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On Normal Graph of a Finite Group

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Abstract. Suppose G is a finite group and $C(G)$ denotes the set of all conjugacy classes of G. The normal graph of G, $\mathcal{N}(G)$, is a finite simple graph such that $V(\mathcal{N}(G)) = C(G)$. Two conjugacy classes A and B in $C(G)$ are adjacent if and only if there is a proper normal subgroup N such that $A \cup B \leq N$. The aim of this paper is to study the normal graph of a finite group G. It is proved, among other things, that the groups with identical character table have isomorphic normal graphs and so this new graph associated to a group has good relationship by its group structure. The normal graphs of some classes of finite groups are also obtained and some open questions are posed.

1. Introduction

Throughout this paper, graph means simple finite graph and all groups are assumed to be finite. Suppose Γ is such a graph on the vertex set $\{1, 2, \ldots, n\}$ and $\mathcal{F} = \{\Gamma_1, \ldots, \Gamma_n\}$ is a family of graphs such that $n_j = |V(\Gamma_j)|$, $1 \leq j \leq n$. The graph $\triangledown = \Gamma[\Gamma_1, \ldots, \Gamma_n]$ is defined as

$$
\begin{aligned}
V(\triangledown) &= \bigcup_{j=1}^{n} V(\Gamma_j),\\
E(\triangledown) &= \left(\bigcup_{j=1}^{n} E(\Gamma_j)\right) \cup \left(\bigcup_{ij \in E(\Gamma)} \{uv \mid u \in V(\Gamma_i), v \in V(\Gamma_j)\}\right).
\end{aligned}
$$

This graph is called the $\Gamma-$**join** of $\mathcal{F}$ [15, p. 396].

Suppose Γ and Δ are two graphs with disjoint vertex sets $V(\Gamma)$ and $V(\Delta)$, respectively. The **union** of Γ and Δ, $\Gamma \cup \Delta$, is a graph with vertex set $V(\Gamma) \cup V(\Delta)$ and edge set $E(\Gamma) \cup E(\Delta)$. Two exceptional cases of the $\Gamma-$join of graphs are usual and sequential joins of graphs. These are defined as follows: The **join** of Γ and Δ is the graph union $\Gamma \cup \Delta$ together with all the edges joining $V(\Gamma)$ and $V(\Delta)$. The **sequential join** $\Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ of graphs $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ with disjoint vertex sets is defined as $P_n[\Gamma_1, \Gamma_2, \ldots, \Gamma_n]$.

A permutation α on the set of all vertices of a graph Γ is called an **automorphism** if and only if α and its inverse preserve adjacency in Γ. The set of all automorphisms of Γ is denoted by $Aut(\Gamma)$. It is well-known that $Aut(\Gamma)$ is a group under composition of functions. This group is named the **full automorphism group**

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of Γ. The **complement** $\overline{\Gamma}$ is a graph with the same vertex set $V(\Gamma)$. Two vertices of $\overline{\Gamma}$ are adjacent if and only if they are not adjacent in Γ. Obviously, $Aut(\Gamma) = Aut(\overline{\Gamma})$.

Suppose G is a finite group and $C(G)$ denotes the set of all conjugacy classes of G. Define $\kappa(G) = |C(G)|$. The **normal graph** of G, $\mathcal{N}(G)$, is a finite graph such that $V(\mathcal{N}(G)) = C(G)$. Two conjugacy classes A and B in $C(G)$ are adjacent if and only if there is a proper normal subgroup N of G with this property that $A \cup B \leq N$. It is easy to see that if G is a simple group then $\mathcal{N}(G)$ is an empty graph.

Suppose G is a finite group and N is a proper normal subgroup of G. If N is a union of n $G-$conjugacy classes then N is called $n-$**decomposable**. The number n is denoted by $ncc(N)$ and if $X = \{ncc(N) \mid N \lhd G\}$ then G is called $X-$**decomposable**. In [3], the authors characterized finite non-perfect groups for which $X = \{1, 2, 3\}$ and in [1] finite non-perfect groups with $X = \{1, 3, 4\}$ are classified.

Throughout this paper, K_n, C_n, P_n and $Star_n$ denote the complete, cycle, path and star graph on n vertices. The **center** of a group G and the set of all positive divisors of an integer n are denoted by $Z(G)$ and $D(n)$, respectively. A group G is said to be **centerless**, if $Z(G) = 1$. An **empty graph** is a graph without edge. Our other notations are standard and can be taken mainly from [6, 12, 13].

2. Examples

In this section, the normal graphs of the dihedral, semi-dihedral, dicyclic and the group V_{8n} will be computed. These groups can be presented as follows:

$$\begin{aligned}
D_{2n} &= \langle a, b \mid a^n = b^2 = e,\ bab = a^{-1}\rangle,\\
SD_{8n} &= \langle a, b \mid a^{4n} = b^2 = e,\ bab = a^{2n-1}\rangle,\\
T_{4n} &= \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1}\rangle,\\
V_{8n} &= \langle a, b \mid a^{2n} = b^4 = e,\ aba = b^{-1},\ ab^{-1}a = b\rangle.
\end{aligned}$$

It is easy to see that $|D_{2n}| = 2n$, $|SD_{8n}| = 8n$, $|T_{4n}| = 4n$ and $|V_{8n}| = 8n$. We start by dihedral groups. The dihedral group D_{2n} has precisely $\frac{1}{2}(n + 3)$ conjugacy classes, when n is odd. These are $\{1\}$, $\{a, a^{-1}\}$, ..., $\{a^{\frac{(n-1)}{2}}, a^{\frac{-(n-1)}{2}}\}$ and $\{b, ab, \ldots, a^{n-1}b\}$. If $n = 2m$ then D_{2n} has exactly $m + 3$ conjugacy classes as follows:

$$\begin{aligned}
&\{1\}, \{a^m\}, \{a, a^{-1}\}, \ldots, \{a^{m-1}, a^{-m+1}\},\\
&\{a^{2j}b \mid 0 \leq j \leq m-1\}, \{a^{2j+1}b \mid 0 \leq j \leq m-1\}.
\end{aligned}$$

Table 1: Non−Trivial Linear Characters of D_{2n}, n is Odd.

Conjugacy Classes Character	1	a^r $1 \leq r \leq (n-1)/2$	b
χ_2	1	1	−1

Table 2: Non−Trivial Linear Characters of D_{2n}, n is Even.

Conjugacy Classes Characters	1	a^m	a^r $1 \leq r \leq m-1$	b	ab
χ_2	1	1	1	−1	−1
χ_3	1	$(-1)^m$	$(-1)^r$	1	−1
χ_4	1	$(-1)^m$	$(-1)^r$	−1	1

Example 2.1. *In this example, the normal graph of dihedral groups are computed. It will be proved that the normal graph of these groups can be described in the following simple form:*

$$\mathcal{N}(D_{2n}) \cong \begin{cases} K_{\frac{n+1}{2}} \cup b^{D_{2n}} & 2 \nmid n \\ S_3[K_1, K_1, K_{\frac{n+2}{4}}, K_{\frac{n+2}{4}}] & 2 \mid n \text{ and } 4 \nmid n \\ S_3[K_1, K_1, K_{\frac{n+4}{4}}, K_{\frac{n}{4}}] & 4 \mid n \end{cases}.$$

To prove, we define $A_1 = \{b^{D_{2n}}\}$, $A_2 = \{(ba)^{D_{2n}}\}$, $A_3 = \{(a^i)^{D_{2n}} \mid i \text{ is even}\}$, $A_4 = \{(a^i)^{D_{2n}} \mid i \text{ is odd}\}$, $B_1 = \{b^{D_{2n}}\}$, $B_2 = \{(ba)^{D_{2n}}\}$, $B_3 = \{(a^i)^{D_{2n}} \mid i \text{ is even}\}$ and $B_4 = \{(a^i)^{D_{2n}} \mid i \text{ is odd}\}$. If $\frac{n}{2}$ is odd then

$$\mathcal{N}(D_{2n}) = S_3[A_1, A_2, A_3, A_4] \cong S_3[K_1, K_1, K_{\frac{n+2}{4}}, K_{\frac{n+2}{4}}]$$

and if $\frac{n}{2}$ is even then

$$\mathcal{N}(D_{2n}) = S_3[B_1, B_2, B_3, B_4] \cong S_3[K_1, K_1, K_{\frac{n+4}{4}}, K_{\frac{n}{4}}],$$

proving the result.

The dicyclic group T_{4n} has order $4n$ and the cyclic subgroup $\langle a\rangle$ of T_{4n} has index 2 [13, p. 420]. This group has exactly $n+3$ conjugacy classes. These are:

$$\{1\}, \{a^n\}, \{a^r, a^{-r}\}, (1 \le r \le n-1), \{a^{2j}b \mid 0 \le j \le n-1\}, \{a^{2j+1}b \mid 0 \le j \le n-1\}.$$

Example 2.2. *The aim of this example is to obtain the graph structure of $\mathcal{N}(T_{4n})$. It will be proved that if n is even then $\mathcal{N}(T_{4n}) \cong S_3[K_1, K_1, K_{n/2+1}, K_{n/2}]$ and if n is odd then $\mathcal{N}(T_{4n}) \cong K_{n+1} \cup K_1 \cup K_1$. To do this, we first assume that n is odd. Then all normal subgroups of T_{4n} are subgroups of $\langle a\rangle$. So, there is no edge connecting $b^{T_{4n}}$ and other vertices of the graph. Since $\langle (a^i)^{T_{4n}}, (a^j)^{T_{4n}}\rangle \subset \langle a\rangle \lhd T_{4n}$, $(a^i)^{T_{4n}}$ and $(a^j)^{T_{4n}}$ are adjacent. Hence the normal graph of T_{4n} has the following structure:*

$$\mathcal{N}(T_{4n}) \cong K_{n+1} \cup K_1 \cup K_1.$$

Next we suppose that n is even. Define:

$A_1 := \{(a^r)^{T_{4n}} \mid 2 \nmid r\}$, $A_2 := \{(a^r)^{T_{4n}} \mid 2 \mid r\} \cup \{e, a^n\}$ and $A_3 := \{b^{T_{4n}}, (ba)^{T_{4n}}\}$. Then the relations

$$\begin{array}{rcl} \langle (a^i)^{T_{4n}}, (a^j)^{T_{4n}}\rangle & \subseteq & \langle a\rangle \lhd T_{4n}, \\ \langle e^{T_{4n}}, b^{T_{4n}}\rangle & \subseteq & \langle a^2, b\rangle \lhd T_{4n}, \\ \langle e^{T_{4n}}, (ba)^{T_{4n}}\rangle & \subseteq & \langle a^2, ba\rangle \lhd T_{4n}, \\ \langle (a^n)^{T_{4n}}, b^{T_{4n}}\rangle & \subseteq & \langle a^2, b\rangle \lhd T_{4n}, \\ \langle (a^n)^{T_{4n}}, (ba)^{T_{4n}}\rangle & \subseteq & \langle a^2, ba\rangle \lhd T_{4n}, \end{array}$$

show that $\mathcal{N}(T_{4n}) \cong S_3[K_1, K_1, K_{n/2+1}, K_{n/2}]$. This completes our argument.

The group V_{8n} and the semidihedral group SD_{8n} have order $8n$ and their character tables computed in [7] and [11], respectively. We first present a notation which is useful in describing the normal graph of the semidihedral group of SD_{8n}. To do this we assume that Δ_1 and Δ_2 are subgraphs of a graph Γ. We write $\Delta_1 \between \Delta_2$, when all vertices of Δ_1 are adjacent with all vertices of Δ_2. Define:

$$\begin{array}{ll} C^{even} = C_1 \cup C_2^{even} \cup C_3^{even}, & C^{odd} = C_1 \cup C_2^{odd} \cup C_3^{odd}, \\ C_1 = \{0, 2, \ldots, 2n\}, & C_2^{even} = \{1, 3, \ldots, n-1\}, \\ C_3^{even} = \{2n+1, 2n+3, \ldots, 3n-1\}, & C_2^{odd} = \{1, 3, 5, \ldots, n\}, \\ C_3^{odd} = \{2n+1, 2n+3, 2n+5, \ldots, 3n\}, & C_{even}^{\dagger} = C_1 \backslash \{0, 2n\}, \\ C_{odd}^{\dagger} = C_2^{even} \cup C_3^{even}, & C_{2,3}^{odd} = C_2^{odd} \cup C_3^{odd}, \\ C_*^{even} = C^{even} \backslash \{0, 2n\}, & C_*^{odd} = C^{odd} \backslash \{0, n, 2n, 3n\}. \end{array}$$

Suppose

$$\begin{array}{ll} A_1 := \{(a^r)^{SD_{8n}} \mid r \in C_1\} & B_1 := \{(a^r)^{SD_{8n}} \mid r \in C_{2,3}^{odd}\} \\ A_2 := \{(a^r)^{SD_{8n}} \mid r \in C_{odd}^{\dagger}\} & B_2 := \{(a^r)^{SD_{8n}} \mid r \in C_1\} \\ A_3 := \{b^{SD_{8n}}, (ba)^{SD_{8n}}\} & B_3 := \{b^{SD_{8n}}, (ba)^{SD_{8n}}, (ba^2)^{SD_{8n}}, (ba^3)^{SD_{8n}}\}. \end{array}$$

In the following example the normal graph of semidihedral groups is described as sequential join of some known graphs.

Example 2.3. *In this example we prove that,*

a. *If n is even, then $\mathcal{N}(SD_{8n}) = A_1 + A_2 + A_3$,*
b. *If n is odd, then $\mathcal{N}(SD_{8n}) = B_1 + B_2 + B_3$.*

By [11], the conjugacy classes of SD_{8n}, $n \geq 2$, are as follows:

If n is even, then there are $2n + 3$ conjugacy classes that can be computed in the following way:
- *Two conjugacy classes of size one as $[1] = \{1\}$ and $[a^{2n}] = \{a^{2n}\}$,*
- *$2n - 1$ conjugacy classes of size two in the form $[a^r] = \{a^r, a^{(2n-1)r}\}$, where $r \in C_*^{even}$,*
- *Two conjugacy classes of size $2n$ as $[b] = \{ba^{2t} \mid t = 0, 1, 2, \ldots, 2n - 1\}$ and $[ba] = \{ba^{2t+1} \mid t = 0, 1, 2, \ldots, 2n - 1\}$.*

If n is odd, then there are $2n + 6$ conjugacy classes as in the following way:
- *Four conjugacy classes of size one as $[1] = \{1\}$, $[a^n] = \{a^n\}$, $[a^{2n}] = \{a^{2n}\}$ and $[a^{3n}] = \{a^{3n}\}$,*
- *$2n - 2$ conjugacy classes of size two as $[a^r] = \{a^r, a^{(2n-1)r}\}$, where $r \in C_*^{odd}$,*
- *Four conjugacy classes of size n as $[b] = \{ba^{4t} \mid t = 0, 1, 2, \ldots, n - 1\}$, $[ba] = \{ba^{4t+1} \mid t = 0, 1, \ldots, n - 1\}$, $[ba^2] = \{ba^{4t+2} \mid t = 0, 1, \ldots, n - 1\}$ and $[ba^3] = \{ba^{4t+3} \mid t = 0, 1, 2, \ldots, n - 1\}$.*

We first assume that n is even. In Table 3, some irreducible characters of the group SD_{8n} are recorded. From this table, one can easily see that $\langle (a^i)^{SD_{8n}}, (a^j)^{SD_{8n}} \rangle \subseteq Ker\chi_1 \trianglelefteq SD_{8n}$. Hence, the induced subgraphs of A_1 and A_2 are complete and we have $A_1 \between A_2$. Therefore for each $r \in C_1$,

$$\begin{array}{rcl} \langle (a^r)^{SD_{8n}}, (b)^{SD_{8n}} \rangle & \subseteq & Ker\chi_2 \trianglelefteq SD_{8n}, \\ \langle (a^r)^{SD_{8n}}, (ba)^{SD_{8n}} \rangle & \subseteq & Ker\chi_3 \trianglelefteq SD_{8n}. \end{array}$$

Hence we have $A_3 \between A_2$. Since $\langle b^{SD_{8n}}, (ba)^{SD_{8n}} \rangle = SD_{8n}$, the induced subgraph on A_3 is empty. On the other hand, for each $r \in C_{odd}^{\dagger}$,

$$\langle (a^r)^{SD_{8n}}, (b)^{SD_{8n}} \rangle = SD_{8n} \text{ and } \langle (a^r)^{SD_{8n}}, (ba)^{SD_{8n}} \rangle = SD_{8n}.$$

This proves that a vertex in A_1 can not be connected to another one in A_3. Therefore, $\mathcal{N}(SD_{8n}) = A_1 + A_2 + A_3$.

Next we suppose that n is odd. Some linear characters of the group SD_{8n} are recorded in Table 4. Since $\langle (a^i)^{SD_{8n}}, (a^j)^{SD_{8n}} \rangle \subseteq Ker\chi_1 \trianglelefteq SD_{8n}$, the induced subgraphs of $\mathcal{N}(SD_{8n})$ on B_1 and B_2 are complete. Furthermore, we have $B_1 \between B_2$. On the other hand, for every $r \in C_1$,

$$\begin{array}{rcl} \langle (a^r)^{SD_{8n}}, (b)^{SD_{8n}} \rangle & \subseteq & Ker\chi_2 \trianglelefteq SD_{8n}, \\ \langle (a^r)^{SD_{8n}}, (ba^2)^{SD_{8n}} \rangle & \subseteq & Ker\chi_2 \trianglelefteq SD_{8n}, \\ \langle (a^r)^{SD_{8n}}, (ba)^{SD_{8n}} \rangle & \subseteq & Ker\chi_3 \trianglelefteq SD_{8n}, \\ \langle (a^r)^{SD_{8n}}, (ba^3)^{SD_{8n}} \rangle & \subseteq & Ker\chi_3 \trianglelefteq SD_{8n}. \end{array}$$

Therefore, we have $B_2 \between B_3$. Since

$$\langle b^{SD_{8n}}, (ba)^{SD_{8n}} \rangle = SD_{8n} \text{ and } \langle b^{SD_{8n}}, (ba^3)^{SD_{8n}} \rangle = SD_{8n},$$

the vertex $b^{SD_{8n}}$ is not adjacent to vertices $(ba)^{SD_{8n}}$ and $(ba^3)^{SD_{8n}}$. In a similar way, the vertex $(ba^2)^{SD_{8n}}$ is not adjacent to $(ba)^{SD_{8n}}$ and $(ba^3)^{SD_{8n}}$, since

$$\langle (ba^2)^{SD_{8n}}, (ba)^{SD_{8n}} \rangle = SD_{8n} \text{ and } \langle (ba^2)^{SD_{8n}}, (ba^3)^{SD_{8n}} \rangle = SD_{8n}.$$

Table 3: Non−Trivial Linear Characters of SD_{8n}, n is Even.

Conjugacy classes Characters	$[a^r]$; $r \in C_1$	$[a^r]$; $r \in C^{\dagger}_{odd}$	$[b]$	$[ba]$
χ_1	1	1	−1	−1
χ_2	1	−1	1	−1
χ_3	1	−1	−1	1

Table 4: Non−Trivial Linear Characters of SD_{8n}, n is Odd.

Conjugacy classes Characters	$[a^r]$; $r \in C_1$	$[a^r]$; $r \in C^{odd}_{2,3}$	$[b]$	$[ba]$	$[ba^2]$	$[ba^3]$
χ_1	1	1	−1	−1	−1	−1
χ_2	1	−1	1	−1	1	−1
χ_3	1	−1	−1	1	−1	1

By our calculations given in Table 4, $\langle (ba^2)^{SD_{8n}}, (b)^{SD_{8n}}\rangle \subseteq Ker\chi_2 \trianglelefteq SD_{8n}$. Hence $(ba^2)^{SD_{8n}}$ and $(b)^{SD_{8n}}$ are adjacent. Also, $\langle (ba)^{SD_{8n}}, (ba^3)^{SD_{8n}}\rangle \subseteq Ker\chi_3 \trianglelefteq SD_{8n}$ and so the vertices $(ba)^{SD_{8n}}$ and $(ba^3)^{SD_{8n}}$ are adjacent.

Finally, for any $r \in C^{odd}_{2,3}$,

$$\begin{aligned} \langle (a^r)^{SD_{8n}}, (b)^{SD_{8n}}\rangle &= SD_{8n}, \\ \langle (a^r)^{SD_{8n}}, (ba)^{SD_{8n}}\rangle &= SD_{8n}, \\ \langle (a^r)^{SD_{8n}}, (ba^2)^{SD_{8n}}\rangle &= SD_{8n}, \\ \langle (a^r)^{SD_{8n}}, (ba^3)^{SD_{8n}}\rangle &= SD_{8n}. \end{aligned}$$

Therefore, there are no vertices in B_1 and B_3 to be adjacent. This proves that $\mathcal{N}(SD_{8n}) = B_1 + B_2 + B_3$.

Suppose

$$\begin{aligned} A_1 &:= \{(a^{2r+1})^{V_{8n}} \mid 0 \le r \le n-1\}, \\ A_2 &:= \{1, (b^2)^{V_{8n}}, (a^{2s})^{V_{8n}}, (a^{2s}b^2)^{V_{8n}} \mid 1 \le s \le \tfrac{n-1}{2}\}, \\ A_3 &:= \{b^{V_{8n}}, (ab)^{V_{8n}}\}, \\ B_1 &:= \{(a^{2r+1})^{V_{8n}} \mid 0 \le r \le n-1\}, \\ B_2 &:= \{1, (b^2)^{V_{8n}}, (a^n)^{V_{8n}}, (a^n b^2)^{V_{8n}}, (a^{2s})^{V_{8n}}, (a^{2s}b^2)^{V_{8n}} \mid 1 \le s \le \tfrac{n}{2} - 1\}, \\ B_3 &:= \{b^{V_{8n}}, (b^{-1})^{V_{8n}}\}, \\ B_4 &:= \{(ab)^{V_{8n}}, (ab^{-1})^{V_{8n}}\}. \end{aligned}$$

In the next example the normal graph of the group V_{8n} is computed.

Example 2.4. *In this example, it is proved that:*

a. If n is odd, then $\mathcal{N}(V_{8n}) = A_1 + A_2 + A_3$.

b. If n is even, then $\mathcal{N}(V_{8n}) = S_3[B_1, B_2, B_3, B_4]$.

Suppose n is odd. By [13, p. 420], the conjugacy classes of V_{8n} are as follows:

$$\{1\}; \{b^2\}; \{a^{2r+1}, a^{-2r-1}b^2\} (0 \le r \le n-1);$$
$$\{a^{2s}, a^{-2s}\}; \{a^{2s}b^2, a^{-2s}b^2\} (1 \le s \le \tfrac{n-1}{2});$$
$$\{a^j b^k \mid k = 1, 3; 2 \mid j\}; \{a^j b^k \mid k = 1, 3; 2 \nmid j\}.$$

Some linear characters for this group are recorded in Table 5.

Table 5: Non−Trivial Linear Characters of V_{8n}, n is Odd.

Conjugacy classes / Characters	1	b^2	a^{2r+1}, $0 \le r \le n-1$	a^{2s}, $1 \le s \le \frac{n-1}{2}$	$a^{2s}b^2$, $1 \le s \le \frac{n-1}{2}$	b	ab
χ_1	1	1	1	1	1	−1	−1
χ_2	1	1	−1	1	1	1	−1
χ_3	1	1	−1	1	1	−1	1

Since $Ker\chi_1 = e^{V_{8n}} \cup (b^2)^{V_{8n}} \cup (a^{2r+1})^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}}$*, the subgraphs induced by* A_1 *and* A_2 *are complete and we have* $A_1 \between A_2$*. On the other hand, for each s,* $1 \le s \le \frac{n-1}{2}$*, we have*

$$\begin{aligned} Ker\chi_2 = &\ e^{V_{8n}} \cup (b^2)^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}} \cup b^{V_{8n}}, \\ Ker\chi_3 = &\ e^{V_{8n}} \cup (b^2)^{V_{8n}} \cup (a^{2s})^{V_{8n}} \cup (a^{2s}b^2)^{V_{8n}} \cup (ab)^{V_{8n}}. \end{aligned}$$

Thus $A_3 \between A_2$*. Since* $\langle b^{V_{8n}}, (ab)^{V_{8n}}\rangle = V_{8n}$*,* $b^{V_{8n}}$ *is not adjacent to* $(ab)^{V_{8n}}$*. On the other hand, for each r,* $0 \le r \le n-1$*, we have* $V_{8n} = \langle (a^{2r+1})^{V_{8n}}, (b)^{V_{8n}}\rangle = \langle (a^{2r+1})^{V_{8n}}, (ab)^{V_{8n}}\rangle$*. So, there is no a vertex in* A_1 *to be adjacent to a vertex in* A_2*. This proves that* $\mathcal{N}(V_{8n}) = A_1 + A_2 + A_3$*. Next we assume that n is even. By [7], the conjugacy classes of* V_{8n} *contained in* $Z(V_{8n})$ *are* $\{e\}$*,* $\{b^2\}$*,* $\{a^n\}$ *and* $\{a^nb^2\}$*. There are also* $2n-3$ *conjugacy classes of length* 2 *as* $\{a^{2r+1}, a^{-(2r+1)}b^2\}$*,* $\{a^{2s}, a^{-2s}\}$ *and* $\{a^{2s}b^2, a^{-2s}b^2\}$*, where* $0 \le r \le n-1$ *and* $1 \le s \le \frac{n}{2}-1$*. We have also four conjugacy classes of length n. These are:*

$$\{a^{2k+1}b^{(-1)^{k+1}} \mid 0 \le k \le n-1\},$$
$$\{a^{2k}b^{(-1)^{k}} \mid 0 \le k \le n-1\},$$
$$\{a^{2k}b^{(-1)^{k+1}} \mid 0 \le k \le n-1\},$$
$$\{a^{2k+1}b^{(-1)^{k}} \mid 0 \le k \le n-1\}.$$

It is well-known that a normal subgroup of a finite group can be written as the intersections of the kernels of some appropriate irreducible characters. To compute normal subgroups, we record in Table 6 some linear characters of V_{8n}*. This information were given in the paper of Darafsheh and Poursalavati [7].*

Table 6: Some Non−Trivial Linear Characters of V_{8n}, n is Even.

Conjugacy classes / Characters	1	b^2	a^n	a^nb^2	a^{4k+1}	a^{4k+3}	a^{4s}
ψ_3	1	1	1	1	−1	−1	1
ψ_5	1	1	1	1	1	1	1
ψ_7	1	1	1	1	−1	−1	1
Conjugacy classes / Characters	a^{4t+2}	$a^{4s}b^2$	$a^{4t+2}b^2$	b	b^{-1}	ab	ab^{-1}
ψ_3	1	1	1	−1	−1	1	1
ψ_5	1	1	1	−1	−1	−1	−1
ψ_7	1	1	1	1	1	−1	−1

From the Table 6, one can easily be seen that,

$$\begin{aligned}
Ker\psi_3 &= 1 \cup (b^2)^{V_{8n}} \cup (a^n)^{V_{8n}} \cup (a^n b^2)^{V_{8n}} \cup (a^{4s})^{V_{8n}} \cup (a^{4t+2})^{V_{8n}} \cup (a^{4s} b^2)^{V_{8n}} \\
&\cup (a^{4t+2} b^2)^{V_{8n}} \cup (ab)^{V_{8n}} \cup (ab^{-1})^{V_{8n}}, \\
Ker\psi_5 &= 1 \cup (b^2)^{V_{8n}} \cup (a^n)^{V_{8n}} \cup (a^n b^2)^{V_{8n}} \cup (a^{4k+1})^{V_{8n}} \cup (a^{4k+3})^{V_{8n}} \cup (a^{4s})^{V_{8n}} \\
&\cup (a^{4t+2})^{V_{8n}} \cup (a^{4s} b^2)^{V_{8n}} \cup (a^{4t+2} b^2)^{V_{8n}}, \\
Ker\psi_7 &= 1 \cup (b^2)^{V_{8n}} \cup (a^n)^{V_{8n}} \cup (a^n b^2)^{V_{8n}} \cup (a^{4s})^{V_{8n}}, (a^{4t+2})^{V_{8n}} \cup (a^{4s} b^2)^{V_{8n}} \\
&\cup (a^{4t+2} b^2)^{V_{8n}} \cup b^{V_{8n}} \cup (b^{-1})^{V_{8n}}.
\end{aligned}$$

By our calculations given above, the induced subgraphs of B_1, B_2, B_3 and B_4 are complete graphs of order n, $n+2$, 2 and 2, respectively. On the other hand, $B_1 \between B_2$, $B_2 \between B_3$, $B_4 \between B_2$ and we have:

$$\begin{aligned}
\langle b^{V_{8n}}, (ab)^{V_{8n}} \rangle &= \langle b^{V_{8n}}, (ab^{-1})^{V_{8n}} \rangle = \langle {b^{-1}}^{V_{8n}}, (ab)^{V_{8n}} \rangle = \langle {b^{-1}}^{V_{8n}}, (ab^{-1})^{V_{8n}} \rangle \\
&= \langle (a^{2r+1})^{V_{8n}}, (b)^{V_{8n}} \rangle = \langle (a^{2r+1})^{V_{8n}}, (ab)^{V_{8n}} \rangle = V_{8n}.
\end{aligned}$$

Therefore, there is no edge connecting a vertex in B_1 and a vertex in $B_3 \cup B_4$ and there is no edge between a vertex of B_3 and a vertex in B_4. This completes our argument.

It is an elementary fact that if p, q are primes and $q|p-1$ then there exists a unique non-abelian group of order pq. By [13, p. 290], this group is the Frobenius group $F_{p,q}$ and can be presented as $F_{p,q} = \langle a, b \mid a^p = b^q = 1, b^{-1}ab = a^u \rangle$, where u has order q modulo p. By [13, Proposition 25.9], the conjugacy classes of $F_{p,q}$ are

$$\begin{aligned}
&\{1\}, \\
&(a^{v_i})^{F_{p,q}} = \{a^{v_i s} \mid s \in S\} (1 \leq i \leq r), \\
&(b^n)^{F_{p,q}} = \{a^m b^n \mid 0 \leq m \leq p-1\} (1 \leq n \leq q-1).
\end{aligned}$$

We end this section by computing the normal graph of $F_{p,q}$ that will be used later.

Example 2.5. *Suppose p, q are primes and $q|p-1$. It is easy to see that $\langle a \rangle$ is the unique non-trivial proper normal subgroup of $F_{p,q}$ containing $1 + \frac{p-1}{q}$ conjugacy classes of $F_{p,q}$. Therefore, $\mathcal{N}(F_{p,q}) = K_{1+\frac{p-1}{q}} \cup \overline{K_{q-1}}$.*

3. Main Properties of Normal Graph

The aim of this section is to obtain the main properties of the normal graph of a finite group. We start this section by the following simple but important lemma:

Lemma 3.1. *Suppose G is a finite group with exactly n conjugacy classes and C_0 denotes the component containing identity element e of G. Then,*

1. *If $\mathcal{N}(G)$ does not have isolated vertex then the degree of e^G is equal to $n-1$,*
2. *All components of $\mathcal{N}(G)$ other than C_0 are isolated vertices of $\mathcal{N}(G)$,*
3. *If $\mathcal{N}(G)$ is connected and $r-$regular, $r \geq 2$, then $\mathcal{N}(G)$ is complete,*
4. *If G and H are two groups that one of them has complete normal graph, then $\mathcal{N}(G \times H)$ is also complete.*

Proof. Our main proof will consider four separate cases as follows:

1. Suppose u is an arbitrary vertex in $\mathcal{N}(G)$. Choose the vertex $v \neq e^G$ in such a way that $\langle u, v \rangle \neq G$. On the other hand $\langle e^G, u \rangle \subseteq \langle u, v \rangle$ and so e^G and u are adjacent. This shows that $deg\ e^G = n-1$.
2. Suppose x and y are adjacent vertices in a component $C \neq C_0$. Hence $\langle x, y \rangle \neq G$ and since $\langle e^G, x \rangle \subseteq \langle x, y \rangle$, $\langle e^G, x \rangle \neq G$ which is impossible.
3. The proof follows from the part (1).

4. Suppose $a^G \times b^H$ and $c^G \times d^H$ are two given vertices of $\mathcal{N}(G \times H)$. We have to prove that $\langle a^G \times b^H, c^G \times d^H \rangle \neq G \times H$. Since $a^G \times b^H \cup c^G \times d^H \subseteq \langle a^G \times e^H \cup c^G \times e^H \cup e^G \times b^H \cup e^G \times d^H \rangle \subseteq \langle a^G \cup c^G \rangle \times \langle b^H \cup d^H \rangle \neq G \times H$. Thus, $a^G \times b^H$ and $c^G \times d^H$ are adjacent, as desired.

This proves our lemma. □

Theorem 3.2. *Let G be a group and $Inn(G)$ be the only maximal normal subgroup of $Aut(G)$. Then $\mathcal{N}(G)$ is a union of a complete graph and an empty graph.*

Proof. If $f \in Aut(G) \setminus Inn(G)$ then $\langle f^{Aut(G)} \rangle \trianglelefteq Aut(G)$. Since $f \notin Inn(G)$, $\langle f^{Aut(G)} \rangle = Aut(G)$ which means that $f^{Aut(G)}$ is an isolated vertex. We now assume that $f^{Aut(G)}$ and $g^{Aut(G)}$ are two vertices of $\mathcal{N}(Aut(G))$ such that $f, g \in Inn(G)$, then $\langle f^{Aut(G)}, g^{Aut(G)} \rangle \subset Inn(G) \trianglelefteq Aut(G)$ and so they are adjacent. □

Lemma 3.3. *Let G be a non-abelian simple group and $|Aut(G) : Inn(G)| = p$, p is prime. Then the normal graph of $Aut(G)$ is a union of a complete and an empty graph.*

Proof. It is easy to see that $Inn(G) \trianglelefteq Aut(G)$. If N is another normal subgroup of $Aut(G)$ then $Aut(G) = N \cdot Inn(G)$. Since $N \cap Inn(G) \trianglelefteq Inn(G)$ and $Inn(G)$ is simple, $N \cap Inn(G) = 1$ and $|N| = p$. This proves that $Aut(G) \cong Inn(G) \times Z_p$. On the other hand, G is simple and so $Aut(G)$ has a unique minimal normal subgroup, which is a contradiction. Therefore, $Aut(G)$ has a unique non-trivial proper normal subgroup and so the result is an immediate consequence of Theorem 3.2. □

Suppose G is a sporadic simple group. It is well-known that $Aut(G) \cong G$ if and only if $G \cong M_{11}$, M_{23}, M_{24}, Co_1, Co_2, Co_3, Th, Fi_{23}, J_1, J_4, Ly, Ru, B or M. Hence, for these groups the normal graph of $Aut(G)$ is an empty graph. In the next result, the normal graph of the automorphism group of other sporadic groups are determined.

Corollary 3.4. *If G is a sporadic simple group isomorphic to M_{12}, M_{22}, HS, J_2, J_3, McL, Suz, He, HN, Fi_{22}, ${}^2F_4(2)'$ or Fi_{24}' then $\mathcal{N}(Aut(G)) \cong K_{12} \cup \overline{K}_9$, $K_{11} \cup \overline{K}_{10}$, $K_{21} \cup \overline{K}_{18}$, $K_{16} \cup \overline{K}_{11}$, $K_{17} \cup \overline{K}_{13}$, $K_{19} \cup \overline{K}_{14}$, $K_{37} \cup \overline{K}_{31}$, $K_{26} \cup \overline{K}_{19}$, $K_{44} \cup \overline{K}_{34}$, $K_{59} \cup \overline{K}_{53}$, $K_{17} \cup \overline{K}_{12}$ or $K_{97} \cup \overline{K}_{86}$, respectively.*

Proof. It is well-known that in each case $|Aut(G) : G| = 2$ and the proof follows from Lemma 3.3. □

Remark 3.5. *Suppose $G = \langle x, y \rangle$ is a non-cyclic two generators finite group. Consider the conjugacy classes x^G and y^G. Since $\langle x^G, y^G \rangle = G$, the vertices x^G and y^G are not adjacent. Thus $\mathcal{N}(G)$ is not complete. As a consequence, the normal graph of a non-abelian simple group is not complete.*

Theorem 3.6. $\mathcal{N}(Z_n) = \Gamma[K_{\phi(d_1)}, \ldots, K_{\phi(d_t)}] \cup \overline{K_{\phi(n)}}$, *where d_i's are divisors of n and Γ is a graph with vertex set $D(n) \setminus \{n\}$ and two vertices d_i and d_j are adjacent if and only if $lcm(d_i, d_j) \neq n$.*

Proof. It is easy to see that each generator of Z_n is an isolated vertex of $\mathcal{N}(Z_n)$ and so the normal graph of the cyclic group of order n has exactly $\phi(n)$ isolated vertices. Since the cyclic group Z_n has a unique subgroup of an order of each divisor of n, all elements of order d_i are in a subgroup of order d_i. This shows that non-generator elements with the same order are adjacent. Suppose d_i and d_j are two divisors of n, $O(x) = d_i$, and $O(y) = d_j$ such that $lcm(d_i, d_j) \neq n$. Since $|\langle x, y \rangle| = lcm(O(x), O(y)) = lcm(d_i, d_j) \neq n$, $\langle x, y \rangle \neq Z_n$. This proves that $\mathcal{N}(Z_n) = \Gamma[K_{\phi(d_1)}, \ldots, K_{\phi(d_t)}] \cup \overline{K_{\phi(n)}}$, proving the result. □

The number of edges in $\mathcal{N}(Z_n)$ can be computed from our previous theorem as follows:

Corollary 3.7. $|E(\mathcal{N}(Z_n))| = \sum_{dd' \in E(\Gamma)} \phi(d)\phi(d') + \frac{1}{2}[\phi(n) - \phi(n)^2 - n] + \frac{1}{2} \sum_{d|n} \phi(d)^2$.

Proof. By Theorem 3.6, we have:

$$\begin{aligned}|E(\mathcal{N}(Z_n))| &= \sum_{n\neq d|n}\binom{\phi(d)}{2} + \sum_{dd'\in E(\Gamma)}\phi(d)\phi(d') \\ &= \frac{1}{2}\left[\sum_{d|n}\phi(d)^2 - \phi(n)^2 - \sum_{d|n}\phi(d) + \phi(n)\right] + \sum_{dd'\in E(\Gamma)}\phi(d)\phi(d') \\ &= \frac{1}{2}\sum_{d|n}\phi(d)^2 + \frac{1}{2}[\phi(n) - n - \phi(n)^2] + \sum_{dd'\in E(\Gamma)}\phi(d)\phi(d'),\end{aligned}$$

proving the result. □

If G is a finite group then the minimum cardinality of a set of generators for G is denoted by $d(G)$.

Theorem 3.8. *The following statements hold:*

1. *The normal graph of an abelian group G is complete if and only if $G \cong Z_{m_1}\times Z_{m_2}\times\cdots\times Z_{m_k}$, where $k \geq 3$ and $2 \leq m_1 \mid m_2 \mid \cdots \mid m_k$.*
2. *The normal graph of a group G is isomorphic to P_n if and only if $|G| = 1$.*
3. *There is no group with a cycle graph $C_n, n \geq 3$, as its normal graph.*

Proof. 1. Let $\mathcal{N}(G)$ be complete. By fundamental theorem of finite abelian groups, $G \cong Z_{m_1}\times Z_{m_2}\times\cdots\times Z_{m_k}$, where $2 \leq m_1 \mid m_2 \mid \cdots \mid m_k$. If $k \leq 2$ then G is cyclic or $d(G) = 2$. In the first case, G has $\phi(n)$ isolated vertices and in the second case the vertices of a 2−generating set are not adjacent which are not possible. Thus $k \geq 3$, as desired. Conversely, we assume that $d(G) = k \geq 3$. So, for any elements x and y in G, $G \neq \langle x, y\rangle$ and so they are adjacent.

2. It is easy to see that $\kappa(G) = 2$ if and only if $G \cong Z_2$ and $\kappa(G) = 3$ if and only if $G \cong Z_3$ or S_3. Thus, there are three normal graphs of orders two or three isomorphic to $\overline{K_2}$, $\overline{K_3}$ or $K_2 \cup K_1$. Therefore, the paths P_2 and P_3 cannot be isomorphic to the normal graph of a group. Suppose $n \geq 4$ and $P_n : v_1e_1v_2e_2\cdots v_{n-1}e_{n-1}v_n$. If $v_1 = e^G$ then it will be adjacent to v_3, a contradiction. If $v_2 \neq e^G$ then $\langle v_1, e^G\rangle \leq \langle v_1, v_2\rangle \neq G$ and so v_1 and e^G are adjacent. This shows that $deg(v_1) \geq 2$ which is impossible. Hence $v_2 = e^G$. The converse is trivial.

3. It is proved in (2) that C_3 cannot be represented as a normal graph. Suppose $n \geq 4$ and G is a finite group with $\mathcal{N}(G) \cong C_n$. If

$$C_n : v_1e_1v_2e_2\ldots v_{n-1}e_{n-1}v_ne_nv_1$$

then there exists i, $1 \leq i \leq n$, such that $v_i = e^G$. By definition of normal graph $\langle v_i, v_{i+1}\rangle \neq G$. Since $\langle v_{i+2}\rangle \leq \langle v_{i+1}, v_{i+2}\rangle \neq G$, v_{i+2} and v_i are adjacent, a contradiction.

This completes our argument. □

By Theorem 3.8, if for a prime p, $p^3|n$ then the group $G = Z_p\times Z_p\times Z_p\times Z_{\frac{n}{p^3}}$ has a complete normal graph. This shows that the maximum edge of a normal graph in the set of all groups with exactly n conjugacy classes is $\frac{n(n-1)}{2}$.

Theorem 3.9. *Let G be a finite group.*

1. *G is simple if and only if $\mathcal{N}(G)$ is an empty graph.*
2. *Let G be abelian. Then $\mathcal{N}(G)$ is bipartite if and only if G is isomorphic to $Z_2\times Z_2$, Z_4 or Z_p, p is prime.*
3. *Let G be a non−abelian group and $G' \neq G$. Then $\mathcal{N}(G)$ is bipartite if and only if G is a simple group, $G \cong S_3$ or G is a Frobenius group of order $|N|(|N| - 1)$, where N is a 2−decomposable normal 2−subgroup of G and $|N| - 1$ is a prime number.*

Proof. We first notice that x^G is an isolated vertex if and only if $\langle x^G \rangle = G$. If $\mathcal{N}(G)$ has an edge connecting two non−trivial conjugacy classes x^G and y^G of G, then the conjugacy classes e^G, x^G and y^G constitute a triangle in $\mathcal{N}(G)$. Our main proof will consider three separate cases as follows:

1. Suppose G is a simple group and $e \neq x \in G$. Then $\langle x^G \rangle$ is a normal subgroup of G and so $\langle x^G \rangle = G$. This proves that the graph $\mathcal{N}(G)$ is empty. Conversely, we assume that $\mathcal{N}(G)$ is an empty graph and N is a non−trivial normal subgroup of G. Choose the non−trivial G−conjugacy class x^G contained in N. So, $\langle x^G, e^G \rangle \subset N$ and so, x^G and e^G are adjacent in $\mathcal{N}(G)$, which is impossible.
2. Suppose $\mathcal{N}(G)$ is bipartite. If the normal graph is empty then by Part (1), G will be simple. This shows that $G \cong Z_p$, p is prime. If $|E(\mathcal{N}(G))| \geq 2$ then by above discussion all edges will be started form e^G. Choose the edges $e^G x^G$ and $e^G y^G$ from $\mathcal{N}(G)$. Hence $N = e^G \cup x^G$ and $M = e^G \cup y^G$ are two distinct non−trivial normal subgroups of G and by our assumption, $MN = G$ and $M \cap N = 1$. This proves that $\kappa(M) = \kappa(N) = 2$ and so $G \cong M \times N \cong Z_2 \times Z_2$. Finally, we assume that $|E(\mathcal{N}(G))| = 1$. If $rank(G) = 1$, then G is a cyclic and since $\mathcal{N}(G)$ dose not have a triangle, $G \cong Z_4$. If $rank(G) = 2$ then $G \cong Z_2 \times Z_2$ and the normal graph of abelian groups with $rank \geq 3$ have at least one triangle, which is not possible. Conversely, it is clear that the normal graph of the abelian groups Z_p, p is prime, $Z_2 \times Z_2$ and Z_4 are bipartite.
3. Suppose the normal graph of a non-abelian and non−perfect finite group G is bipartite. By a similar argument as Part (2), we can assume that $|E(\mathcal{N}(G)| = 1$. Choose the conjugacy class x^G such that $x^G e^G$ is an edge in $\mathcal{N}(G)$. Then $N = e^G \cup x^G$ is a normal subgroup of G. If G is centerless then by [16, Theorem 2.1.](a), G is a Frobenius group with kernel N and its complement is abelian and by [16, Theorem 2.1.](d), $|G| = |N|(|N| - 1)$. Since G is centerless, x is not a central element of G and so $|N| > 2$. On the other hand, by our assumption $N = G'$ is an elementary abelian 2−subgroup of order 2^n and $2^n - 1$ is a Mersenne prime. This proves that $G \cong S_3$ or
$$G \cong \underbrace{Z_2 \times \cdots \times Z_2}_{n\ times} : Z_{2^n - 1},$$
as desired. Finally, if $Z(G) \neq 1$ then a simple argument leads to another contradiction.
Conversely, it is clear that the normal graph of the symmetric group S_3 and all finite simple groups are bipartite. Suppose G is a Frobenius group of order $|N|(|N| - 1)$, where N is a 2−decomposable normal 2−subgroup of G and $|N| - 1$ is a prime number. Since N is 2−decomposable normal 2−subgroup of G, it is elementary abelian group of order 2^α. If G has another proper non-trivial normal subgroup M. It is clear that $G \cong M \times N$ and since M, N are abelian subgroup of G, G is abelian. This contradiction shows that G has a unique proper non-trivial normal subgroup. Therefore, $\mathcal{N}(G)$ has a unique edge and some isolated vertices and so it is bipartite.

This proves the result. □

It is possible to find finite groups with bipartite normal graphs which are not simple, abelian and centerless. As an example, we consider the finite groups $SL(n, q)$. These groups are perfect except in the cases that $(n, q) = (2, 2)$ or $(2, 3)$. On the other hand, the special linear groups $SL(n, q)$ are simple if and only if $(n, q - 1) = 1$. By [14, Theorem 5.13 and 5.14], proper normal subgroups of $SL(n, q)$, for $n \geq 3$ or $n = 2$ and $q \geq 4$ are central. It is clear that $|Z(SL(n, q))| = (n, q - 1) = 2$ if and only if one of the following conditions are satisfied:

a. $2|n$, $4 \nmid n$ and $2 \nmid q$,
b. $4|n$, $2 \nmid q$ and $4 \nmid q - 1$.

This proves that if $(n, q - 1) = 1$ or the pair (n, q) satisfies one of the conditions a or b then the normal graph of the special linear groups $SL(n, q)$ will be bipartite.

Let n be a natural number with $n \geq 2$, and let q be a prime power such that $(n, q) \notin \{(2, 2), (2, 3), (3, 2)\}$. Then $SU(n, q^2)$ is perfect. By [8, Theorem 5, p. 70], all proper normal subgroups of $SU(n, q^2)$ are central. On the other hand, $|Z(SU(n, q^2))| = (n, q + 1) = 2$ if and only if one of the following conditions are satisfied:

c. $2|n$, $4 \nmid n$ and $2 \nmid q$,
d. $4|n$, $2 \nmid q$ and $4 \nmid q+1$.

Therefore, if $(n, q+1) = 1$ or the pair (n, q) satisfies one of the conditions c or d then the normal graph of the special unitary groups $SU(n, q^2)$ will be bipartite.

Question 3.10. *Is there any classification of perfect non-simple groups with bipartite normal graphs?*

Suppose G_1 and G_2 are finite groups and $\alpha : C(G_1) \longrightarrow C(G_2)$, $\beta : Irr(G_1) \longrightarrow Irr(G_2)$ are two bijections. We say that G_1 and G_2 have **identical character table** if the value of $\beta(\chi)$ on all the elements of the class $\alpha(K)$ is equal to $\chi(x)$, where $x \in K$. We shall also say that (α, β) is a **character table isomorphism** from G_1 to G_2. It is easy to see that if (α, β) is a character table isomorphism from G_1 to G_2 then $(\alpha^{-1}, \beta^{-1})$ is a character table isomorphism from G_2 to G_1.

Theorem 3.11. *Let G and H be finite groups with identical character table. Then $\mathcal{N}(G) \cong \mathcal{N}(H)$.*

Proof. Suppose G and H have identical character table and the pair (α, β) is a character table isomorphism from G to H. To prove the theorem, we show that the map $\alpha : V(\mathcal{N}(G)) \longrightarrow V(\mathcal{N}(H))$ defines a graph isomorphism from $\mathcal{N}(G)$ to $\mathcal{N}(H)$. To do this, we assume that x^G and y^G are adjacent in $\mathcal{N}(G)$. By definition $N_1 = \langle x^G, y^G \rangle \lhd G$. Suppose $N_1 = x_1^G \cup x_2^G \cup \cdots \cup x_r^G$ with $x_1^G = x^G$ and $x_r^G = y^G$, where x_i^G, $1 \leq i \leq r < \kappa(G)$, are distinct conjugacy classes of G. Define $N_2 = \alpha(x_1^G) \cup \cdots \cup \alpha(x_r^G)$. It is then obvious that N_2 is a normal subset of H and that $|N_2| = |N_1|$. We still must show that N_2 is a subgroup. There is a character ψ_1 of G (not necessarily irreducible) such that $N_1 = Ker(\psi_1)$, so the classes K of G in N_1 are exactly the classes such that if x is in K, then $\psi_1(x) = \psi_1(1)$. Now H has a character ψ_2 corresponding to ψ_1. To construct ψ_2, we assume that $\psi_1 = a_1\chi_1 + \cdots + a_t\chi_t$ such that $t = \kappa(G)$ and $\chi_1, \ldots, \chi_t \in Irr(G)$. Then $\psi_2 = a_1\beta(\chi_1) + \cdots + a_t\beta(\chi_t)$. We show that $\psi_1(x) = \psi_2(1)$. Since $x \in K \subset N_1$,

$$\begin{aligned} \psi_2(1) &= a_1\beta(\chi_1)(1) + \cdots + a_t\beta(\chi_t)(1) \\ &= a_1\chi_1(1) + \cdots + a_t\chi_t(1) \\ &= \psi_1(1) = \psi_1(x), \end{aligned}$$

as desired. Then because the character tables are identical, we see that N_2 is exactly the set of elements of H that lie in $Ker(\psi_2)$. Thus $N_2 = Ker(\psi_2)$ and this is a subgroup.

Since G and H have identical character table, $|G| = |H|$. Hence, $|N_2| = |N_1| < |G| = |H|$ which proves that $\alpha(x^G)$ and $\alpha(y^G)$ are adjacent in $\mathcal{N}(H)$. Finally, since $(\alpha^{-1}, \beta^{-1})$ is a character table isomorphism from H to G, the map α^{-1} preserves adjacency in $\mathcal{N}(H)$. This completes the proof. □

The converse of the previous theorem is not generally correct. For example, the simple groups $PSL(2, 8)$ and $PSL(2, 13)$ have exactly nine conjugacy classes and $\mathcal{N}(PSL(2, 8)) \cong \mathcal{N}(PSL(2, 13)) \cong \overline{K_9}$. On the other hand, these groups have different orders and so they don't have identical character tables. For non-simple groups, we can choose $G = SmallGroup(57, 1) \cong Z_{19} : Z_3$ and $H = SmallGroup(60, 7) \cong Z_{15} : Z_4$, where $SmallGroup(n, i)$ denotes the $i-th$ group of order n in the small group library of GAP and $H : K$ is the semi-direct product of a group H by the group K [17].

Question 3.12. *Are there finite groups G and H such that $|G| = |H|$, $\mathcal{N}(G) \cong \mathcal{N}(H)$ but G and H don't have identical character table?*

Example 3.13. *Suppose p is prime. In this example the normal graph of a non-abelian group of order p^3 is considered into account. The normal graph of dihedral group D_8 and quaternoin group Q_8 are obtained in Examples 2.1 and 2.2, respectively. So, it is enough to consider the case that p is odd.*

Following Conrad [5], we define:

$$
\begin{aligned}
Heis(Z_p) &= \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in Z_p \right\}, \\
G_p &= \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in Z_{p^2}, a \equiv 1 \mod p \right\} \\
&= \left\{ \begin{pmatrix} 1 + pm & b \\ 0 & 1 \end{pmatrix} : m, b \in Z_{p^2} \right\},
\end{aligned}
$$

where m has to be calculated in modulo p. By [5, Theorem 4.] every non-abelian group of order p^3 is isomorphic to G_p or $Heis(Z_p)$. Since these groups have identical character table, by Theorem 3.11, it is enough to obtain the normal graph of $Heis(Z_p)$. The group $Heis(Z_p)$ has exponent p containing a unique normal subgroup of order p, the center of $Heis(Z_p)$, and $p + 1$ normal subgroups of order p^2. Suppose H is a subgroup of order p^2 in $Heis(Z_p)$ and $Z = Z(Heis(Z_p))$. Then $Z \leqslant H$ and so H has exactly p $Heis(Z_p)-$conjugacy classes of length 1 and $p - 1$ $Heis(Z_p)-$conjugacy classes of length p. This proves that $\mathcal{N}(Heis(Z_p)) \cong Star_{p+2}[K_p, K_{p-1}, \ldots, K_{p-1}]$, where K_p is corresponding to the central vertex of $Star_{p+2}$.

The relationship between $X-$decomposable finite groups and the structure of normal graphs are investigated. Here, the notation $\omega(\Gamma)$ stands for the clique number of Γ which is defined as the number of vertices in a maximal clique and $E(p^n) \cong \underbrace{Z_p \times \cdots \times Z_p}_{n\ times}$.

Theorem 3.14. *Suppose G is a non-perfect finite group. Then the following hold:*

1. *If $\omega(\mathcal{N}(G)) = 3$ then the group G is isomorphic to one of the following groups:*
   (a) *One of the groups Z_6, D_8, Q_8, $Z_3 \times Z_3$, Z_9, S_4, $SmallGroup(20, 3)$, $SmallGroup(24, 3)$, $SmallGroup(36, 9)$,*
   (b) *A non-abelian group of order pq, p, q are primes and $q = \frac{p-1}{2}$,*
   (c) *The semi-direct product $Z_q \rtimes E(3^n)$ in which $q = \frac{3^n-1}{2}$ is prime.*
2. *If $\omega(\mathcal{N}(G)) = 4$ then G is isomorphic to one of the following groups:*
   (a) *One of the groups Z_8, $Z_2 \times Z_4$, S_5, Q_{12}, $Z_2 \times A_4$, D_{12}, $((Z_3 \times Z_3) : Q_8) : Z_3 = SmallGroup(216, 153)$, $((Z_5 \times Z_5) : Q_8) : Z_3 = SmallGroup(600, 150)$ and $(Z_7 : Z_3) : Z_2 = SmallGroup(42, 1)$.*
   (b) *A non-abelian group of order pq, p, q are primes and $q = \frac{p-1}{3}$,*
   (c) *A metabelian group of order $2^n(2^{\frac{n-1}{2}} - 1)$ in which n is odd positive integer and $2^{\frac{n-1}{2}} - 1$ is a Mersenne prime,*
   (d) *A metabelian group of order $2^n(2^{\frac{n}{3}} - 1)$, where $3|n$ and $\frac{n}{3} - 1$ is a Mersenne prime,*
   (e) *The semi-direct product $Z_q \rtimes E(2^n)$ in which $q = \frac{2^n-1}{3}$ is prime.*

Proof. Our main proof will consider two separate cases as follows:

1. $\omega(\mathcal{N}(G)) = 3$. In this case $\mathcal{N}(G) \cong K_3$, G is $\{1, 3\}-$decomposable or $\{1, 2, 3\}-$ decomposable. By Theorem 3.8(3), the case of $\mathcal{N}(G) \cong K_3$ cannot be occured and if G is $\{1, 3\}-$decomposable or $\{1, 2, 3\}-$decomposable then by [2, Theorem 4] and [3, Theorem], the proof will be completed.
2. $\omega(\mathcal{N}(G)) = 4$. In this case $\mathcal{N}(G) \cong K_4$, G is $\{1, 4\}-$decomposable, $\{1, 2, 4\}-$ decomposable, $\{1, 3, 4\}-$decomposable or $\{1, 2, 3, 4\}-$decomposable. If $\mathcal{N}(G) \cong K_4$ then G is isomorphic to D_{10}, Z_4, $Z_2 \times Z_2$ or A_4 which are not possible. Other cases follow from [2, Theorem 5], [9, Theorems 3.1 and 3.2], [1, Theorem] and [10, Main Theorem].

Hence the result. □

To characterize finite non-perfect groups in which the clique number of its normal graph is 5 we have to first characterize all $\{1,5\}-$, $\{1,2,5\}-$, $\{1,3,5\}-$, $\{1,4,5\}-$, $\{1,2,3,5\}-$, $\{1,2,4,5\}-$, $\{1,3,4,5\}-$ and $\{1,2,3,4,5\}-$decomposable non-perfect finite groups. The $\{1,5\}-$ decomposable non-perfect finite groups are characterized in [4], but with the best of our knowledge there is no characterization of $X-$decomposable non-perfect finite groups, where $\{1,5\} \subset X \subseteq \{1,2,3,4,5\}$. Therefore, the characterization of finite non-perfect groups G with $\omega(\mathcal{N}(G)) = 5$ will remain an open question. We end this paper by recording this open question.

Question 3.15. *Is there a characterization of finite groups with* $\omega(\mathcal{N}(G)) = 5$.

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References

[1] A. R. Ashrafi, On decomposability of finite groups, Journal of the Korean Mathematical Society 41 (2004) 479–487.
[2] A. R. Ashrafi, H. Sahraei, On finite groups whose every normal subgroup is a union of the same number of conjugacy classes, Vietnam Journal of Mathematics 30(3) (2002) 289–294.
[3] A. R. Ashrafi, G. Venkataraman, On finite groups whose every proper normal subgroup is a union of a given number of conjugacy classes, Proceedings of the Indian Academy of Sciences: Mathematical Sciences 114(3) (2004) 217–224.
[4] A. R. Ashrafi, Z. Yaoqing, On 5− and 6−decomposable finite groups, Mathematica Slovaca 53(4) (2003) 373–383.
[5] K. Conrad, Groups of order p^3, Preprint 2014.
[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, Atlas of Finite Groups, Oxford University Press, Eynsham, 1985.
[7] M. R. Darafsheh, N. S. Poursalavati, On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups, SUT Journal of Mathematics 37 (2001) 1–17.
[8] J. Dieudonné, Sur les Groupes Classiques, Hermann, Paris, 1973.
[9] X. Guo, J. Li, K. P. Shum, On finite X-decomposable groups for $X = \{1,2,4\}$, Siberian Mathematical Journal 53(3) (2012) 444–449.
[10] X. Guo, R. Chen, On finite X-decomposable groups for $X = \{1,2,3,4\}$, Bulletin of the Iranian Mathematical Society 40(5) (2014) 1243–1262.
[11] M. Hormozi, K. Rodtes, Symmetry classes of tensors associated with the semi-dihedral groups SD_{8n}, Colloquium Mathematicum 131 (2013) 59–67.
[12] I. M. Isaacs, Character Theory of Finite Groups, AMS Chelsea Publishing, Providence, RI, 2006.
[13] G. James, M. Liebeck, Representations and Characters of Groups, Second edition, Cambridge University Press, New York, 2001.
[14] C. Lanski, Concepts in Abstract Algebra, Books/Cole Series in Advanced Mathematics, Thompson Books/Cole, Belmont, CA, 2005.
[15] G. Sabidussi, Graph derivatives, Mathematische Zeitschrift 76 (1961) 385–401.
[16] M. Shahryari, M. A. Shahabi, Subgroups which are the union of two conjugacy classes, Bulletin of the Iranian Mathematical Society 25(1) (1999) 59–72.
[17] The GAP Team, GAP – Groups, Algorithms, and Programming, Version 4.7.5; 2014.