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On Some Deddens Subspaces of Banach Algebras

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Abstract. Let $\mathcal{A}$ be a Banach algebra with a unit e, and let $a \in \mathcal{A}$ be an invertible element. We define the following algebra:

$$\mathcal{B}_a^{loc} := \left\{x \in \mathcal{A} : \left\|a^n x a^{-n}\right\| \leq c_x n^{\alpha(x)} \text{ for some } \alpha(x) \geq 0 \text{ and } c_x > 0\right\}.$$

In this article we study some properties of this algebra; in particular, we prove that $\mathcal{B}_{e+p}^{loc} = \{x \in \mathcal{A} : px(e-p) = 0\}$, where p is an idempotent in $\mathcal{A}$. We also investigate the following Deddens subspace. Let $a, b \in \mathcal{A}$ be two elements. Fix any number α, $0 \leq \alpha < 1$, and consider the following subspace of $\mathcal{A}$:

$$D_{a,b}^{\alpha} := \{x \in \mathcal{A} : \|a^n x b^n\| = O(n^{\alpha}) \text{ as } n \to \infty\}.$$

Here we study some properties of the subspaces $D_{a,b}^{\alpha}$ and $D_{b,a}^{\alpha}$.

1. Introduction

In this article, the Deddens subspaces of Banach algebras with unit are introduced and their some properties are studied.

Let $\mathcal{A}$ be a Banach algebra with a unit e, and let $a, b \in \mathcal{A}$ be two elements such that b is invertible. We define the following subspace in $\mathcal{A}$, which we call following by Karaev and Pehlivan [8] the Deddens subspace:

$$D_{a,b} := \left\{x \in \mathcal{A} : \sup_{n \geq 0} \left\|a^n x b^{-n}\right\| =: c_x < +\infty\right\}.$$

Note that, when $a \in \mathcal{A}$ is also invertible, our definition coincides with the definition of Karaev and Pehlivan in [8], where was firstly introduced the notion of Deddens subspace. We also remark that when a is invertible and $b = a$, the concept of Deddens subspace coincides with the concept of Deddens algebra, introduced firstly by Karaev and Mustafayev in their paper [9].

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Let $\mathcal{L}(H)$ denote the Banach algebra of all bounded linear operators on the complex separable infinite-dimensional Hilbert space H. Recall that given an invertible operator $T \in \mathcal{L}(H)$, the study of operators A whose conjugation orbit $\{T^n A T^{-n}\}$ is bounded, that is, for $A \in \mathcal{L}(H)$ and T invertible,

$$\sup_{n\geq 0} \left\| T^n A T^{-n} \right\| < +\infty, \tag{1}$$

was initiated by Deddens in the 1970s when he gave a characterization of nest algebras in terms of the so-called Deddens algebra $\mathcal{B}_T$ (see Karaev and Mustafayev [9]), where $\mathcal{B}_T$ is the set of operators A in $\mathcal{L}(H)$ satisfying (1). It is easy to see actually that $\mathcal{B}_T$ is an algebra.

Recall that a family of subspaces of H is called a nest if it is totally ordered by inclusion.

Recall that given a nest $\mathcal{R}$ of subspaces on a separable Hilbert space H, Ringrose [15] introduced the concept of the associated nest algebra

$$\mathcal{N}_{\mathcal{R}} := \{X \in \mathcal{L}(H) : \text{ every } N \in \mathcal{R} \text{ is invariant for } X, \text{ i.e. } XN \subset N\}.$$

Nest algebras have been studied by many authors, e.g., [1], [2], [6], [10] and [11]. In particular, in [2], Deddens studied the following set and relationship to nest algebras: given an invertible operator A, let

$$\mathcal{B}_A := \left\{X \in \mathcal{L}(H) : \left\| A^k X A^{-k} \right\| \text{ bounded for } k = 0, 1, 2, ...\right\}.$$

For $X \in \mathcal{B}_A$, let $C_A(X) = \sup_{k\geq 0} \left\| A^k X A^{-k} \right\| < +\infty$. In general, $\mathcal{B}_A$ is an algebra that contains the commutant

$$\{A\}' := \{Y \in \mathcal{L}(H) : YA = AY\}$$

of A. If A is a positive invertible operator, then Deddens proved that $\mathcal{B}_A$ is equal to the nest algebra associated with the nest $\{E([0, a]) : a \geq 0\}$, where E is the spectral measure of A, and that $C_A(X) = \|X\|$. Conversely, every nest algebra arises in this manner. Note that the algebra $\mathcal{B}_A$ has been investigated previously in [3]: If $A \in \mathcal{L}(H)$ is of the form $A = \lambda I + N$, where $0 \neq \lambda \in \mathbb{C}$ is a complex number and N is a nilpotent operator (i.e., $N^n = 0$, $N^{n-1} \neq 0$ with some integer $n \geq 2$), then $\mathcal{B}_A = \{A\}'$. Furthermore, if H is finite dimensional, then the converse holds.

More general results are obtained by Karaev and Mustafayev [9], Karaev and Pehlivan [8], Drissi and Mbekhta [4, 5] and Mustafayev [12, 13], see also references therein.

In this paper, we consider the following two subspaces of a Banach algebra $\mathcal{A}$. Let $a, b \in \mathcal{A}$ be two elements. Fix any number $\alpha \in [0, 1)$. We define the following Deddens subspace of $\mathcal{A}$:

$$D^{\alpha}_{a,b} = \{x \in \mathcal{A} : \|a^n x b^n\| = O(n^{\alpha}) \text{ as } n \to \infty\}.$$

In some special situations, we give a complete description of the subspace $D^{\alpha}_{a,b}$, which improves the similar result in [8].

For any fixed invertible element $a \in \mathcal{A}$, we also introduce the following set:

$$\mathcal{B}^{loc}_a = \left\{x \in \mathcal{A} : \left\| a^n x a^{-n} \right\| \leq c_x n^{\alpha(x)} \text{ for some } \alpha(x) \geq 0 \text{ and some } c_x > 0\right\}.$$

It is easy to see that actually B^{loc}_a is an algebra. In case of $a = e + p$, where p is an idempotent element in $\mathcal{A}$, we prove that $\mathcal{B}^{loc}_{e+p} = \{x \in \mathcal{A} : px(e - p) = 0\}$, which improves a result in [8].

2. Some Properties of Deddens Algebras

If A is a nonzero operator in $\mathcal{L}(H)$, then a complex number λ is an extended eigenvalue of A (see Gürdal [7], and references therein) if there is a nonzero operator T such that

$$AT = \lambda TA. \tag{2}$$

We will denote by the symbol $ext(A)$ the set of extended eigenvalues of A, and the set of all extended eigenvectors corresponding to $\lambda \in ext(A)$ will be denoted as $Ext_\lambda(A)$. It is clear that always $1 \in ext(A)$ for every operator $A \in \mathcal{L}(H)$. In fact, one can take T being self of A, or the identity operator I on H.

Here we present a new simple property, and also only for completeness, some known properties of Deddens algebras. For more details the reader can be see and consult in [2].

Proposition 2.1. *Let $A \in \mathcal{L}(H)$ be an invertible operator, and let $\lambda \neq 0$ be an extended eigenvalue of A. Then $\mathcal{B}_A$ and $\mathcal{B}_{A^{-1}}$ are algebras such that $Ext_\lambda(A) \subset \mathcal{B}_A \cup \mathcal{B}_{A^{-1}}$, where $Ext_\lambda(A)$ is the set of extended eigenvectors corresponding to λ.*

Proof. As is shown already in Section 1, $\mathcal{B}_A$ is an algebra. Similarly $\mathcal{B}_{A^{-1}}$ is also an algebra.

Now let X be an arbitrary element of the set $Ext_\lambda(A)$. Then, in view of (2), $AX = \lambda XA$. Hence

$$A^n X = \lambda^n X A^n \tag{3}$$

for any integer $n > 0$. First, assume that $|\lambda| \leq 1$. Then by using (3), we have for any n that

$$\left\|A^n X A^{-n}\right\| = \|\lambda^n X\| \leq \|X\|,$$

that is $X \in \mathcal{B}_A$. Now assume that $|\lambda| > 1$. Then we have from (3) that

$$\left(\frac{1}{\lambda}\right)^n A^n X = X A^n,$$

and hence $\left(\frac{1}{\lambda}\right)^n X = A^{-n} X A^n$ for all $n \geq 0$. Therefore

$$\left\|A^{-n} X A^n\right\| = \left|\frac{1}{\lambda}\right|^n \|X\| \to 0 \text{ as } n \to \infty,$$

which implies that $X \in \mathcal{B}_{A^{-1}}$. The obtained inclusions shows that $X \in \mathcal{B}_A \cup \mathcal{B}_{A^{-1}}$, as desired. □

Corollary 2.2. $\{A\}' \subset \mathcal{B}_A \cup \mathcal{B}_{A^{-1}}$.

Proposition 2.3 ([2]). *If $A, A_1 \in \mathcal{L}(H)$ are two invertible operators such that $A = TA_1T^{-1}$ for some invertible operator $T \in \mathcal{L}(H)$ (i.e., A is similar to A_1), then $\mathcal{B}_A = \mathcal{B}_{TA_1T^{-1}} = T\mathcal{B}_{A_1}T^{-1}$. Also $\mathcal{B}^*_A = \mathcal{B}_{A^{*-1}}$.*

We remark that the same proof can be given for the algebras $\mathcal{B}^{loc}_A$ and $\mathcal{B}^{loc}_{A_1}$; we omit it.

Proposition 2.4 ([2]). *Suppose A is an invertible operator on $\overset{n}{\underset{i=1}{\oplus}} H$ of the form $A = A_1 \oplus A_2 \oplus ... \oplus A_n$. If $\|A_i\| \left\|A^{-1}_{i+1}\right\| < 1$ for $i = 1, 2, ..., n-1$, then*

$$\mathcal{B}_A = \left\{X = \left(X_{ij}\right) \in \mathcal{L}\left(\overset{n}{\underset{i=1}{\oplus}} H\right) : X_{ii} \in \mathcal{B}_{A_i} \text{ and } X_{ij=0} \text{ whenever } i > j\right\}.$$

That is, $\mathcal{B}_A$ is all upper triangular matrices whose diagonal entries belong to the corresponding $\mathcal{B}_{A_i}$.

Proposition 2.5 ([2]). *There is a projection π of $\mathcal{B}_A$ onto $\{A\}'$ that satisfies $\pi(A_1XA_2) = A_1\pi(X)A_2$ for $A_1, A_2 \in \{A\}'$. If there is a constant C such that $c_A(X) \leq C\|X\|$ for all $X \in \mathcal{B}_A$, then π is bounded in norm by C.*

The following theorem due to Deddens [2, Theorem 5], which characterizes the nest algebras in terms of the Deddens algebra $\mathcal{B}_A$.

Theorem 2.6 ([2]). *Let A be a positive invertible operator and let $\mathcal{R}(A)$ be the completion of the nest $\{E[0,\lambda] : \lambda \geq 0\}$. Then $\mathcal{B}_A = \mathcal{N}_{\mathcal{R}(A)}$. Conversely, if $\mathcal{R}$ is any complete nest of subspaces, then there exists a positive invertible operator A with $\mathcal{B}_A = \mathcal{N}_{\mathcal{R}}$. Furthermore, for $X \in \mathcal{B}_A$ one has that $\left\|A^k X A^{-k}\right\| \leq \|X\|$ for all $k = 0, 1, 2, ...$*

3. Description of Some Special Deddens Subspaces

Let $\mathcal{A}$ be a complex Banach algebra with unit e. An element $a \in \mathcal{A}$ is called quadratic if it satisfies

$$a^2 + \lambda a + \mu e = 0$$

for some complex numbers $\lambda, \mu \in \mathbb{C}$. Recall that an element $p \in \mathcal{A}$ is called an idempotent if $p^2 = p$, and p is called nilpotent if $p^n = 0$ for some $n \geq 2$; in particular, if $n = 2$, an element p is nilpotent of nilpotency order 2. It is trivial then that all idempotents and nilpotent elements of order 2 are quadratic elements.

In this section, we prove some new results for the special type of Deddens subspaces associated with the idempotent and nilpotent elements. Our results essentially improve the main results of the paper [8, Theorems 1 and 3]. Before giving the results, we define the following Deddens subspace. Let $a, b \in \mathcal{A}$ be two elements. Fix any number α, $0 \leq \alpha < 1$, and consider the following set:

$$D^{\alpha}_{a,b} := \{x \in \mathcal{A} : \|a^n x b^n\| = O(n^{\alpha}) \text{ as } n \to \infty\}. \tag{4}$$

It is easy to see that the sets $D^{\alpha}_{a,b}$ and $D^{\alpha}_{b,a}$ are subspaces of $\mathcal{A}$. We call these subspaces the Deddens subspaces. Note that when a and b are invertible, the notion of Deddens subspace coincides with the notion of Deddens algebra, introduced in [9].

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra with unit e. Let p be any idempotent and q a nilpotent of order* 2*. Let $D^{\alpha}_{e+p,e+q}$ be the Deddens subspace defined by* (4)*. Then we have:*
(i) $D^{\alpha}_{e+p,e+q} = \{x \in \mathcal{A} : px = xq\}$;
(ii) $D^{\alpha}_{e+q,e+p} = \{x \in \mathcal{A} : qx = xp\}$.

Proof. *(i)* For the proof, we will essentially use the method in [8]. Let us set

$$Int\{p, q\} := \{x \in \mathcal{A} : px = xq\}.$$

We will prove that $Int\{p, q\} = D^{\alpha}_{e+p,e+q}$. In fact, the inclusion $Int\{p, q\} \subset D^{\alpha}_{e+p,e+q}$ is trivial. Let us prove the reverse inclusion $D^{\alpha}_{e+p,e+q} \subset Int\{p, q\}$. Let $x \in D^{\alpha}_{e+p,e+q}$ be arbitrary. Denoting $p_1 := e + p$, $q_1 := e + q$ and

$$c_n := p_1^n x q_1^{-n},\ n \geq 0,$$

we have that

$$\|c_n\| \leq c_x n^{\alpha}. \tag{5}$$

Then we obtain

$$c_n q_1 = p_1^n x q_1^{-n} q_1 = p_1 \left(p_1^{n-1} x q_1^{-(n-1)}\right) = p_1 c_{n-1},$$

that is

$$c_n q_1 = p_1 c_{n-1}\ \ (n \geq 1). \tag{6}$$

It follows from (2) that

$$c_n q_2^n = p_1 c_0\ \ (n \geq 1), \tag{7}$$

and hence

$$c_n (e + q)^n = (e + p)^n c_0.$$

Since $p^2 = p$, it is easy to verify that $(e + p)^n = e + (2^n - 1) p$ for all $n \geq 1$. Also, it is easy to see that $(e + q)^{-n} = e - nq$ because $q^2 = 0$. By considering these in (7), we have

$$c_n = (e + (2^n - 1) p) x (e - nq),\ n = 1, 2, ...,$$

and therefore

$$c_n - x = (2^n - 1)\, px - nxq - n\,(2^n - 1)\, pxq \tag{8}$$

for all $n \geq 1$. By considering inequality (5), we have from (8) that

$$\begin{aligned}\left\|pxq\right\| &\leq \frac{\|c_n - x\|}{n\,(2^n - 1)} + \frac{\left\|px\right\|}{n} + \frac{\left\|xq\right\|}{2^n - 1}\\ &\leq \frac{\|c_n\| + \|x\|}{n\,(2^n - 1)} + \frac{\left\|px\right\|}{n} + \frac{\left\|xq\right\|}{2^n - 1}\\ &\leq c_x \frac{n^\alpha}{n\,(2^n - 1)} + \frac{\|x\|}{n\,(2^n - 1)} + \frac{\left\|px\right\|}{n} + \frac{\left\|xq\right\|}{2^n - 1}\\ &= c_x \frac{1}{n^{1-\alpha}\,(2^n - 1)} + \frac{\|x\|}{n\,(2^n - 1)} + \frac{\left\|px\right\|}{n} + \frac{\left\|xq\right\|}{2^n - 1} \to 0\end{aligned}$$

as $n \to \infty$, because $1 - \alpha > 0$. Thus, we have that $pxq = 0$, which implies that (see (8))

$$c_n - x = (2^n - 1)\, px - nxq.$$

Form this, by using again (5), we have by similar arguments that

$$\left\|px\right\| \leq \frac{\|c_n - x\|}{2^n - 1} + \frac{n}{2^n - 1}\left\|xq\right\| \to 0 \quad (n \to \infty)\,,$$

which implies obviously that $px = 0$. Hence, we have

$$c_n - x = -nxq,$$

from which

$$\left\|xq\right\| = \frac{\|c_n - x\|}{n} \leq c_x \frac{n^\alpha}{n} + \frac{\|x\|}{n} \to 0 \quad (n \to \infty)\,.$$

Thus $c_n - x = 0$ for all $n \geq 1$. In particular, $c_1 = x$, which means that

$$x = (e + p)\, x\, (e - q)\,.$$

Hence

$$(e + p)\, x = x\, (e + q)\,.$$

Therefore $px = xq$, which shows that $x \in Int\,\{p, q\}$, and hence $D^\alpha_{e+p,e+q} \subset Int\,\{p, q\}$ as desired.

(ii) The proof of *(ii)* is the same as in the case (*i*), and therefore we omit it.

The theorem is proven. □

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra with unit e. Let p be any idempotent and q be any nilpotent element of order* 2*. Then*

$$\left(D^\alpha_{e+p,e+q} \cap D^\alpha_{e+q,e+p}\right) \cap \{p\}' = \left(D^\alpha_{e+p,e+q} \cap D^\alpha_{e+q,e+p}\right) \cap \{q\}'\,;$$

here $\{t\}' := \{x \in \mathcal{A} : xt = tx\}$ is the commutant of the element $t \in \mathcal{A}$.

The proof is immediate from Theorem 3.1.

For any Banach algebra $\mathcal{A}$ with the idempotent p and with a unit e, we define the following set (see [8])

$$S_p := \{x \in \mathcal{A} : px(e-p) = 0\}.$$

It is easy to see that S_p is an algebra (S_p is a proper subalgebra of $\mathcal{A}$ if p is a nontrivial idempotent element, i.e., $p \neq 0$ and $p \neq e$). Indeed, let $x, y \in S_p$ be two arbitrary elements. Then we have

$$\begin{aligned} pxy(e-p) &= px(e-p+p)y(e-p) \\ &= [px(e-p) + pxp]\, y(e-p) \\ &= (px(e-p))\, y(e-p) + px(py(e-p)) = 0, \end{aligned}$$

because $px(e-p) = py(e-p) = 0$. This implies that $xy \in S_p$, as desired.

Let $a \in \mathcal{A}$ be any fixed invertible element. We define the following set:

$$\mathcal{B}_a^{loc} := \left\{x \in \mathcal{A} : \left\|a^n x a^{-n}\right\| \leq c_x n^{\alpha} \text{ for some } \alpha = \alpha(x) \geq 0 \text{ and some } c_x > 0\right\}.$$

It is elementary to show that actually $\mathcal{B}_a^{loc}$ is an algebra in $\mathcal{A}$. Indeed, let $x, y \in \mathcal{B}_a^{loc}$ and $\lambda, \mu \in \mathbb{C}$ be arbitrary. Then we have:

$$\begin{aligned} \left\|a^n(\lambda x + \mu y)a^{-n}\right\| &= \left\|a^n(\lambda x)a^{-n} + a^n(\mu y)a^{-n}\right\| \\ &\leq |\lambda|\, c_x n^{\alpha(x)} + \left|\mu\right| c_y n^{\beta(y)} \\ &\leq \left(|\lambda|\, c_x + \left|\mu\right| c_y\right) n^{\max\{\alpha(x),\beta(y)\}} \\ &= c_{\lambda x+\mu y} n^{\max\{\alpha(x),\beta(y)\}}, \end{aligned}$$

where $c_{\lambda x+\mu y} := |\lambda|\, c_x + \left|\mu\right| c_y$; this shows that $\mathcal{B}_a^{loc}$ is a (linear) subspace in $\mathcal{A}$. Now we verify that $\mathcal{B}_a^{loc}$ is closed with respect to the multiplication operation. Indeed, we have :

$$\begin{aligned} \left\|a^n xy a^{-n}\right\| &= \left\|(a^n x a^{-n})(a^n y a^{-n})\right\| \\ &\leq (c_x n^{\alpha})\left(c_y n^{\beta}\right) = c_x c_y n^{\alpha+\beta}, \end{aligned}$$

and hence, since $c_x c_y > 0$ and $\alpha + \beta = \alpha(x) + \beta(x) \geq 0$, this inequality means that $xy \in \mathcal{B}_a^{loc}$. Thus, $\mathcal{B}_a^{loc}$ is an algebra.

Our next result describes the algebra S_p in terms of the Deddens type algebra $\mathcal{B}_{e+p}^{loc}$.

Theorem 3.3. *Let $\mathcal{A}$ be a Banach algebra with an idempotent p and with a unit e. Then $\mathcal{B}_{e+p}^{loc} = S_p$.*

Proof. It is easy to check that

$$(e+p)^{-1} = e - \frac{1}{2}p.$$

Then we have that

$$(e+p)^n = e + (2^n - 1)\, p \tag{9}$$

and

$$(e+p)^{-n} = e + \left(\frac{1}{2^n} - 1\right) p \tag{10}$$

for all $n \geq 0$. By setting for any element $x \in \mathcal{A}$

$$c_n := (e+p)^n x (e+p)^{-n} \quad (n \geq 0),$$

and using equalities (9) , (10), we obtain

$$c_n = x + \left(\frac{1}{2^n} - 1\right) xp + (2^n - 1)\, px + (2^n - 1)\left(\frac{1}{2^n} - 1\right) pxp \tag{11}$$

for all $n = 0, 1, 2, ...$ Since for every $x \in \mathcal{B}^{loc}_{e+p}$ $\|c_n\| \leq c_x n^{\alpha}$ for some $c_x > 0$ and $\alpha = \alpha(x) \geq 0$, by using an elementary fact that $\frac{n^{\alpha}}{2^n} \to 0$ $(n \to \infty)$ for any $\alpha \geq 0$, we have for every $x \in \mathcal{B}^{loc}_{e+p}$ that

$$\lim_{n\to\infty} \frac{1}{(2^n - 1)\left(\frac{1}{2^n} - 1\right)} (c_n - x) = 0.$$

Therefore we obtain from (11) that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \left(\frac{1}{2^{-n} - 1} px + \frac{1}{2^n - 1} xp + pxp\right) \\ &= pxp - px, \end{aligned}$$

and hence $px(e - p) = 0$, that is $x \in S_p$.

Conversely,let $x \in S_p$ be arbitrary. Then, by considering that $px = pxp$, we have from (11) that

$$\begin{aligned} c_n &= x + \left(\frac{1}{2^n} - 1\right) xp + (2^n - 1)\, px + \frac{2^n - 1}{2^n} pxp - (2^n - 1)\, pxp \\ &= x + \left(\frac{1}{2^n} - 1\right) xp + \frac{2^n - 1}{2^n} pxp. \end{aligned}$$

Whence

$$\|c_n\| \leq \|x\| + \left\|xp\right\| + \left\|pxp\right\| =: c_x < +\infty$$

for all $n \geq 0$, which means that $x \in \mathcal{B}^{loc}_{e+p}$ (with $\alpha(x) = 0$). This proves the theorem. □

Let $\mathcal{A} = \mathcal{B}(H)$, the Banach algebra of all bounded linear operators acting in the Hilbert space H, and let $Q \subset B(H)$ be a proper subset. We denote by $LatQ$ the lattice of (closed) subspaces E of H invariant under Q, i.e.,

$$LatQ := \{E \subset H : AE \subset E \text{ for all } A \in Q\}.$$

Recall also that $AlgLatQ$ is the following algebra in $B(H)$, which is important in the theory of invariant subspaces:

$$AlgLatQ := \{A \in B(H) : AE \subset E \text{ for all } E \in LatQ\}.$$

By considering the well known (and simple) fact that $AE \subset E$ if and only if $(I - P_E)\, AP_E = 0$ (see, for instance, Radjavi and Rosenthal [14]), the following is an immediate corollary of Theorem 3.3

Corollary 3.4. *$AlgLatQ = \underset{E \in LatQ}{\cap} \mathcal{B}^{loc}_{2I - P_E}$, where $P_E : H \to E$ is the orthogonal projector.*

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