1} \in L^{\frac{p}{p-q}}(\mu)$.

(b) Assume that $\sum_{i=1}^{n} h_i E^i(|u_i|^q) \circ \varphi_i^{-1} \in L^{\frac{p}{p-q}}(\mu)$. By using Holder's inequality we get that for every $f \in L^p(\mu)$,

$$
\begin{aligned}
\|Wf\|_q^q &= \int_X |\sum_{i=1}^{n} u_i f \circ \varphi_i|^q d\mu \\
&\leq n^{q-1} \int_X \sum_{i=1}^{n} |u_i f \circ \varphi_i|^q d\mu \\
&= n^{q-1} \sum_{i=1}^{n} \int_X h_i E^i(|u_i|^q) \circ \varphi_i^{-1} |f|^q d\mu \\
&\leq n^{q-1} (\int_X (\sum_{i=1}^{n} h_i E^i(|u_i|^q) \circ \varphi_i^{-1})^{\frac{p}{p-q}} d\mu)^{\frac{p-q}{p}} (\int_X (|f|^q)^{\frac{p}{q}} d\mu)^{\frac{q}{p}} \\
&= n^{q-1} \| \sum_{i=1}^{n} h_i E^i(|u_i|^q) \circ \varphi_i^{-1} \|_{\frac{p}{p-q}} \|f\|_p^q.
\end{aligned}
$$

So W is a bounded operator from $L^p(\mu)$ into $L^q(\mu)$ with

$$\|W\| \leq n^{\frac{q-1}{q}} \| \sum_{i=1}^{n} h_i E^i(|u_i|^q) \circ \varphi_i^{-1} \|_{\frac{p}{p-q}}^{\frac{1}{q}}.$$

(c) It is a direct consequence of (a) and (b). □

Now in the next theorem we give some necessary and sufficient conditions for W to be bounded as an operator from $L^p(\mu)$ into $L^q(\mu)$ when $1 < p < q < \infty$.

Theorem 2.2. *Let $1 < p < q < \infty$ and $J = \sum_{k=1}^{n} h_k E^k(|u_k|^q) o \varphi_k^{-1}$. Then the following assertions hold.*

(a) If W is a bounded operator from $L^p(\mu)$ into $L^q(\mu)$ and u_i's are non-negative, then $M = \sup_{i\in\mathbb{N}} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} < \infty$ and $J(B) = 0$ $\mu - a.e.$

(b) If $M = \sup_{i\in\mathbb{N}} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} < \infty$ and $J(B) = 0$ $\mu - a.e$, then W is a bounded operator from $L^p(\mu)$ into $L^q(\mu)$.

(c) If u_i's are non-negative, then W is a bounded operator from $L^p(\mu)$ into $L^q(\mu)$ if and only if $M = \sup_{i\in\mathbb{N}} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} < \infty$ and $J(B) = 0$ $\mu - a.e.$ In this case $M \leq \|W\|^q \leq n^{q-1} M$.

Proof. (a) Let $i \in \mathbb{N}$ and $f_i := \chi_{A_i}$. Then $f_i \in L^p(\mu)$ and $\|f_i\|_p = \mu(A_i)^{\frac{1}{p}} < \infty$. Also we have

$$
\begin{aligned}
\frac{\|Wf_i\|_q^q}{\|f_i\|_p^q} &= \frac{\int_X |\sum_{k=1}^{n} u_k f_i \circ \varphi_k|^q d\mu}{\mu(A_i)^{\frac{q}{p}}} \\
&\geq \frac{\int_X \sum_{k=1}^{n} |u_k f_i \circ \varphi_k|^q d\nu}{\mu(A_i)^{\frac{q}{p}}} \\
&= \frac{\int_X \sum_{k=1}^{n} h_k E^k(u_k^q) \circ \varphi_k^{-1} \chi_{A_i} d\mu}{\mu(A_i)^{\frac{q}{p}}} \\
&= \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}}.
\end{aligned}
$$

Thus $M = \sup_{i\in\mathbb{N}} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} \leq \|W\|^q < \infty$. Now; we prove that $J(B) = 0$ $\mu - a.e.$

Suppose on the contrary, we can find some $\delta > 0$ such that $\mu(\{x \in B; J(x) \geq \delta\}) > 0$. Put $D = \{x \in B; J(x) \geq \delta\}$. Since (X, Σ, μ) is a $\sigma-$finite measure space, we may assume that $\mu(D) < \infty$. As D is non-atomic, there exists $D_i \in \Sigma$ with $D_i \subseteq D$ and $\mu(D_i) = \frac{\mu(D)}{i}$. It is easy to see that $\chi_{D_i} \in L^p(\mu)$ and $\|\chi_{D_i}\|_p = (\frac{\mu(D)}{i})^{\frac{1}{p}}$. Moreover, $\|W\chi_{D_i}\|_q^q \geq \int_{D_i} J d\mu \geq \frac{\delta\mu(D)}{i}$. Since $\frac{q}{p} - 1 > 0$, we have

$$\frac{\|W\chi_{D_i}\|_q^q}{\|\chi_{D_i}\|_p^q} \geq \frac{\int_{D_i} J d\mu}{(\frac{\mu(D)}{i})^{\frac{q}{p}}} \geq \delta(\frac{i}{\mu(D)})^{\frac{q}{p}-1} \to \infty \ \ as \ \ i \to \infty.$$

This implies that W is not bounded and it is a contradiction.

(b) Assume that $J(B) = 0 \quad \mu - a.e$ and $M < \infty$. Then for any $f \in L^p(\mu)$,

$$\begin{aligned} \|Wf\|_q^q &= \int_X |\sum_{k=1}^n u_k f o\varphi_k|^q d\mu \\ &\leq n^{q-1} \int_X \sum_{k=1}^n |u_k f o\varphi_k|^q d\mu \\ &= n^{q-1} \sum_{i\in\mathbb{N}} J(A_i)|f(A_i)|^q \mu(A_i) \\ &\leq n^{q-1} M \sum_{i\in\mathbb{N}} (|f(A_i)|^p \mu(A_i))^{\frac{q}{p}}. \end{aligned}$$

If $\|f\|_p = 1$, then $|f(A_i)|^p \mu(A_i) = \int_{A_i} |f|^p d\mu \leq \int_X |f|^p d\mu = \|f\|_p^p = 1$. Since $\frac{q}{p} > 1$, we have $(|f(A_i)|^p \mu(A_i))^{\frac{q}{p}} \leq |f(A_i)|^p \mu(A_i)$. And so

$$\begin{aligned} \sum_{i\in\mathbb{N}} (|f(A_i)|^p \mu(A_i))^{\frac{q}{p}} &\leq \sum_{i\in\mathbb{N}} |f(A_i)|^p \mu(A_i) \\ &< \int_X |f|^p d\mu \\ &= 1. \end{aligned}$$

Hence we get that $\|W\|^q \leq n^{q-1}M$.

(c) It is a direct consequence of (a) and (b). □

Here we give some necessary and sufficient conditions for W to be bounded as an operator on $L^\infty(\mu)$.

Theorem 2.3. *Suppose that $W : L^\infty(\mu) \to L^\infty(\mu)$ and $u_i : X \to \mathbb{C}$. Then the following assertions hold.*

(a) If u_i's are non-negative and $\sum_{i=1}^n u_i \in L^\infty(\mu)$, then W is a bounded operator.

(b) If W is a bounded operator, then $\sum_{i=1}^n u_i \in L^\infty(\mu)$.

(b) If u_i's are non-negative, then W is a bounded operator if and only if $\sum_{i=1}^n u_i \in L^\infty(\mu)$ and in this case $\|W\| = \|\sum_{i=1}^n u_i\|_\infty$.

Proof. (a) Suppose that $\sum_{i=1}^n u_i \in L^\infty(\mu)$. Then for any $f \in L^\infty(\mu)$ we have

$$|Wf| \leq \sum_{i=1}^n u_i |f \circ \varphi_i| \leq \|\sum_{i=1}^n u_i\|_\infty \|f\|_\infty.$$

This means that $\|W\| \leq \|\sum_{i=1}^n u_i\|_\infty$.

(b) We may assume that $\mu(X) > 0$ and so $\|W\chi_X\|_\infty \leq \|W\|\|\chi_X\|_\infty$, this implies that $\|\sum_{i=1}^n u_i\|_\infty \leq \|W\|$. □

In the sequel, we provide some necessary and sufficient conditions for W to be bounded as an operator from $L^\infty(\mu)$ into $L^q(\mu)$, when $1 < q < \infty$.

Theorem 2.4. *Let $1 < q < \infty$ and W be a bounded operator from $L^\infty(\mu)$ into $L^q(\mu)$.*

(a) If W is a bounded operator from $L^\infty(\mu)$ into $L^q(\mu)$, then $\sum_{i=1}^n u_i \in L^q(\mu)$.

(b) If $\sum_{i=1}^n |u_i| \in L^q(\mu)$ then W is a bounded operator from $L^\infty(\mu)$ into $L^q(\mu)$.

(c) If u_i's are non-negative, then W is bounded if and only if $\sum_{i=1}^n u_i \in L^q(\mu)$. In this case $\|W\| = \|\sum_{i=1}^n u_i\|_q$.

Proof. (a) We may assume that $\mu(X) > 0$. Since W is a bounded operator from $L^\infty(\mu)$ into $L^q(\mu)$, $\|W\chi_X\|_q \le \|W\|$ and

$$\begin{aligned}\|W\chi_X\|_q^q &= \int_X |W\chi_X|^q d(\mu) \\ &= \int_X |\sum_{i=1}^n u_i\chi_X|^q d(\mu) \\ &= \|\sum_{i=1}^n u_i\|_q^q.\end{aligned}$$

Hence $\sum_{i=1}^n u_i \in L^q(\mu)$ and $\|\sum_{i=1}^n u_i\|_q \le \|W\|$.
(b) Suppose that $\sum_{i=1}^n |u_i| \in L^q(\mu)$. Then for every $f \in L^\infty(\mu)$, we have

$$\begin{aligned}\|Wf\|_q^q &= \int_X |\sum_{i=1}^n u_i f\circ\varphi_i|^q d\mu \\ &\le \int_X (\sum_{i=1}^n |u_i||f\circ\varphi_i|)^q d\mu \\ &\le \|\sum_{i=1}^n |u_i|\|_q^q \|f\|_\infty^q.\end{aligned}$$

Therefore, W is a bounded operator from $L^\infty(\mu)$ into $L^q(\mu)$ and $\|W\| \le \|\sum_{i=1}^n |u_i|\|_q$.
(c) If u_i's are non-negative, then by (a) and (b) we have $\|W\| = \|\sum_{i=1}^n u_i\|_q$. □

Here, we provide some examples to illustrate the results of this section.

Example 2.5. *(a) Let $X = [0,1]$, $d\mu = dx$ and Σ be the σ-algebra of Lebesgue sets. Take $u_1(x) = x^3 + 3x$, $u_2(x) = 3x^2 + 1$ and $\varphi_1(x) = \varphi_2(x) = x$. If $W : L^\infty(\mu) \to L^q(\mu)$, then $u_1 + u_2 \in L^q(\mu)$ and so W is bounded with $\|W\| \le \|u_1 + u_2\|_{L^q(\nu)}$.*

(b) Let $X = [0,1]$, $d\mu = dx$ and Σ be the Lebesgue sets. Take $u_1(x) = x^2+1$, $u_2(x) = 2x$ and $\varphi_1(x) = x$, $\varphi_2(x) = x^2$. If $W : L^\infty(\mu) \to L^\infty(\mu)$, then $u_1 + u_2 \in L^\infty(\mu)$ and W is bounded with $\|W\| \le \|u_1 + u_2\|_\infty$.

(c) Let $X = [1,2]$, $d\mu = dx$ and Σ be the Lebesgue sets. Take $u_1(x) = x^2$, $u_2(x) = \frac{1}{2}(x+1)$ and $\varphi_1(x) = x$, $\varphi_2(x) = \frac{1}{2}x$. If $W : L^p(\mu) \to L^q(\mu)$, then $J_1 = x^2$, $J_2 = 2x+1$. Direct computations show that $J_1 + J_2 \in L^{\frac{p}{p-q}}$ and $\|W\| \le 2^{q-1}\|J_1 + J_2\|_{\frac{p}{p-q}}$.

Example 2.6. *Let $(w_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers, we define a measure μ on $P(\mathbb{N})$ (the power set of $\mathbb{N}$) by $\mu(E) = \sum_{n\in E} w_n$ for any $E \in P(\mathbb{N})$. Let $u_1(n) = \alpha_n, \varphi_1(n) = n, u_2(n) = \beta_n, \varphi_2(n) = n$ a direct computation yields $h_1(n) = 1 = h_2(n), J_1(n) = \alpha_n, J_2(n) = \beta_n$, in which $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real or complex numbers. If $W : L^p(\mathbb{N}) \to L^q(\mathbb{N})$, the followings hold:*

(a) If $\sup_{n\in\mathbb{N}} |\alpha_n + \beta_n| < \infty$, then $\|W\| \le \sup_{n\in\mathbb{N}} |\alpha_n + \beta_n|$.

(b) If $\sum_{n\in\mathbb{N}} w_n|\alpha_n + \beta_n|^{\frac{p}{p-q}} < \infty$, then

$$\|W\| \le 2^{\frac{q-1}{q}} (\sum_{n\in\mathbb{N}} w_n|\alpha_n + \beta_n|^{\frac{p}{p-q}})^{\frac{p-q}{pq}} \quad (1 < q < p < \infty).$$

(c) *If* $\sup_{n\in\mathbb{N}} \frac{|\alpha_n+\beta_n|}{w_n^{\frac{q-p}{p}}} < \infty$, *then* $\|W\| \le 2^{\frac{q-1}{q}} (\sup_{n\in\mathbb{N}} \frac{|\alpha_n+\beta_n|}{w_n^{\frac{q-p}{p}}})^{\frac{1}{q}}$ $(1 < p < q < \infty)$.

(d) *If* $\|(\alpha_n + \beta_n)\|_{L^q(\nu)} < \infty$, *then* $\|W\| \le 2^{\frac{q-1}{q}} \|(\alpha_n + \beta_n)\|_{L^q(\nu)}$.

Example 2.7. *Let* $X = [0, 1]$, $d\mu = dx$ *and* Σ *be the Lebesgue sets. Take* $u_1(x) = 2x - 4$, $u_2(x) = x^2 + 4$ *and* $\varphi_1(x) = \varphi_2(x) = x^2$. *Direct computation shows that* $h_1(x) = h_2(x) = \frac{1}{2\sqrt{x}}$ *and* $J_1 = 4x - 6\sqrt{x} + 48 - \frac{4^3}{2\sqrt{x}}$, $J_2 = \frac{x^2\sqrt{x}}{2} + 6x\sqrt{x} + 24\sqrt{x} + \frac{4^3}{2\sqrt{x}}$, *where* $W : L^p(\mu) \to L^q(\mu)$ $q = 3, 5, 7, \ldots$ *and* $p > q$. *Hence W is bounded, but it's summands are not bounded.*

3. Compactness

In this section, we characterize compactness of W as a bounded operator between different L^p-spaces. First we provide some necessary and sufficient conditions for W to be compact in case $1 \le q < p < \infty$.

Theorem 3.1. *Let* $1 \le q < p < \infty$ *and* $J = \sum_{i=1}^{n} h_i E^i(|u_i|^q) o\varphi_i^{-1}$. *Then the followings hold.*

(a) *If W is a compact operator from* $L^p(\mu)$ *into* $L^q(\mu)$ *and* u_i*'s are non-negative, then* $J(B) = 0$ $\mu - a.e$ *and* $\sum_{i\in\mathbb{N}} J(A_i)^{\frac{p}{p-q}} \mu(A_i) < \infty$.

(b) *If* $J(B) = 0$ $\mu - a.e$ *and* $\sum_{i\in\mathbb{N}} J(A_i)^{\frac{p}{p-q}} \mu(A_i) < \infty$, *then W is a compact operator from* $L^p(\mu)$ *into* $L^q(\mu)$.

Proof. (a) First, we show that $J(B) = 0$ $\mu - a.e$. Suppose on the contrary, then we can find some $\delta > 0$ such that $S = \{x \in B; J(x) \ge \delta\}$ has positive measure. We may also assume $\mu(S) < \infty$. Since S is non-atomic, we can find a sequence $\{S_n\}_{n\in\mathbb{N}} \subset \Sigma$ such that $S_{n+1} \subset S_n \subset S$ with $0 < \mu(S_n) < \frac{\mu(S)}{n}$. For each $n \in \mathbb{N}$, we define $f_n := \frac{\chi_{S_n}}{\sqrt[p]{\mu(S_n)}}$. Clearly, $\|f\|_p = 1$.

Let $A \subset \Sigma$ with $0 < \mu(A) < \infty$. Since $p > 1$, we get that $|\int_S f_n \chi_A d\mu| = \frac{\mu(S_n \cap A)}{\sqrt[p]{\mu(S_n)}} \le \mu(S_n)^{1-\frac{1}{p}} < (\frac{\mu(s)}{n})^{1-\frac{1}{p}} \to 0$, as $n \to \infty$ and so $f_n \longrightarrow 0$ weakly. Since W is compact, we have $\|Wf_n\|_q \longrightarrow 0$. On the other hand,

$$\|Wf_n\|_q^q \ge \sum_{i=1}^{n} \int_X u_i^q \frac{\chi_{S_n} o\varphi_i}{\mu(S_n)} d\mu \ge \sum_{i=1}^{n} \int_X \frac{J_i \chi_{S_n}}{\mu(S_n)} > \delta.$$

A contradiction, hence $J(B) = 0$ $\mu - a.e.$ Moreover, since W is compact, then is bounded and so we have $\int_X J^{\frac{p}{p-q}} d\mu < \infty$. Therefore,

$$\sum_{i\in\mathbb{N}} J(A_i)^{\frac{p}{p-q}} \mu(A_i) < \infty.$$

(b) Since $\sum_{i\in\mathbb{N}} J(A_i)^{\frac{p}{p-q}} \mu(A_i) < \infty$, there exists some $k_m \in \mathbb{N}$ such that

$$\sum_{i>k_m} J(A_i)^{\frac{p}{p-q}} \mu(A_i) < \frac{1}{m}.$$

Let $A = \cup_{i=1}^{k_m} A_i$ and define $W' = WM\chi_A$. Clearly, W' is a finite rank operator. Moreover, for every

$f \in L^p(\mu)$ we have

$$\begin{aligned}
\|Wf - W'f\|_q^q &= \int_X |Wf - W'f|^q d\mu \\
&\leq n^{q-1} \int_X \sum_{i=1}^{n} |u_i f \circ \varphi_i|^q (1 - \chi_A \circ \varphi_i) d\mu \\
&= n^{q-1} \int_{\bigcup_{i>k_m} A_i} J|f|^q d\mu \\
&\leq n^{q-1} (\sum_{i>k_m} J(A_i)^{\frac{p}{p-q}} \mu(A_i))^{\frac{p-q}{p}} (\sum_{i>k_m} |f(A_i)|^p \mu(A_i))^{\frac{q}{p}} \\
&\leq n^{q-1} (\frac{1}{m})^{\frac{p-q}{p}} \|f\|_p^q.
\end{aligned}$$

Therefore W is the limit of finite rank operators, and so W is compact. □

In the second theorem of this section, we give some necessary and sufficient conditions for W to be compact in case $1 \leq p < q < \infty$.

Theorem 3.2. *Suppose that* $1 \leq p < q < \infty$.

(a) Let u_i*'s are non-negative and the sequence* $\{\mu(A_j)\}_{j\in\mathbb{N}}$ *has no subsequence that converges to zero. If* W *is a compact operator from* $L^p(\mu)$ *into* $L^q(\mu)$*, then* $J(B) = 0 \quad \mu - a.e$ *and* $\lim_{i\to\infty} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} = 0$.

(b) If $J(B) = 0 \quad \mu - a.e$ *and* $\lim_{i\to\infty} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} = 0$*, then* W *is a compact operator from* $L^p(\mu)$ *into* $L^q(\mu)$.

Proof. (a) Since W is compact, it is bounded and so we have $J(B) = 0 \quad \mu - a.e.$ Now; we show that $\lim_{i\to\infty} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} = 0$. Suppose on the contrary, then there exists constant $\epsilon_0 > 0$ and we have $\{j_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\frac{J(A_{j_k})}{\mu(A_{j_k})^{\frac{q-p}{p}}} \geq \epsilon_0$ for all $k \in \mathbb{N}$, define $f_k := \frac{\chi_{A_{j_k}}}{\sqrt[p]{\mu(A_{j_k})}}$.

Obviously, $f_k \in L^p(\mu)$ and $\|f_k\|_p = 1$. Let $A \subset \Sigma$ with $0 < \mu(A) < \infty$. Since the sequence $\{\mu(A_j)\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero, $\{k \in \mathbb{N}; A_{j_k} \subset A\}$ is finite and so $\mu(A_{j_k} \cap A) = 0$ for sufficiently large k. Thus, $|\int_X f_k \chi_A d\mu| = \frac{\mu(A_{j_k} \cap A)}{\sqrt[p]{\mu(A_{j_k})}} \to 0$ as $k \to \infty$, it follows that $f_k \longrightarrow 0$ weakly. Since W is a compact operator, $\|Wf_k\|_q \longrightarrow 0$.

On the other hand

$$\|Wf_k\|_q^q \geq \sum_{i=1}^{n} \int_X |u_i \frac{\chi_{A_{j_k}} o \varphi_i}{\sqrt[p]{\mu(A_{j_k})}}|^q d\mu = \mu(A_{j_k})^{-\frac{q}{p}} \int_X J\chi_{A_{j_k}} d\mu > \epsilon_0.$$

A contradiction. Hence, we have that $\lim_{i\to\infty} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} = 0$.

(b) Since $\lim_{i\to\infty} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} = 0$, for every $m \in \mathbb{N}$, there exists some $k_m \in \mathbb{N}$ such that $\frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} < \frac{1}{m}$, for all $i > k_m$. Let $A = \cup_{j=1}^{k_m} A_j$ and $W' = WM_{\chi_A}$. Easily, we get that W' is a finite rank operator. Similar to the proof of the previous theorem we have

$$\begin{aligned} \|Wf - W'f\|_q^q &= \int_X |Wf - W'f|^q d\mu \\ &\leq n^{q-1} \int_{X\setminus A} J|f|^q d\mu \\ &= n^{q-1} \sum_{i>k_m} J(A_i)|f(A_i)|^q \mu(A_i) \\ &= n^{q-1} \sum_{i>k_m} \frac{J(A_i)}{\mu(A_i)^{\frac{q-p}{p}}} |f(A_i)|^q \mu(A_i)^{\frac{q}{p}} \\ &\leq n^{q-1}(\frac{1}{m}) \sum_{i>k_m} (|f(A_i)|^p \mu(A_i))^{\frac{q}{p}}. \end{aligned}$$

Let $f \in L^p(\mu)$ with $\|f\|_p < 1$. For any $i \in \mathbb{N}$,

$$|f(A_i)|^p \mu(A_i) \leq \int_X |f|^p d\mu = \|f\|_p^p < 1.$$

Since $\frac{q}{p} > 1$, we have that $(|f(A_i)|^p \mu(A_i))^{\frac{q}{p}} \leq |f(A_i)|^p \mu(A_i)$ and so

$$\begin{aligned} \|Wf - W'f\|_q^q &\leq n^{q-1}(\frac{1}{m}) \sum_{i\in\mathbb{N}} |f(A_i)|^p \mu(A_i) \\ &= n^{q-1}(\frac{1}{m}) \int_{\bigcup_{i\in\mathbb{N}} A_i} |f|^p d\mu \\ &< n^{q-1}(\frac{1}{m}) \int_X |f|^p d\mu \\ &< \frac{n^{q-1}}{m}. \end{aligned}$$

Consequently, $\|W - W'\| \to 0$ as $m \to \infty$. Therefore, W is compact. □

In the last theorem of this section, we give some necessary and sufficient conditions for W to be compact as an operator from $L^\infty(\mu)$ into $L^q(\mu)$.

Theorem 3.3. *If $1 < q < \infty$ and $J = \sum_{i=1}^n h_i E^i(|u_i|^q) o\varphi_i^{-1}$, then the followings hold:*

(a) If u_i's are non-negative and W is a compact operator from $L^\infty(\mu)$ into $L^q(\mu)$, then $J(B) = 0 \quad \mu - a.e$ and $\sum_{i\in\mathbb{N}} J(A_i)\mu(A_i) < \infty$.

(b) If $J(B) = 0 \quad \mu - a.e$ and $\sum_{i\in\mathbb{N}} J(A_i)\mu(A_i) < \infty$, then W is compact a operator from $L^\infty(\mu)$ into $L^q(\mu)$.

Proof. (*a*) First we prove $J(B) = 0$. Suppose on the contrary, then there exists some $\delta > 0$ such that $S = \{x \in B; J(x) \geq \delta\}$ has positive measure, we may also assume that $\mu(S) < \infty$. Since S is non-atomic, we can find a sequence $\{S_n\}_{n\in\mathbb{N}} \subset \Sigma$ such that $S_{n+1} \subset S_n \subset S$ with $0 < \mu(S_n) < \frac{\mu(S)}{n}$. For each $n \in \mathbb{N}$, define $f_n = \chi_{S_n}$. Obviously, $\|f\|_\infty = 1$. Let $A \subset \Sigma$ with $0 < \mu(A) < \infty$.

Since $q > 1$, we get that $|\int_X f_n \chi_A d\mu| = \mu(S_n \cap A) < \frac{\mu(S)}{n} \to 0$ as $n \to \infty$ and so $f_n \longrightarrow 0$ weakly. Since W is a compact operator, $\|Wf_n\|_q \longrightarrow 0$. On the other hand $\|Wf_n\|_q^q \geq \sum_{i=1}^n \int_X u_i^q \chi_{S_n} \circ \varphi_i d\mu > \delta\mu(S_n) > 0$, a contradiction.

Since W is compact, it is bounded and so

$$\int_X J d\mu = \sum_{i=1}^n \int_X J_i d\mu = \sum_{i=1}^n \int_{\varphi_i^{-1}(X)} u_i^q d\mu \leq \int_X (\sum_{i=1}^n u_i)^q < \infty.$$

Hence $\sum_{i\in\mathbb{N}} J(A_i)\mu(A_i) < \infty$.

(b) Since $\sum_{i\in\mathbb{N}} J(A_i)\mu(A_i) < \infty$, there exists some $k_m \in \mathbb{N}$ such that

$$\sum_{i>k_m} J(A_i)\mu(A_i) < \frac{1}{m}.$$

Let $A = \cup_{i=1}^{k_m} A_i$ and set $W' = WM_{\chi_A}$. Clearly, W' is a finite rank operator. Moreover, for any $f \in L^\infty(\mu)$ we have

$$\begin{aligned}
\|Wf - W'f\|_q^q &= \int_X |Wf - W'f|^q d\mu \\
&\leq n^{q-1} \int_X \sum_{i=1}^{n} |u_i f o\varphi_i|^q (1 - \chi_A o\varphi_i) d\mu \\
&= n^{q-1} \sum_{i>k_m} J(A_i)\mu(A_i)|f(A_i)|^q \\
&\leq n^{q-1} (\frac{1}{m}) \|f\|_\infty^q.
\end{aligned}$$

This implies that $\|W - W'\| \to 0$ as $m \to \infty$ and so W is compact. □

Finally we provide some examples to illustrate the main results of this section.

Example 3.4. *Let $X = \bigcup_{n\in\mathbb{N}}(n-1, n) \bigcup \mathbb{N}$, μ be the Lebesque measure on $\bigcup_{n\in\mathbb{N}}(n-1, n)$, $\mu(\{n\}) = n$, for $n \in \mathbb{N}$ and $u_1 = u_2 = 1$. Define $\varphi_i : X \to X$ as:*

$$\varphi_1(x) = \begin{cases} x^2 & x = 2k, k \in \mathbb{N} \\ x + 1 & x; x = 2k-1, k \in \mathbb{N} \\ n & If n-1 < x < n (n \in \mathbb{N}) \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} (x-1)^2 & x = 2k, k \in \mathbb{N} \\ x & x = 2k-1, k \in \mathbb{N} \\ n & If n-1 < x < n (n \in \mathbb{N}) \end{cases}$$

Direct computations show that

$$h_1 = \begin{cases} 1 + \frac{1}{\sqrt{n}} & n \text{ is even and } n = k^2 \text{ for some } k \in \mathbb{N} \\ 1 & n \text{ is even and } n \neq k^2,\ \forall k \in \mathbb{N} \\ \frac{1}{n} & n \text{ is odd} \end{cases}$$

and

$$h_2 = \begin{cases} \frac{1}{\sqrt{n}} + \frac{2}{n} + 1 & n \text{ is odd and } n = k^2 \text{ for some } k \in \mathbb{N} \\ 1 + \frac{1}{n} & n \text{ is odd and } n \neq k^2,\ \forall k \in \mathbb{N} \\ \frac{1}{n} & n \text{ is even} \end{cases}.$$

It follows that $J(\bigcup_{n\in\mathbb{N}}(n-1, n)) = 0$ and for $p < q$, $\lim_{n\to\infty} \frac{J(n)}{\mu(n)^{\frac{q-p}{p}}} = 0$. By [3.2] W is a compact operator from $L^p(X)$ into $L^q(X)$.

Example 3.5. *Let $X = \mathbb{N}$ and $(w_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers, we define a measure μ on $P(\mathbb{N})$ by $\mu(E) = \sum_{n\in E} w_n$ for any $E \in P(\mathbb{N})$. Let $u_1(n) = \alpha_n, \varphi_1(n) = n, u_2(n) = \beta_n, \varphi_2(n) = n$ for any $n \in \mathbb{N}$. Direct computations yield that $h_1(n) = 1, h_2(n) = 1, J_1(n) = \alpha_n, J_2(n) = \beta_n$. Hence we have the followings:*

(a) *If $\sum_{n\in\mathbb{N}} w_n|\alpha_n + \beta_n|^{\frac{p}{p-q}} < \infty$, then W is a compact operator from $L^p(X)$ into $L^q(X)$ for $q < p$.*
(b) *If $\lim_{n\to\infty} \frac{\alpha_n+\beta_n}{w_n^{\frac{q-p}{p}}} = 0$, then W is a compact operator from $L^p(X)$ into $L^q(X)$ for $q > p$.*
(c) *If $\sum_{n\in\mathbb{N}} w_n|\alpha_n + \beta_n| < \infty$, then for $1 < q < \infty$, W is a compact operator from $L^\infty(X)$ into $L^q(X)$.*

Example 3.6. *Let $\mu(n) = n$, $u_1(n) = \frac{1}{n^3}$, $\varphi_1(n) = n$, $u_2(n) = \frac{1}{n^4}$, $\varphi_2(n) = n$, a direct computation yields $h_1(n) = 1 = h_2(n), J_1(n) = \frac{1}{n^{3p}}, J_2(n) = \frac{1}{n^{4p}}$*

(a) *For $1 < q < p < \infty$, W is compact, since $\sum_{n\in\mathbb{N}} J(n)^{\frac{p}{p-q}}\mu(n) < \infty$.*
(b) *For $1 < p < q < \infty$, W is compact, since $\lim_{n\to\infty} \frac{J(n)}{n^{\frac{q-p}{p}}} = 0$.*
(c) *For $1 < q < \infty$, $J(n) = \frac{1}{n^{3q}} + \frac{1}{n^{4q}}$ and W is compact, since $\sum_{n\in\mathbb{N}} J(n)\mu(n) < \infty$.*

4. Essential Norm

In the current section we are going to estimate the essential norm of W as an operator between different L^p-spaces. First, we consider W as a bounded operator from $L^\infty(\mu)$ into $L^q(\mu)$ in case $1 < q < p < \infty$.

Theorem 4.1. *Let $1 < q < p < \infty$, $\mu(X) < \infty$ and $W = \sum_{i=1}^n u_iC_{\varphi_i}$ be a bounded operator from $L^\infty(\mu)$ into $L^q(\mu)$. Put*

$$\alpha = \inf\{r > 0 : \ N_r(\sqrt[q]{J}) \text{ consists of finitely many atoms}\},$$

where $J = \sum_{i=1}^n h_iE^i(|u_i|^q)o\varphi_i^{-1}$ and $N_r(\sqrt[q]{J}) = \{x \in X : \sqrt[q]{J}(x) \geq r\}$. Then

$$\|W\|_e \leq \sqrt[q]{n^{q-1}\mu(X)}\alpha.$$

Proof. (a) Let $\epsilon > 0$. Put $K = N_{\alpha+\epsilon}(\sqrt[q]{J}), u_i^{'} = u_i\chi_{\varphi_i^{-1}(K)}$ and $W^{'} = \sum_{i=0}^n u_i^{'}C_{\varphi_i}$. Since by definition of α, K consists of finitely many atoms, $W^{'} = \sum_{i=0}^n u_iC_{\varphi_i}M_{\chi_K}$ is a finite rank operator on $L^q(\mu)$. Hence for every $f \in L^\infty(\mu)$ we get that

$$\begin{aligned}
\|Wf - W^{'}f\|_q^q &= \int_X |Wf - W^{'}f|^q d\mu \\
&\leq n^{q-1}\sum_{i=1}^n \int_X |u_i|^q\chi_{X\setminus K}o\varphi_i|f|^q \circ \varphi_i d\mu \\
&= n^{q-1}\sum_{i=1}^n \int_{X\setminus K} J_i|f|^q d\mu \\
&\leq n^{q-1}(\alpha+\epsilon)^q\mu(X)\|f\|_\infty^q.
\end{aligned}$$

Thus, we have

$$\|Wf - W^{'}f\|_q \leq n^{\frac{q-1}{q}}(\alpha+\epsilon)\mu(X)^{\frac{1}{q}}\|f\|_\infty,$$

and

$$\|W - W^{'}\| \leq n^{\frac{q-1}{q}}(\alpha+\epsilon)\mu(X)^{\frac{1}{q}}.$$

The compactness of $W^{'}$ implies that $\|W\|_e \leq \|W - W^{'}\| \leq \sqrt[q]{n^{q-1}\mu(X)}(\alpha+\epsilon)$. Consequently, $\|W\|_e \leq \sqrt[q]{n^{q-1}\mu(X)}\alpha$. □

Here, we provide a lower bound for the essential norm of W as a bounded operator from $L^p(\mu)$ into $L^\infty(\mu)$.

Theorem 4.2. *Let $1 < p < \infty$, $\mu(X) < \infty$ and $W = \sum_{i=1}^{n} u_i C_{\varphi_i}$ be a bounded operator from $L^p(\mu)$ into $L^\infty(\mu)$. Put*

$$\alpha = \inf\{r > 0 : \; N_r(\sqrt[p]{J}) \text{ consists of finitely many atoms}\},$$

where $J = \sum_{i=1}^{n} h_i E^i(|u_i|^p) o\varphi_i^{-1}$. Then If $u_i' s$ are non-negative and the sequence $\{\mu(A_i)\}_{i\in\mathbb{N}}$ has no subsequence that converges to zero, then $\|W\|_e \geq \frac{\alpha}{\mu(X)^{\frac{1}{p}}}$.

Proof. For every $0 < \epsilon < \alpha$, the set $N_{\alpha-\epsilon}(\sqrt[p]{J})$ either contains a non-atomic subset or has infinitely many atoms. Since $\|W\|_e = \inf_{S\in\mathcal{K}} \|W - S\|$. Then we can find a compact operator T such that $\|W - T\| \leq \|W\|_e + \epsilon$.

If $N_{\alpha-\epsilon}(\sqrt[p]{J})$ contains a non-atomic subset B, then we can find a sequence $\{B_j\}_{j\in\mathbb{N}} \subseteq \Sigma$ such that $B_{j+1} \subseteq B_j \subseteq B$ with $0 < \mu(B_j) < \frac{\mu(B)}{j}$. For each $j \in \mathbb{N}$, define $f_j := \frac{\chi_{B_j}}{\mu(B_j)^{\frac{1}{p}}}$, obviously $\|f_j\|_p = 1$. Let $A \subset \Sigma$ with $0 < \mu(A) < \infty$. Since $p > 1$ we have

$$|\int_X f_j \chi_A d\mu| = \frac{\mu(A \cap B_j)}{\mu(B_j)^{\frac{1}{p}}} < \mu(B_j)^{1-\frac{1}{p}} < (\frac{\mu(B)}{j})^{1-\frac{1}{p}} \to 0 \;\text{ as} j \to \infty.$$

So $f_j \to 0$ weakly and Since T is compact, $\|Tf_j\| \to 0$.

Now; we assume that $N_{\alpha-\epsilon}(\sqrt[p]{J})$ contains infinitely many atoms. Let $\{A_j\}_{j\in\mathbb{N}}$ be disjoint atoms in $N_{\alpha-\epsilon}(\sqrt[p]{J})$. Put $g_j := \frac{\chi_{A_j}}{\mu(A_j)^{\frac{1}{p}}}$ and let $A \subset \Sigma$ with $0 < \mu(A) < \infty$. Since the sequence $\{\mu(A_j)\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero, $\{j \in \mathbb{N} \,;\, A_j \subseteq A\}$ is finite and so $\mu(A_j \cap A) = 0$ for sufficiently large j and

$$|\int_X g_j \chi_A| = \frac{\mu(A \cap A_j)}{\mu(A_j)^{\frac{1}{p}}} \to 0 \;\; \text{as} j \to \infty.$$

It follows that $g_j \to 0$ weakly. Since T is compact, we have $\|Tg_j\| \to 0$. Therefore.

$$\begin{aligned}
\|Wf_j\|_\infty^p \mu(X) &\geq \int_X |\sum_{i=1}^{n} u_i f_j \circ \varphi_i|^p d\mu \\
&\geq \sum_{i=1}^{n} \int_X u_i^p \frac{\chi_{B_j} \circ \varphi_i}{\mu(B_j)} d\mu \\
&= \frac{1}{\mu(B_j)} \sum_{i=1}^{n} \int_X h_i E(u_i^p) \circ \varphi_i^{-1} \chi_{B_j} d\mu \\
&= \frac{1}{\mu(B_j)} \int_{B_j} J \chi_{B_j} d\mu \\
&\geq (\alpha - \epsilon)^p.
\end{aligned}$$

Thus we get that $\|Wf_j\|_\infty \geq \frac{\alpha-\epsilon}{\mu(X)^{\frac{1}{p}}}$. Similarly, we have $\|Wg_j\|_\infty \geq \frac{\alpha-\epsilon}{\mu(X)^{\frac{1}{p}}}$. It follows that

$$\begin{aligned}
\|W\|_e &> \|W - T\| - \epsilon \\
&\geq \begin{cases} \|Wf_j - Tf_j\|_\infty - \epsilon \geq \|Wf_j\|_\infty - \|Tf_j\|_\infty - \epsilon \\ \|Wg_j - Tg_j\|_\infty - \epsilon \geq \|Wg_j\|_\infty - \|Tg_j\|_\infty - \epsilon \end{cases} \\
&\geq \frac{\alpha - \epsilon}{\mu(X)^{\frac{1}{p}}} - \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, then we obtain $\|W\|_e \geq \frac{\alpha}{\mu(X)^{\frac{1}{p}}}$. □

In the next theorem, we find an upper bound for the essential norm of W as a bounded operator from $L^p(\mu)$ into $L^q(\mu)$ for $1 < q < p < \infty$.

Theorem 4.3. *Let $1 < q < p < \infty$, $\mu(X) < \infty$ and $W = \sum_{i=1}^{n} u_i C_{\varphi_i}$ be a bounded operator from $L^p(\mu)$ into $L^q(\mu)$. Put*

$$\alpha = \inf\{r > 0 \ ; \ N_r(\sqrt[q]{J}) \ consists \ of \ finitely \ many \ atoms\},$$

where $J = \sum_{i=1}^{n} h_i E^i(|u_i|^q) o\varphi_i^{-1}$. Then $\|W\|_e \le n^{\frac{1}{p}} \mu(X)^{\frac{1}{q}-\frac{1}{p}} \alpha$.

Proof. Take $\epsilon > 0$, put $K = N_{\alpha+\epsilon}(\sqrt[q]{J}), u_i' = u_i \chi_{\varphi_i^{-1}(K)}$ and $W' = \sum_{i=0}^{n} u_i' C_{\varphi_i}$. Since by definition of α, K consists of finitely many atoms, $W' = \sum_{i=0}^{n} u_i C_{\varphi_i} M_{\chi_K}$ is a finite rank operator on $L^q(\mu)$. Hence, for every $f \in L^p(\mu)$ we get that

$$\begin{aligned}
\|Wf - W'f\|_q^q &\le n^{q-1} \sum_{i=1}^{n} \int_X |u_i|^q \chi_{X \setminus K} o\varphi_i |f|^q \circ \varphi_i d\mu \\
&= n^{q-1} \sum_{i=1}^{n} \int_{X \setminus K} J_i |f|^q d\mu \\
&\le n^{q-1} (\alpha + \epsilon)^q (\mu(X)^{\frac{1}{q}-\frac{1}{p}} \|f\|_p)^q.
\end{aligned}$$

Thus

$$\|Wf - W'f\|_q \le n^{\frac{1}{p}} (\alpha + \epsilon) \mu(X)^{\frac{1}{q}-\frac{1}{p}} \|f\|_p$$

and so

$$\|W - W'\|_q \le n^{\frac{1}{p}} (\alpha + \epsilon) \mu(X)^{\frac{1}{q}-\frac{1}{p}}.$$

Compactness of W' implies that $\|W\|_e \le \|W - W'\|_q \le n^{\frac{1}{p}} (\alpha + \epsilon) \mu(X)^{\frac{1}{q}-\frac{1}{p}}$. Consequently, $\|W\|_e \le n^{\frac{1}{p}} \mu(X)^{\frac{1}{q}-\frac{1}{p}} \alpha$. □

Now we provide some upper and lower bounds for the essential norm of W as an operator from $L^p(\mu)$ into $L^q(\mu)$ in case $1 < p < q < \infty$.

Theorem 4.4. *Let $1 < p < q < \infty$, $\mu(X) < \infty$ and $W = \sum_{i=1}^{n} u_i C_{\varphi_i}$ be a bounded operator from $L^p(\mu)$ into $L^q(\mu)$. Put*

$$\alpha = \inf\{r > 0 \ ; \ N_r(\sqrt[q]{J}) \ consists \ of \ finitely \ many \ atoms\},$$

where $J = \sum_{i=1}^{n} h_i E^i(|u_i|^q) o\varphi_i^{-1}$. Then the followings hold.

(a) If $M = \sup_{i \in \mathbb{N}} \frac{1}{\mu(A_i)^{\frac{q-p}{p}}} < \infty$, then $\|W\|_e \le n^{\frac{1}{p}} M^{\frac{1}{q}} \alpha$

(b) If u_i's are non-negative and the sequence $\{\mu(A_i)\}_{i \in \mathbb{N}}$ has no subsequence that converges to zero, then $\|W\|_e \ge \frac{\alpha}{\mu(X)^{\frac{1}{p}-\frac{1}{q}}}$.

Proof. (a) Take $\epsilon > 0$, put $K = N_{\alpha+\epsilon}(\sqrt[q]{J}), u_i' = u_i \chi_{\varphi_i^{-1}(K)}$ and $W' = \sum_{i=0}^{n} u_i' C_{\varphi_i}$. Since by definition of α, K consists of finitely many atoms, $W' = \sum_{i=0}^{n} u_i C_{\varphi_i} M_{\chi_K}$ is a finite rank operator on $L^q(\mu)$. Hence, for every $f \in L^p(\mu)$ we have

$$\begin{aligned}
\|Wf - W'f\|_q^q &\le n^{q-1} \sum_{i=1}^{n} \int_X |u_i|^q \chi_{X \setminus K} \circ \varphi_i |f|^q \circ \varphi_i d\mu \\
&= n^{q-1} \sum_{i=1}^{n} \int_{X \setminus K} J_i |f|^q d\mu \\
&\le n^{q-1} (\alpha + \epsilon)^q \int_X |f|^q d\mu \\
&\le n^{q-1} (\alpha + \epsilon)^q M \sum_{i \in \mathbb{N}} (|f(A_i)|^p \mu(A_i))^{\frac{q}{p}}.
\end{aligned}$$

If $\|f\|_p = 1$, then $|f(A_i)|^p\mu(A_i) = \int_{A_i}|f|^p d\mu \le \int_X |f|^p d\mu = \|f\|_p^p = 1$. Since $\frac{q}{p} > 1$, we have $(|f(A_i)|^p\mu(A_i))^{\frac{q}{p}} \le |f(A_i)|^p\mu(A_i)$. And so

$$\begin{aligned}\sum_{i\in\mathbb{N}}(|f(A_i)|^p\mu(A_i))^{\frac{q}{p}} &\le \sum_{i\in\mathbb{N}}|f(A_i)|^p\mu(A_i)\\ &< \int_X |f|^p d\mu\\ &= 1.\end{aligned}$$

Hence, we get that $\|W - W'\|^q \le n^{q-1}M(\alpha+\epsilon)^q$. The compactness of W' implies that $\|W\|_e \le \|W - W'\| \le n^{\frac{1}{p}}M^{\frac{1}{q}}(\alpha+\epsilon)$. Consequently, $\|W\|_e \le n^{\frac{1}{p}}M^{\frac{1}{q}}\alpha$.
(b) For every $0 < \epsilon < \alpha$ the set $N_{\alpha-\epsilon}(\sqrt[q]{J})$ either contains a non-atomic subset or has infinitely many atoms. Since $\|W\|_e = \inf_{S\in\mathcal{K}}\|W - S\|$, take compact operator T such that $\|W - T\| \le \|W\|_e + \epsilon$.

If $N_{\alpha-\epsilon}(\sqrt[q]{J})$ contains a no-atomic subset B, then we can find a sequence $\{B_j\}_{j\in\mathbb{N}} \subseteq \Sigma$ such that $B_{j+1} \subseteq B_j \subseteq B$ with $0 < \mu(B_j) < \frac{\mu(B)}{j}$. For each $j \in \mathbb{N}$, define $f_j = \frac{\chi_{B_j}}{\mu(B_j)^{\frac{1}{p}}}$. Obviously, $\|f_j\|_p = 1$. Let $A \subset \Sigma$ with $0 < \mu(A) < \infty$. Since $p > 1$, we have $|\int_X f_j\chi_A| = \frac{\mu(A\cap B_j)}{\mu(B_j)^{\frac{1}{p}}} < \mu(B_j)^{1-\frac{1}{p}} < (\frac{\mu(B)}{j})^{1-\frac{1}{p}} \to 0$ as $j \to \infty$. So $f_j \to 0$ weakly. Since T is compact, we have $\|Tf_j\| \to 0$.

Now, assume that $N_{\alpha-\epsilon}(\sqrt[q]{J})$ contains infinitely many atoms. Let $\{A_j\}_{j\in\mathbb{N}}$ be disjoint atoms in $N_{\alpha-\epsilon}(\sqrt[q]{J})$. Put $g_j = \frac{\chi_{A_j}}{\mu(A_j)^{\frac{1}{p}}}$ and let $A \subset \Sigma$ with $0 < \mu(A) < \infty$. Since sequence $\{\mu(A_j)\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero, then $\{j \in \mathbb{N}\,;\, A_j \subseteq A\}$ is finite and so $\mu(A_j \cap A) = 0$ for sufficiently large j and $|\int_X g_j\chi_A| = \frac{\mu(A\cap A_j)}{\mu(A_j)^{\frac{1}{p}}} \to 0$ as $j \to \infty$. It follows that $g_j \to 0$ weakly. Since T is compact, we get that $\|Tg_j\| \to 0$. Thus,

$$\begin{aligned}\|Wf_j\|_q^q &= \int_X |\sum_{i=1}^n u_i\frac{\chi_{B_j}\circ\varphi_i}{\mu(B_j)^{\frac{1}{p}}}|^q d\mu\\ &\ge \sum_{i=1}^n\int_X u_i^q\frac{\chi_{B_j}\circ\varphi_i}{\mu(B_j)^{\frac{q}{p}}}d\mu\\ &\ge \int_{B_j}\frac{J}{\mu(B_j)^{\frac{q}{p}}}d\mu\\ &\ge \frac{(\alpha-\epsilon)^q}{\mu(X)^{\frac{q-p}{p}}}.\end{aligned}$$

Similarly, $\|Wg_j\|_q \ge \frac{\alpha-\epsilon}{\mu(X)^{\frac{1}{p}-\frac{1}{q}}}$. It follows that

$$\begin{aligned}\|W\|_e &> \|W - T\| - \epsilon\\ &\ge \begin{cases}\|Wf_j - Tf_j\|_q - \epsilon \ge \|Wf_j\|_q - \|Tf_j\|_q - \epsilon\\ \|Wg_j - Tg_j\|_q - \epsilon \ge \|Wg_j\|_q - \|Tg_j\|_q - \epsilon\end{cases}\\ &\ge \frac{\alpha-\epsilon}{\mu(X)^{\frac{1}{p}-\frac{1}{q}}} - \epsilon.\end{aligned} \tag{1}$$

Since $\epsilon > 0$ was arbitrary, we obtain $\|W\|_e \ge \frac{\alpha}{\mu(X)^{\frac{1}{p}-\frac{1}{q}}}$. □

In the last theorem we give some upper and lower bounds for the essential norm of W as a bounded operator on $L^\infty(\mu)$.

Theorem 4.5. *Let $1 < p < \infty$, $\mu(X) < \infty$ and $W = \sum_{i=1}^{n} u_i C_{\varphi_i}$ be a bounded operator from $L^\infty(\mu)$into $L^\infty(\mu)$. Put $u = \sum_{i=1}^{n} u_i$ and*

$$\alpha = \inf\{r > 0 \ ; \ N_r(u) \text{ consists of finitely many atoms}\}.$$

Then the followings hold.

(a) $\|W\|_e \leq \alpha$.
(b) If u_i's are non-negative and the sequence $\{\mu(A_i)\}_{i\in\mathbb{N}}$ has no subsequence that converges to zero, then $\|W\|_e \geq \alpha$

Proof. (a) Take $\epsilon > 0$, put $K = N_{\alpha+\epsilon}(u)$, $u_i' = u_i \chi_{\varphi_i^{-1}(K)}$ and $W' = \sum_{i=0}^{n} u_i' C_{\varphi_i}$. Since by definition of α, K consists of finitely many atoms, $W' = \sum_{i=0}^{n} u_i C_{\varphi_i} M_{\chi_K}$ is a finite rank operator on $L^\infty(\mu)$. Hence, for every $f \in L^\infty(\mu)$, we have

$$\begin{aligned} |Wf - W'f| &= |\sum_{i=1}^{n} u_i f \circ \varphi_i \chi_{X\setminus K}| \\ &\leq \sum_{i=1}^{n} |u_i||f \circ \varphi_i||\chi_{X\setminus K}| \\ &\leq (\alpha + \epsilon)\|f\|_\infty. \end{aligned}$$

Thus, we have $\|Wf - W'f\|_\infty \leq (\alpha + \epsilon)\|f\|_\infty$ and $\|W - W'\| \leq \alpha + \epsilon$. Compactness of W' implies that $\|W\|_e \leq \|W - W'\| \leq \alpha + \epsilon$. Consequently $\|W\|_e \leq \alpha$.

(b) For every $0 < \epsilon < \alpha$, the set $N_{\alpha-\epsilon}(u)$ either contains a non-atomic subset or has infinitely many atoms. Since $\|W\|_e = \inf_{S\in\mathcal{K}} \|W - S\|$, one can find a compact operator T such that $\|W - T\| \leq \|W\|_e + \epsilon$.

If $N_{\alpha-\epsilon}(u)$ contains a non-atomic subset B, then, we can find a sequence $\{B_j\}_{j\in\mathbb{N}} \subseteq \Sigma$ such that $B_{j+1} \subseteq B_j \subseteq B$ with $0 < \mu(B_j) < \frac{1}{j}$. For each $j \in \mathbb{N}$, define $f_j = \chi_{B_j}$. Obviously, $\|f_j\|_\infty = 1$. Let $A \subset \Sigma$ with $0 < \mu(A) < \infty$. We have

$$|\int_X f_j \chi_A| = \mu(A \cap B_j) < \mu(B_j) < \frac{1}{j} \to 0 \ \text{ as } \ j \to \infty.$$

So $f_j \to 0$ weakly. Since T is compact, we have $\|Tf_j\| \to 0$. Now; assume that $N_{\alpha-\epsilon}(u)$ contains infinitely many atoms. Let $\{A_j\}_{j\in\mathbb{N}}$ be disjoint atoms in $N_{\alpha-\epsilon}(u)$. Put $g_j = \chi_{A_j}$. Let $A \subset \Sigma$ with $0 < \mu(A) < \infty$. Since the sequence $\{\mu(A_j)\}_{j\in\mathbb{N}}$ has no subsequence that converges to zero, $\{j \in \mathbb{N} \ ; \ A_j \subseteq A\}$ is finite and so $\mu(A_j \cap A) = 0$ for sufficiently large j and $|\int_X g_j \chi_A| = \mu(A \cap A_j) \to 0$ as $j \to \infty$. It follows that $g_j \to 0$ weakly. Since T is compact, we have that $\|Tg_j\| \to 0$ and

$$\begin{aligned} \|Wf_j\|_\infty &= |\sum_{i=1}^{n} u_i \chi_{B_j} \circ \varphi_i| \\ &= \sum_{i=1}^{n} u_i \chi_{B_j} \circ \varphi_i \\ &\geq \alpha - \epsilon. \end{aligned}$$

Thus, we have $\|Wf_j\|_\infty \geq \alpha - \epsilon$. Similarly, $\|Wg_j\|_\infty \geq \alpha - \epsilon$. It follows that

$$\begin{aligned} \|W\|_e &> \|W - T\| - \epsilon \\ &\geq \begin{cases} \|Wf_j - Tf_j\|_\infty - \epsilon \geq \|Wf_j\|_\infty - \|Tf_j\|_\infty - \epsilon \\ \|Wg_j - Tg_j\|_\infty - \epsilon \geq \|Wg_j\|_\infty - \|Tg_j\|_\infty - \epsilon \end{cases} \\ &\geq \alpha - 2\epsilon. \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we obtain $\|W\|_e \geq \alpha$. □

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