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The Morita Contexts and Galois Extensions for Weak Hopf Group Coalgebras

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Abstract. In this paper, we mainly construct a Morita context associated to a weak Hopf group coalgebra and as application we also compute the Morita contexts of weak Hopf group Galois extensions.

1. The first section

Recently, Turaev introduced the notion of a Hopf π-coalgebra which played important role in homotopy quantum field theory and generalized the notion of a Hopf algebra. In [7] the authors proved that a Hopf group coalgebra is essentially a Hopf algebra in a symmetric monoidal category called Turaev module category. Another important generalization of Hopf algebras is in the introduction of weak Hopf algebras in [4] and [5]. The axioms are the same as ones for Hopf algebras, but the multiplicativity of the counit and comultiplicativity of the unit are replaced by weaker axioms. These naturally lead to the introduction of weak Hopf algebras in Turaev module category called weak Hopf group coalgebras in [9-12].

The basic idea of Morita contexts is to find a relationship between two rings via their modules, which is weaker than Morita equivalence, but is still strong enough to enalbe the rings to share some properties. Cohen and Fischman formalized the relationship between $A\sharp H$ and A^H via a Morita context [3] which extended earlier works of [1] and [2]. Using the theory of Hopf group coalgebras, [13] generalized Morita contexts in the setting of group corings. Dually, the authors also investigated some properties of (weak)Hopf group algebras in [14-15].

The aim of this paper is to establish a Morita context for a weak Hopf group coalgebra, as example we also compute the Morita contexts associated to weak Hopf Galois extensions which generalize the result in [6]. The organization of this paper is as follows. In section 1, we recall some basic notions and present a summary of important properties concerning weak Hopf group coalgebras. In section 2 we construct a Morita context for weak Hopf group coalgebras. In section 3, as example we compute a Morita context related to weak Hopf group Galois extensions.

Conventions We work over field k. We denote by i the unit of the group π. The identity map from any k-space V to itself is denoted by id_V. We follow Virelizier's paper for the terminologies on π-coalgebras and π-comodules. (see ref.[7])

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Definition 1.1. *([7]) A π-coalgebra* over k is a family $C = \{C_\alpha\}_{\alpha\in\pi}$ of k-spaces endowed with a family k-linear maps $\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta,\ c \mapsto c_{1\alpha} \otimes c_{2\beta}\}_{\alpha,\beta\in\pi}$ and a k-linear map $\varepsilon : C_i \to k$ such that

$$(\Delta_{\alpha,\beta} \otimes id_{C_\gamma})\Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma})\Delta_{\alpha,\beta\gamma},\ \alpha,\beta,\gamma \in \pi; \tag{1}$$

$$(id_{C_\alpha} \otimes \varepsilon)\Delta_{\alpha,i} = (\varepsilon \otimes id_{C_\alpha})\Delta_{i,\alpha} = id_{C_\alpha},\ \alpha \in \pi. \tag{2}$$

Definition 1.2. *([9]) Let $C = \{C_\alpha\}_{\alpha\in\pi}$ be a π-coalgebra and M a k-space with a family of k-linear maps*

$$\rho = \{\rho_\alpha : M \to M \otimes C_\alpha,\ \rho_\alpha(m) = m_{[0]} \otimes m_{([1],\alpha)},\ m \in M\}_{\alpha\in\pi}.$$

We call M a right C-π-comodulelike object if the following conditions are satisfied:

$$(\rho_\alpha \otimes id_{C_\beta}) \circ \rho_\beta = (id_M \otimes \Delta_{\alpha,\beta}) \circ \rho_{\alpha\beta},\ \alpha,\beta \in \pi; \tag{3}$$

$$(id_M \otimes \varepsilon) \circ \rho_i = id_M. \tag{4}$$

Definition 1.3. *([10]) A weak Hopf π-coalgebra* is a family of algebras $\{H_\alpha, \mu_\alpha, \eta_\alpha\}_{\alpha\in\pi}$ and also a π-coalgebra $\{H, \Delta = \{\Delta_{\alpha,\beta}\}, \varepsilon\}_{\alpha,\beta\in\pi}$ endowed with a family $S = \{S_\alpha : H_{\alpha^{-1}} \to H_\alpha\}_{\alpha\in\pi}$ of k-linear maps called an antipode satisfying the following conditions for any $\alpha,\beta \in \pi$,

$$\Delta_{\alpha,\beta}(hg) = \Delta_{\alpha,\beta}(h)\Delta_{\alpha,\beta}(g), \varepsilon(1_i) = 1_i,\ h,g \in H_{\alpha\beta}; \tag{5}$$

$$\Delta^2(1_{\alpha\beta\gamma}) = 1_{\alpha\beta\gamma 1\alpha} \otimes 1_{\alpha\beta\gamma 2\beta} \otimes 1_{\alpha\beta\gamma 3\gamma} = 1_{\alpha\beta 1\alpha} \otimes 1_{\beta\gamma 1\beta}1_{\alpha\beta 2\beta} \otimes 1_{\beta\gamma 2\gamma} = 1_{\alpha\beta 1\alpha} \otimes 1_{\alpha\beta 2\beta}1_{\beta\gamma 1\beta} \otimes 1_{\beta\gamma 2\gamma}; \tag{6}$$

$$\varepsilon(xyz) = \varepsilon(xy_{1i})\varepsilon(y_{2i}z) = \varepsilon(xy_{2i})\varepsilon(y_{1i}z),\ x,y,z \in H_i; \tag{7}$$

$$x_{1\alpha}S_\alpha(x_{2\alpha^{-1}}) = \varepsilon(1_{1i}x)1_{2\alpha},\ x \in H_\alpha; \tag{8}$$

$$S_{\alpha^{-1}}(x_{1\alpha})x_{2\alpha^{-1}} = 1_{1\alpha^{-1}}\varepsilon(x1_{2i}),\ x \in H_\alpha; \tag{9}$$

$$S_{\alpha^{-1}}(x_{1\alpha})x_{2\alpha^{-1}}S_{\alpha^{-1}}(x_{3\alpha}) = S_{\alpha^{-1}}(x_\alpha),\ x \in H_\alpha. \tag{10}$$

We call H finite-dimensional if for any $\alpha \in \pi$, H_α is finitely dimensional. In this paper we always assume that H is finite-dimensional.

Let H be a weak Hopf π-coalgebra, we define

$$\sqcap^r_\alpha : H_i \to H_\alpha, \sqcap^r_\alpha(h) = h^{r_\alpha} = 1_{1\alpha}\varepsilon(h1_{2i}),\ h \in H_i;$$

$$\sqcap^l_\alpha : H_i \to H_\alpha, \sqcap^l_\alpha(h) = h^{l_\alpha} = 1_{2\alpha}\varepsilon(1_{1i}h),,\ h \in H_i.$$

Let H be a weak Hopf π-coalgebra and M a right H- π-comodulelike object. The *coinvants of H on M* are the elements of the k-space

$$M^{coH} = \{m = (m_\alpha)_{\alpha\in\pi} \mid m_{[0]} \otimes m_{([1],\alpha)} = m_{[0]} \otimes m_{([1],i)}{}^{l_\alpha}\}.$$

Lemma 1.4. *Let H be a weak Hopf π-coalgebra, then for any $\alpha,\beta \in \pi$, $h,g \in H_i$, $l \in H_{\alpha\beta}$,*

$$(h^{l_i}g)^{l_\alpha} = h^{l_\alpha}g^{l_\alpha};\ \ (hg^{r_i})^{r_\alpha} = h^{r_\alpha}g^{r_\alpha}. \tag{11}$$

$$l_{1\alpha}g^{r_\alpha} \otimes l_{2\beta} = l_{1\alpha} \otimes l_{2\beta}S_\beta(g^{r_{\beta^{-1}}}),\ l_{1\alpha} \otimes g^{l_\beta}l_{2\beta} = S_\alpha(g^{l_{\alpha^{-1}}})l_{1\alpha} \otimes l_{2\beta}. \tag{12}$$

$$h^{l_\alpha}g^{r_\alpha} = g^{r_\alpha}h^{l_\alpha},\ \varepsilon(h^{r_i}g) = \varepsilon(hg),\ \varepsilon(hg^{l_i}) = \varepsilon(hg). \tag{13}$$

Proof. In fact, for any $\alpha,\beta\in\pi,\ h,g\in H_i,\ l\in H_{\alpha\beta},\ h^*\in H^*_\alpha,\ g^*\in H^*_\beta$,

$$h^{l_\alpha}g^{l_\alpha}=\varepsilon(1_{1i}h)1_{2\alpha}\varepsilon(1'_{1i}g)1'_{2\alpha}\overset{(14)^{[11]}}{=}\varepsilon(S_i^{-1}(1_{2i}{}^{l_i})h)\varepsilon(1_{1i}g)1_{3\alpha}=\varepsilon(S_i^{-1}(1_{2i})h)\varepsilon(1_{1i}g)1_{3\alpha}$$
$$=\varepsilon(1_{2i}h)\varepsilon(S_i(1_{3i})g)S_\alpha(1_{1\alpha^{-1}})\overset{(8)^{[11]}}{=}\varepsilon(1_{2i}h^{l_i}g)S_\alpha(1_{1\alpha^{-1}})\overset{(11)^{[11]}}{=}(h^{l_i}g)^{l_\alpha};$$

$$(h^*\otimes g^*)(l_{1\alpha}g^{r_\alpha}\otimes l_{2\beta})=h^*(l_{1\alpha}g^{r_\alpha})g^*(l_{2\beta})=h^*(l_{1\alpha}g^{r_\alpha}{}_{1\alpha})g^*(l_{3\beta})\varepsilon(l_{2i}g^{r_\alpha}{}_{2i})\overset{(16)^{[11]}}{=}h^*(l_{1\alpha}1_{1\alpha})g^*(l_{3\beta})\varepsilon(l_{2i}1_{2i}g^{r_i})$$
$$=h^*(l_{1\alpha})g^*(l_{3\beta})\varepsilon(l_{2i}g^{r_i})\overset{(8)^{[11]}}{=}h^*(l_{1\alpha})g^*(l_{2\beta}S_i(g^{r_i})^{l_\beta})=h^*(l_{1\alpha})g^*(l_{2\beta}S_\beta(g^{r_{\beta^{-1}}}))=(h^*\otimes g^*)(l_{1\alpha}\otimes l_{2\beta}S_\beta(g^{r_{\beta^{-1}}}));$$

$$h^{l_\alpha}g^{r_\alpha}=\varepsilon(1_{1i}h)1_{2\alpha}\varepsilon(g1'_{2i})1'_{1\alpha}\overset{(6)}{=}\varepsilon(1_{1i}h)1'_{1\alpha}1_{2\alpha}\varepsilon(g1'_{2i})=g^{r_\alpha}h^{l_\alpha};$$

$$\varepsilon(h^{r_i}g)=\varepsilon(1_{1i}g)\varepsilon(h1_{2i})\overset{(7)}{=}\varepsilon(hg).$$

Similarly one can obtain the other equlities. □

Definition 1.5. *([10]) Let H be a weak Hopf π-coalgebra. An algebra A is called a weak right H-π-comodulelike algebra* if A is a right H-π-comodulelike object via $\rho=\{\rho_\alpha\}_{\alpha\in\pi}$ such that the following conditions are satisfied:

$$\rho_\alpha(ab)=\rho_\alpha(a)\rho_\alpha(b),\ a,b\in A,\ \alpha\in\pi; \tag{14}$$

$$1_{[0]}\otimes 1_{([1],\alpha\beta)1\alpha}\otimes 1_{([1],\alpha\beta)2\beta}=1_{[0]}\otimes 1'_{1\alpha}1_{([1],\alpha)}\otimes 1'_{2\beta},\ \alpha,\beta\in\pi. \tag{15}$$

Lemma 1.6. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. Then*

$$a_{[0]}\otimes a_{([1],i)}{}^{l_\alpha}=1_{[0]}a\otimes 1_{([1],\alpha)};\ a_{[0]}\otimes a_{([1],i)}{}^{r_\alpha}=a1_{[0]}\otimes S_\alpha(1_{([1],\alpha^{-1})}),\ \alpha\in\pi,\ a\in A. \tag{16}$$

Proof. In fact, for any $\alpha\in\pi$ and $a\in A$, we compute

$$a_{[0]}\otimes a_{([1],i)}{}^{l_\alpha}=a_{[0]}\otimes\varepsilon(1_{1i}a_{([1],i)})1_{2\alpha}=1'_{[0]}a_{[0]}\otimes\varepsilon(1_{1i}1'_{([1],i)}a_{([1],i)})1_{2\alpha}\overset{(15)}{=}1'_{[0]}a_{[0]}\otimes\varepsilon(1'_{([1],i)1i}a_{([1],i)})1'_{([1],i)2\alpha}$$
$$=1'_{[0]}a\otimes 1'_{([1],\alpha)}.$$

Similarly we can obtain the other. □

Remark 1.7. *Eq. (15) is equivalent to the first part of Eq. (16).*

In fact, if $a_{[0]}\otimes a_{([1],i)}{}^{l_\beta}=1_{[0]}a\otimes 1_{([1],\beta)}$, *then we compute*

$$1_{[0]}\otimes 1_{([1],\beta\gamma)1\beta}\otimes 1_{([1],\beta\gamma)2\gamma}\overset{(3)}{=}1_{[0][0]}\otimes 1_{[0]([1],\beta)}\otimes 1_{([1],\gamma)}\overset{(16)}{=}1_{[0][0]}\otimes 1_{[0]([1],\beta)}\otimes 1_{([1],i)}{}^{l_\gamma}$$
$$=1_{[0][0]}\otimes 1_{[0]([1],\beta)}\varepsilon(1'_{1i}1_{([1],i)})\otimes 1'_{2\gamma}\overset{(3)}{=}1_{[0]}\otimes 1_{([1],\beta)1\beta}\varepsilon(1'_{1i}1_{([1],\beta)2i})\otimes 1'_{2\gamma}\overset{(8)^{[11]}}{=}1_{[0]}\otimes 1'_{1i}1_{([1],\beta)i}\otimes 1'_{2\gamma}.$$

Definition 1.8. *Let H be a weak Hopf π-coalgebra. A left (resp. right) π-integral for H* is a family of *k*-linear forms $\lambda=(\lambda_\alpha)_{\alpha\in\pi}\in\prod_{\alpha\in\pi}H_\alpha{}^*$ such that

$$f\lambda_\alpha=f^{l_\beta}\lambda_{\beta\alpha},\ (\text{resp. }\lambda_\alpha f=\lambda_{\alpha\beta}f^{r_\beta}),\ \alpha,\beta\in\pi,\ f\in H_\beta{}^*, \tag{17}$$

where $f^{l_\beta},f^{r_\beta}\in H_i{}^*,\ f^{l_\beta}(h)=f(h^{l_\beta}),\ f^{r_\beta}(h)=f(h^{r_\beta}),\ h\in H_i$.

In addition, if for all $\alpha\in\pi,\ S^*_\alpha(\lambda_\alpha)=\lambda_{\alpha^{-1}}$, then we call $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ *a S-fixed left (resp. right) integral for H.*

Lemma 1.9. *Let H be a weak Hopf π-coalgebra. Then the following assertions are satisfied for all $\alpha\in\pi,\ f\in H_\alpha{}^*,\ g\in H_\beta{}^*$:*

$$\begin{aligned}&\varepsilon_{1i}f^{l_\alpha}\otimes\varepsilon_{2i}=f^{l_\alpha}\varepsilon_{1i}\otimes\varepsilon_{2i};\\&\varepsilon_{1i}\otimes\varepsilon_{2i}f^{r_\alpha}=\varepsilon_{1i}\otimes f^{r_\alpha}\varepsilon_{2i}.\end{aligned} \tag{18}$$

$$f^{l_\alpha}{}_{1i} \otimes f^{l_\alpha}{}_{2i} = \varepsilon_{1i} f^{l_\alpha} \otimes \varepsilon_{2i};$$
$$f^{r_\alpha}{}_{1i} \otimes f^{r_\alpha}{}_{2i} = \varepsilon_{1i} \otimes \varepsilon_{2i} f^{r_\alpha}. \tag{19}$$

$$f_{1\alpha} S_i^{*-1}(g^{l_\beta}) \otimes f_{2\alpha} = f_{1\alpha} \otimes f_{2\alpha} g^{l_\beta}. \tag{20}$$

$$f_{1\alpha}{}^{l_\alpha} \otimes f_{2\alpha} = S_i^*(\varepsilon_{1i}) \otimes \varepsilon_{2i} f;$$
$$f_{1\alpha} \otimes f_{2\alpha}{}^{l_\alpha} = \varepsilon_{1i} f \otimes \varepsilon_{2i};$$
$$f_{1\alpha}{}^{r_\alpha} \otimes f_{2\alpha} = \varepsilon_{1i} \otimes f \varepsilon_{2i};$$
$$f_{1\alpha} \otimes f_{2\alpha}{}^{r_\alpha} = f \varepsilon_{1i} \otimes S_i^*(\varepsilon_{2i}). \tag{21}$$

$$f_{1\alpha}{}^{l_\alpha} f_{2\alpha} = f;$$
$$f_{1\alpha} f_{2\alpha}{}^{r_\alpha} = f. \tag{22}$$

$$S_i^{*-1}(f^{l_\alpha}{}_{2i}) f^{l_\alpha}{}_{1i} = f^{l_\alpha};$$
$$S_i^{*-1}(f^{l_\alpha}) = S_{\alpha^{-1}}^{*-1}(f)^{r_{\alpha^{-1}}};$$
$$S_i^{*-1}(f^{r_\alpha}) = S_{\alpha^{-1}}^{*-1}(f)^{l_{\alpha^{-1}}}. \tag{23}$$

Proof. The first parts of (18)-(21) are obtained in the following way. Applying $x \otimes y \in H_i \otimes H_i$ to both sides of them, we get:

$$(18)\ LHS = \varepsilon_{1i}(x_{1i}) f^{l_\alpha}(x_{2i}) \varepsilon_{2i}(y) = \varepsilon(x_{1i} y) f(x_{2i}{}^{l_\alpha}) \overset{(8)^{[11]}}{=} f((xy^{l_i})^{l_\alpha}) \overset{(7)^{[11]}}{=} f((xy)^{l_\alpha}) = f(1_{2\alpha}) \varepsilon(1_{1i} xy)$$
$$\overset{(5)}{=} f(1_{2\alpha}) \varepsilon(1_{1i} x_{1i}) \varepsilon(x_{2i} y) = f(x_{1i}{}^{l_\alpha}) \varepsilon_{1i}(x_{2i}) \varepsilon_{2i}(y) = RHS;$$

$$(19)\ LHS = f^{l_\alpha}{}_{1i}(x) f^{l_\alpha}{}_{2i}(y) = f((xy)^{l_\alpha}) \overset{(7)^{[11]}}{=} f((xy^{l_i})^{l_\alpha}) \overset{(8)^{[11]}}{=} \varepsilon(x_{1i} y) f(x_{2i}{}^{l_\alpha}) = \varepsilon_{1i}(x_{1i}) f^{l_\alpha}(x_{2i}) \varepsilon_{2i}(y) = RHS;$$

$$(20)\ LHS = f(x_{1i} y) g(S_i^{-1}(x_{2i})^{l_\beta}) \overset{Th.1.8^{[11]}}{=} f(x_{1i} y) g(S_{\beta^{-1}}^{-1}(x_{2i}{}^{r_\beta^{-1}})) \overset{(15)^{[11]}}{=} f(x 1_{1i} y) g(1_{2\beta})$$
$$\overset{(14)^{[11]}}{=} f(x y_{1i}) g(y_{2i}{}^{l_\beta}) = RHS;$$

$$(21)\ LHS = f(x^{l_\alpha} y) = \varepsilon(1_{1i} x) f(1_{2\alpha} y) \overset{(12)^{[11]}}{=} \varepsilon(S_i^{-1}(y_{1i}{}^{l_i}) x) f(y_{2\alpha}) \overset{Th.1.8^{[11]}}{=} \varepsilon(S_i^{-1}(y_{1i}) x) f(y_{2\alpha})$$
$$= \varepsilon(S_i(x) y_{1i}) f(y_{2\alpha}) = RHS,$$

by similar computations we can establish the rest of (18)-(21).

We end by proving the first parts of (22)-(23). Applying $x \in H_i$ to both sides of (22) and (23), we get

$$(22)\ \ LHS = (f_{1\alpha}{}^{l_\alpha} f_{2\alpha})(x) = f(x_{1i}{}^{l_\alpha} x_{2i}) = \varepsilon(1_{1i} x_{1i}) f(1_{2\alpha} x_{2\alpha}) = f(x) = RHS;$$
$$(23)\ \ LHS = f(S_i^{-1}(x)^{l_\alpha}) \overset{Th.1.8^{[11]}}{=} f(S_{\alpha^{-1}}^{-1}(x^{r_{\alpha^{-1}}}) = RHS,$$

the rest of (22)-(23) can be analogously checked. □

Lemma 1.10. *Let H be a weak Hopf π-coalgebra and $\lambda = (\lambda_\alpha)_{\alpha \in \pi}$ a left integral for H (resp. $\zeta = (\zeta_\alpha)_{\alpha \in \pi}$ a right integral). Then the following conclusions hold for all $\alpha, \beta \in \pi$, $f \in H_\beta{}^*$, $m \in H_\alpha{}^*$, $g \in H_{\alpha^{-1}}{}^*$ and $h \in H_{\alpha\beta}$:*

$$\zeta_{\alpha 1\alpha} f_{1\beta} \otimes \zeta_{\alpha 2\alpha} f_{2\beta} = \zeta_{\alpha\beta 1\alpha\beta} \otimes \zeta_{\alpha\beta 2\alpha\beta} f^{r_\beta}. \tag{24}$$

$$(m\lambda_\beta)_{1\alpha\beta} \otimes (m\lambda_\beta)_{2\alpha\beta} = m^{l_\alpha} \lambda_{\alpha\beta 1\alpha\beta} \otimes \lambda_{\alpha\beta 2\alpha\beta}. \tag{25}$$

$$\lambda_{\alpha\beta 1\alpha\beta} \otimes g \lambda_{\alpha\beta 2\alpha\beta} = S_{\alpha^{-1}}^*(g) \lambda_{\beta 1\beta} \otimes \lambda_{\beta 2\beta},\ (resp.\ \zeta_{\alpha 1\alpha} f \otimes \zeta_{\alpha 2\beta} = \zeta_{\alpha\beta 1\alpha\beta} \otimes \zeta_{\alpha\beta 2\alpha\beta} S_{\beta^{-1}}^*(f)). \tag{26}$$

$$(S_{\alpha^{-1}}^*(\lambda_{\alpha^{-1}}))(h_{1\alpha}) f(h_{2\beta}) = S_{(\alpha\beta)^{-1}}^* \lambda_{(\alpha\beta)^{-1}}(h_{1\alpha\beta}) f(h_{2i}{}^{r_\beta}), \tag{27}$$

that is, $S^*(\lambda) = (S_{\alpha^{-1}}^*(\lambda_{\alpha^{-1}}))_{\alpha \in \pi}$ is a right integral.

Proof. Applying $x \otimes y \in H_{\alpha\beta} \otimes H_{\alpha\beta}$ into two sides of (24)-(25), we have

$$(24)\ LHS = \zeta_\alpha(x_{1\alpha}y_{1\alpha})f(x_{2\beta}y_{2\beta}) \stackrel{(17)}{=} \zeta_{\alpha\beta}(x_{1\alpha\beta}y_{1\alpha\beta})f((x_{2i}y_{2i})^{r_\beta}) \stackrel{(15)^{[11]}}{=} \zeta_{\alpha\beta}(xy1_{1\alpha\beta})f(S_\beta(1_{2\beta^{-1}}))$$
$$\stackrel{(15)^{[11]}}{=} \zeta_{\alpha\beta}(xy_{1\alpha\beta})f({y_{2i}}^{r_\beta}) = RHS.$$

$$(25)\ LHS = m(x_{1\alpha}y_{1\alpha})\lambda_\beta(x_{2\beta}y_{2\beta}) \stackrel{(17)}{=} m((x_{1i}y_{1i})^{l_\alpha})\lambda_{\alpha\beta}(x_{2\alpha\beta}y_{2\alpha\beta}) \stackrel{(7)^{[11]}}{=} m((x_{1i}{y_{1i}}^{l_i})^{l_\alpha})\lambda_{\alpha\beta}(x_{2\alpha\beta}y_{2\alpha\beta})$$
$$\stackrel{(8)^{[11]}}{=} \varepsilon(x_{1i}y_{1i})m({x_{2i}}^{l_\alpha})\lambda_{\alpha\beta}(x_{3\alpha\beta}y_{2\alpha\beta}) = m^{l_\alpha}(x_{1i})\lambda_{\alpha\beta 1\alpha\beta}(x_{2\alpha\beta})\lambda_{\alpha\beta 2\alpha\beta}(y) = RHS.$$

$$(26)\ LHS = \varepsilon_{1i}\lambda_{\alpha\beta 1\alpha\beta} \otimes g\varepsilon_{2i}\lambda_{\alpha\beta 2\alpha\beta} \stackrel{(21)}{=} {g_{1\alpha^{-1}}}^{r_{\alpha^{-1}}}\lambda_{\alpha\beta 1\alpha\beta} \otimes g_{2\alpha^{-1}}\lambda_{\alpha\beta 2\alpha\beta} = S^*_{\alpha^{-1}}(g_{1\alpha^{-1}})g_{2\alpha^{-1}}\lambda_{\alpha\beta 1\alpha\beta} \otimes g_{3\alpha^{-1}}\lambda_{\alpha\beta 2\alpha\beta}$$
$$\stackrel{(25)}{=} S^*_{\alpha^{-1}}(g_{1\alpha^{-1}}){g_{2\alpha^{-1}}}^{l_{\alpha^{-1}}}\lambda_{\beta 1\beta} \otimes \lambda_{\beta 2\beta} = S^*_{\alpha^{-1}}(g)\lambda_{\beta 1\beta} \otimes \lambda_{\beta 2\beta} = RHS.$$

$$(27)\ LHS = \lambda_{\alpha^{-1}}(S_{\alpha^{-1}}(h_{1\alpha}))f(h_{2\beta}) = \lambda_{\alpha^{-1}}(S_{(\alpha\beta)^{-1}}(h_{\alpha\beta})_{2\alpha^{-1}})S^{*-1}_{\beta^{-1}}(f)(S_{(\alpha\beta)^{-1}}(h_{\alpha\beta})_{1\beta^{-1}})$$
$$\stackrel{(17)}{=} \lambda_{(\alpha\beta)^{-1}}(S_{(\alpha\beta)^{-1}}(h_{1\alpha\beta}))f(S^{-1}_{\beta^{-1}}(S_i(h_{2i})^{l_{\beta^{-1}}})) \stackrel{Th.1.8^{[11]}}{=} \lambda_{(\alpha\beta)^{-1}}(S_{(\alpha\beta)^{-1}}(h_{1\alpha\beta}))f({h_{2i}}^{r_\beta}) = RHS.$$

□

Definition 1.11. *Let $H = \oplus_{\alpha\in\pi}H_\alpha$ be a weak Hopf algebra such that H is a π-graded algebra and each component $(H_\alpha, \Delta_\alpha, \varepsilon_\alpha)$ is a coalgebra verifying $\Delta_{\alpha\beta}(hg) = \Delta_\alpha(h)\Delta_\beta(g)$ for any $h \in H_\alpha$ and $g \in H_\beta$. An algebra A is called a left H-π-module algebra* if A is a left H-module with a module structure decomposition H_α-module via $\bullet_\alpha : H_\alpha \otimes A \to A,\ h \otimes a \mapsto h\bullet_\alpha a$ for all $\alpha \in \pi$, such that the following assertions are satisfied:

$$h\bullet_\alpha(ab) = (h_{1\alpha}\bullet_\alpha a)(h_{2\alpha}\bullet_\alpha b),\ h \in H_\alpha,\ a, b \in A. \tag{28}$$

$$h\bullet_\alpha 1 = h^{l_\alpha}\bullet_\alpha 1,\ h \in H_\alpha. \tag{29}$$

Example 1.12. *(1) Let H be a finitely dimensional weak Hopf π-coalgebra. Then it is easy to prove that $H^* = \oplus_{\alpha\in\pi}{H_\alpha}^*$ is a weak Hopf algbra such that each component $({H_\alpha}^*, {\mu_\alpha}^*, {\eta_\alpha}^*)$ is a coalgebra verifying ${\mu_{\alpha\beta}}^*(fg) = ({\mu_\alpha}^*f)({\mu_\beta}^*g)$ for any $f \in {H_\alpha}^*$ and $g \in {H_\beta}^*$.*

(2) Let H be a finitely dimensional weak Hopf π-coalgebra. Then it is easy to show that an algebra A is a left H^*-π-module algebra if and only if it is a right H-π-comodulelike algebra. In fact, if A is a left H^*-π-module algebra, as in the weak Hopf case, A becomes a right weak H-π-comdulelike algebra via $\rho = \{\rho_\alpha(a) = u^*_\alpha\bullet_\alpha a \otimes u_\alpha\}_{\alpha\in\pi}$, where u^*_α and u_α are dual basis in ${H_\alpha}^*$ and H_α. Firstly we have the fact $u^*_{\alpha\beta} \otimes u_{\alpha\beta 1\alpha} \otimes u_{\alpha\beta 2\beta} = u^*_\alpha u^*_\beta \otimes u_\alpha \otimes u_\beta$ and $u^*_{\alpha 1\alpha} \otimes u^*_{\alpha 2\alpha} \otimes u_\alpha = u^*_\alpha \otimes u^*_\alpha \otimes u_\alpha u_\alpha$. Now we show that A is a right weak H-π-comdulelike algebra. Indeed, for any $a, b \in A,\ \alpha, \beta \in \pi$, we have

$$a_{[0]} \otimes a_{([1],\alpha\beta)1\alpha} \otimes a_{([1],\alpha\beta)2\beta} = u^*_{\alpha\beta}\bullet_{\alpha\beta} a \otimes u_{\alpha\beta 1\alpha} \otimes u_{\alpha\beta 2\beta} = u^*_\alpha\bullet_\alpha(u^*_\beta\bullet_\beta a) \otimes u_\alpha \otimes u_\beta = a_{[0][0]} \otimes a_{[0]([1],\alpha)} \otimes a_{([1],\beta)};$$
$$\varepsilon(a_{([1],i)})a_{[0]} = u^*_i\bullet_i a\varepsilon(u_i) = \varepsilon\bullet_i a = a;$$
$$(ab)_{[0]} \otimes (ab)_{([1],\alpha)} = u^*_\alpha\bullet_\alpha(ab) \otimes u_\alpha = (u^*_{\alpha 1\alpha}\bullet_\alpha a)(u^*_{\alpha 2\alpha}\bullet_\alpha b) \otimes u_\alpha = (u^*_\alpha\bullet_\alpha a)(u^*_\alpha\bullet_\alpha b) \otimes u_\alpha u_\alpha = a_{[0]}b_{[0]} \otimes a_{([1],\alpha)}b_{([1],\alpha)};$$
$$1_{[0]}a \otimes 1_{([1],\alpha)} = (u^*_\alpha\bullet_\alpha 1)a \otimes u_\alpha \stackrel{(29)}{=} ({u^*_\alpha}^{l_\alpha}\bullet_\alpha 1)a \otimes u_\alpha \stackrel{(28)}{=} (\varepsilon_{1\alpha}{u^*_\alpha}^{l_\alpha}\bullet_\alpha 1)(\varepsilon_{2\alpha}\bullet_\alpha a) \otimes u_\alpha = ({u^*_\alpha}^{l_\alpha}{}_{1\alpha}\bullet_\alpha 1)({u^*_\alpha}^{l_\alpha}{}_{2\alpha}\bullet_\alpha a) \otimes u_\alpha$$
$$\stackrel{(28)}{=} {u^*_\alpha}^{l_\alpha}\bullet_\alpha a \otimes u_\alpha = u^*_i\bullet_i a \otimes {u_i}^{l_\alpha} = a_{[0]} \otimes {a_{([1],i)}}^{l_\alpha},$$

where the last equlity comes from the fact that ${u^*_\alpha}^{l_\alpha} \otimes u_\alpha = u^*_i \otimes {u_i}^{l_\alpha}$ for all $\alpha \in \pi$. In fact, applying $h \otimes f \in H_i \otimes {H_\alpha}^*$ into the two sides we have $LHS = {u^*_\alpha}^{l_\alpha}(h)f(u_\alpha) = u^*_\alpha(h^{l_\alpha})f(u_\alpha) = f(h^{l_\alpha}) = RHS$.

Conversely, if A is a right weak H-π-comdulelike algebra, then A becomes a left H^*-π-module algebra via

$$\bullet = \{f\bullet_\alpha a = f(a_{([1],\alpha)})a_{[0]},\ f \in {H_\alpha}^*,\ a \in A\}_{\alpha\in\pi}.$$

In fact, for any $f \in H_\alpha{}^*$, $g \in H_\beta{}^*$, $a, b \in A$, $\alpha, \beta \in \pi$ we have,

$$f \bullet_\alpha (g \bullet_\beta a) = g(a_{([1],\beta)}) f(a_{[0]([1],\alpha)}) a_{[0][0]} \stackrel{(3)}{=} g(a_{([1],\alpha\beta)2\beta}) h(a_{([1],\alpha\beta)1\alpha}) a_{[0]} = (fg)(a_{([1],\alpha\beta)}) a_{[0]} = (fg) \bullet_{\alpha\beta} a;$$
$$\varepsilon \bullet_i a = \varepsilon(a_{([1],i)}) a_{[0]} = a;$$
$$f \bullet_\alpha (ab) = f((ab)_{([1],\alpha)})(ab)_{[0]} \stackrel{(14)}{=} f(a_{([1],\alpha)} b_{([1],\alpha)}) a_{[0]} b_{[0]} = f_{1\alpha}(a_{([1],\alpha)}) a_{[0]} f_{2\alpha}(b_{([1],\alpha)}) a_{[0]} = (f_{1\alpha} \bullet_\alpha a)(f_{2\alpha} \bullet_\alpha b);$$
$$f \bullet_\alpha 1 = f(1_{([1],\alpha)}) 1_{[0]} \stackrel{(16)}{=} f(1_{([1],i)}{}^{l_\alpha}) 1_{[0]} = f^{l_\alpha} \bullet_\alpha 1.$$

Proposition 1.13. *Let H be a weak Hopf π-coalgebra and A a right weak H-π-comdulelike algebra. Then $A \sharp_{H^{*l}} H^* = \oplus_{\alpha \in \pi} (A \sharp_{H_\alpha{}^{*l}} H_\alpha{}^*)$ is an associative algebra with the unit $1 \sharp \varepsilon$, where the multiplication defined by $(a \sharp f)(b \sharp g) = a(f_{1\alpha} \bullet_\alpha b) \sharp f_{2\alpha} g$ and A is a right $H_\alpha{}^{*l}$-module via the action given by $a \leftharpoonup f^{l_\alpha} = S_i^{*-1}(f^{l_\alpha}) \bullet_i a$, $a, b \in A$, $f \in H_\alpha{}^*$, $g \in H_\beta{}^*$.*

Proof. Firstly we have the following identity

$$\begin{aligned} a \leftharpoonup f^{l_\alpha} &= S_i^{*-1}(f^{l_\alpha}) \bullet_i a \stackrel{(28)}{=} (S_i^{*-1}(f^{l_\alpha})_{1i} \bullet_i a)(S_i^{*-1}(f^{l_\alpha})_{2i} \bullet_i 1) \stackrel{(19)}{=} (\varepsilon_{1i} \bullet_i a)(\varepsilon_{2i} S_i^{*-1}(f^{l_\alpha}) \bullet_i 1) \\ &\stackrel{(28)}{=} a(S_i^{*-1}(f^{l_\alpha}) \bullet_i 1) \stackrel{Th.1.8^{[11]}}{=} a(f^{l_\alpha} \bullet_i 1). \end{aligned} \tag{30}$$

Now we prove that the multiplication is reasonable. In fact, for any $f \in H_\alpha{}^*$, $g \in H_\beta{}^*$, $l \in H_\gamma{}^*$, $a, b \in A$, $\alpha, \beta \in \pi$ we have,

$$\begin{aligned} &(a \leftharpoonup f^{l_\alpha} \sharp l)(b \sharp g) = a(f^{l_\alpha} \bullet_i 1)(l_{1\gamma} \bullet_\gamma b) \sharp l_{2\gamma} g \stackrel{(28)}{=} a(\varepsilon_{1i} f^{l_\alpha} \bullet_i 1)(\varepsilon_{2i} l_{1\gamma} \bullet_\gamma b) \sharp l_{2\gamma} g \stackrel{(19)}{=} a(f^{l_\alpha}{}_{1i} \bullet_i 1)(f^{l_\alpha}{}_{2i} l_{1\gamma} \bullet_\gamma b) \sharp l_{2\gamma} g \\ &\stackrel{(28)}{=} a(f^{l_\alpha} l_{1\gamma} \bullet_\gamma b) \sharp l_{2\gamma} g = (a \sharp f^{l_\alpha} l)(b \sharp g); \end{aligned}$$

$$\begin{aligned} &(a \sharp l)(b \sharp f^{l_\alpha} g) = a(l_{1\gamma} \bullet_\gamma b) \sharp l_{2\gamma} f^{l_\alpha} g \stackrel{(20)}{=} a(l_{1\gamma} S_i^{*-1}(f^{l_\alpha}) \bullet_\gamma b) \sharp l_{2\gamma} g \stackrel{(23)}{=} a(l_{1\gamma} S_{\alpha^{-1}}^{*-1}(f)^{r_{\alpha^{-1}}} \bullet_\gamma b) \sharp l_{2\gamma} g \\ &\stackrel{(28)}{=} a(l_{1\gamma} S_{\alpha^{-1}}^{*-1}(f)^{r_{\alpha^{-1}}}{}_{1i} \bullet_\gamma b)(l_{2\gamma} S_{\alpha^{-1}}^{*-1}(f)^{r_{\alpha^{-1}}}{}_{2i} \bullet_\gamma 1) \sharp l_{3\gamma} g \stackrel{(19)}{=} a(l_{1\gamma} \varepsilon_{1i} \bullet_\gamma b)(l_{2\gamma} \varepsilon_{2i} S_{\alpha^{-1}}^{*-1}(f)^{r_{\alpha^{-1}}} \bullet_\gamma 1) \sharp l_{3\gamma} g \\ &= a(l_{1\gamma} \bullet_\gamma b)(l_{2\gamma} S_{\alpha^{-1}}^{*-1}(f)^{r_{\alpha^{-1}}} \bullet_\gamma 1) \sharp l_{3\gamma} g \stackrel{(29)}{=} a(l_{1\gamma} \bullet_\gamma b)(l_{2\gamma} f^{l_\alpha} \bullet_\gamma 1) \sharp l_{3\gamma} g = (a \sharp l)(b \leftharpoonup f^{l_\alpha} \sharp g). \end{aligned}$$

Next we have to show that $A \sharp H^*$ is a unital associative algebra. In fact, for any $f \in H_\alpha{}^*$, $g \in H_\beta{}^*$, $l \in H_\gamma{}^*$, $a, b, c \in A$, $\alpha, \beta \in \pi$,

$$[(a \sharp f)(b \sharp g)](c \sharp l) = a(f_{1\alpha} \bullet_\alpha b)(f_{2\alpha} g_{1\beta} \bullet_{\alpha\beta} c) \sharp f_{3\alpha} g_{2\beta} l \stackrel{(17)}{=} a(f_{1\alpha} \bullet_\alpha (b g_{1\beta} \bullet_\beta c)) \sharp f_{2\alpha} g_{2\beta} l = (a \sharp f)[(b \sharp g)(c \sharp l)];$$

$$(a \sharp f)(1 \sharp \varepsilon) = a(f_{1\alpha} \bullet_\alpha 1) \sharp f_{2\alpha} = a \sharp f_{1\alpha}{}^{l_\alpha} f_{2\alpha} \stackrel{(22)}{=} a \sharp f;$$

$$(1 \sharp \varepsilon)(a \sharp f) = \varepsilon_{1i} \bullet_i a \sharp \varepsilon_{2i} f \stackrel{(21)'}{=} \varepsilon_{1i} \bullet_i a \sharp \varepsilon_{2i}{}^{l_i} f = (\varepsilon_{1i} \bullet_i a)(\varepsilon_{2i} \bullet_i 1) \sharp f \stackrel{(28)}{=} a \sharp f.$$

Thus we complete the proof. □

For simplicity we denote $a \sharp f$ by af for any $a \in A$ and $f \in H_\alpha{}^*$, and hence we have the following assertions.

Lemma 1.14. *Let H be a weak Hopf π-coalgebra and A a right weak H-π-comdulelike algebra. Then for all $a \in A$, $b \in A^{coH}$ and $f \in H_\alpha{}^*$,*

$$fa = (f_{1\alpha} \bullet_\alpha a) f_{2\alpha}, \tag{31}$$

$$fb = bf, \tag{32}$$

$$f_{2\alpha}(S_{\alpha^{-1}}^{*-1}(f_{1\alpha}) \bullet_\alpha a) = af. \tag{33}$$

Proof. In fact, for any $a \in A,\ b \in A^{coH}$ and $f \in H_\alpha{}^*$, we have

$$fa = (1\sharp f)(a\sharp\varepsilon) = (f_{1\alpha}\bullet_\alpha a)\sharp f_{2\alpha} = (f_{1\alpha}\bullet_\alpha a)f_{2\alpha};$$

$$fb = (f_{1\alpha}\bullet_\alpha b)f_{2\alpha} = f(b_{([1],\alpha)})b_{[0]}f_{2\alpha} = f_{1\alpha}{}^{l_\alpha}(b_{([1],i)})b_{[0]}f_{2\alpha} = (f_{1\alpha}{}^{l_\alpha}\bullet_i b)f_{2\alpha} \overset{(21)}{=} (S_i^*(\varepsilon_{1i})\bullet_i b)\varepsilon_{2i}f_{2\alpha} \overset{(21)}{=} (\varepsilon_{1i}{}^{l_i}\bullet_i b)\varepsilon_{2i}f_{2\alpha}$$
$$= (\varepsilon_{1i}\bullet_i b)\varepsilon_{2i}f_{2\alpha} = bS_i^*(\varepsilon_{1i})\varepsilon_{2i}f = bf;$$

$$f_{2\alpha}(S^{*-1}_{\alpha^{-1}}(f_{1\alpha})\bullet_{\alpha^{-1}}a) = ((f_{2\alpha}S^{*-1}_{\alpha^{-1}}(f_{1\alpha}))\bullet_i a)f_{3\alpha} \overset{(23)}{=} (S^{*-1}_{\alpha^{-1}}(f_{1\alpha})^{r_{\alpha^{-1}}}\bullet_i a)f_{2\alpha} = af_{1\alpha}{}^{l_\alpha}f_{2\alpha} \overset{(22)}{=} af.$$

□

Let H be a weak Hopf π-coalgebra and A a right weak H-π-comdulelike algebra. Then A^{coH} is a subalgebra of A and so A is both a left A^{coH}-module and right A^{coH}-module via the multiplication. Directly from Lemma 1.6 it is clear that $1 \in A^{coH}$. In fact, for any $a, b \in A^{coH}$,

$$(ab)_{[0]} \otimes (ab)_{([1],\alpha)} = a_{[0]}b_{[0]} \otimes a_{([1],i)}{}^{l_\alpha}b_{([1],i)}{}^{l_\alpha} \overset{(11)}{=} a_{[0]}b_{[0]} \otimes (a^{l_i}_{([1],i)}b_{([1],i)})^{l_\alpha} = a_{[0]}b_{[0]} \otimes (a_{([1],i)}b_{([1],i)})^{l_\alpha}$$
$$= (ab)_{[0]} \otimes (ab)_{([1],i)}{}^{l_\alpha},$$

thus $ab \in A^{coH}$.

Lemma 1.15. *Let H be a weak Hopf π-coalgebra and A a right weak H-π-comdulelike algebra. Then*

(1) A is a left and right π-graded $A\sharp H^*$-module under the graded action given by for all $a, b \in A,\ f \in H_\alpha{}^*$,

$$(a\sharp f) \rhd b = a(f\bullet_\alpha b), \quad b \lhd (a\sharp f) = S^{*-1}_{\alpha^{-1}}(f)\bullet_{\alpha^{-1}}(ba).$$

(2) A is both A^{coH}-$A\sharp H^*$-bimodule and $A\sharp H^*$-A^{coH}-bimodule, where A is A^{coH}-module via the multiplication.

Proof. To prove the statement *(1)* we firstly claim that the above actions are reasonable. In fact, for any $a, b \in A,\ f \in H_\alpha{}^*,\ g \in H_\beta{}^*$, we have

$$(a \leftharpoonup f^{l_\alpha}\sharp g) \rhd b = (a \leftharpoonup f^{l_\alpha})(g\bullet_\beta b) \overset{(30)}{=} a(f^{l_\alpha}\bullet_i 1)(g\bullet_\beta b) \overset{(28)}{=} a(\varepsilon_{1i}f^{l_\alpha}\bullet_i 1)(\varepsilon_{2i}g\bullet_\beta b) \overset{(19)}{=} a(f^{l_\alpha}{}_{1i}\bullet_i 1)(f^{l_\alpha}{}_{2i}g\bullet_\beta b)$$
$$\overset{(28)}{=} a(f^{l_\alpha}g\bullet_\beta b) = (a\sharp f^{l_\alpha}g) \rhd b;$$

$$b \lhd (a \leftharpoonup f^{l_\alpha}\sharp g) = g(S^{-1}_{\beta^{-1}}(b_{([1],\beta^{-1})}a_{([1],\beta^{-1})}1_{[0]([1],\beta^{-1})}))f(1_{([1],i)}{}^{l_\alpha})b_{[0]}a_{[0]}1_{[0][0]}$$
$$\overset{(15)}{=} g(S^{-1}_{\beta^{-1}}(b_{([1],\beta^{-1})}a_{([1],\beta^{-1})}1_{([1],\beta^{-1})}1'_{1\beta^{-1}}))f(1'_{2i}{}^{l_\alpha})b_{[0]}a_{[0]}1_{[0]} = g(S^{-1}_{\beta^{-1}}(b_{([1],\beta^{-1})}a_{([1],\beta^{-1})}1'_{1\beta^{-1}}))f(1'_{2i}{}^{l_\alpha})b_{[0]}a_{[0]}$$
$$\overset{(15)[11]}{=} g(S^{-1}_{\beta^{-1}}((b_{([1],\beta^{-1})}a_{([1],\beta^{-1})})_{1\beta^{-1}}))f(S^{-1}_{\alpha^{-1}}((b_{([1],\beta^{-1})}a_{([1],\beta^{-1})})_{2i}{}^{r_{\alpha^{-1}}}))b_{[0]}a_{[0]}$$
$$\overset{(Th.1.8)[11]}{=} g(S^{-1}_{\beta^{-1}}(b_{([1],\beta^{-1})}a_{([1],\beta^{-1})})_{2\beta^{-1}})f(S^{-1}_{\beta^{-1}}(b_{([1],\beta^{-1})}a_{([1],\beta^{-1})})_{1i}{}^{l_\alpha})b_{[0]}a_{[0]}$$
$$= (f^{l_\alpha}g)(S^{-1}_{\beta^{-1}}(b_{([1],\beta^{-1})}a_{([1],\beta^{-1})}))b_{[0]}a_{[0]} = b \lhd (a\sharp f^{l_\alpha}g).$$

Secondly we prove that A exactly is a left and right $A\sharp H^*$-module. In fact, for any $a, b, c \in A,\ f \in H_\alpha{}^*,\ g \in H_\beta{}^*$, we have

$$[(a\sharp f)(b\sharp g)] \rhd c = (a(f_{1\alpha}\bullet_\alpha b)\sharp f_{2\alpha}g) \rhd c = af_{1\alpha}(b_{([1],\alpha)})f_{2\alpha}(c_{([1],\alpha\beta 1\alpha)})g(c_{([1],\alpha\beta 2\beta)})b_{[0]}c_{[0]}$$
$$\overset{(3)}{=} af_{1\alpha}(b_{([1],\alpha)})f_{2\alpha}(c_{[0]([1],\alpha)})g(c_{([1],\beta)})b_{[0]}c_{[0][0]} = (a\sharp f) \rhd ((b\sharp g)] \rhd c);$$

$$(1\sharp\varepsilon) \rhd c = \varepsilon(c_{([1],i)})c_{[0]} = c;$$

$$c \lhd [(a\sharp f)(b\sharp g)] = f_{2\alpha}(S^{-1}_{\alpha^{-1}}(c_{([1],(\alpha\beta)^{-1}2\alpha^{-1})}a_{([1],(\alpha\beta)^{-1}2\alpha^{-1})}b_{[0]([1],(\alpha\beta)^{-1}2\alpha^{-1})}))f_{1\alpha}(b_{([1],\alpha)})$$
$$\times g(S^{-1}_{\beta^{-1}}(c_{([1],(\alpha\beta)^{-1}1\beta^{-1})}a_{([1],(\alpha\beta)^{-1}1\beta^{-1})}b_{[0]([1],(\alpha\beta)^{-1})1\beta^{-1}}))c_{[0]}a_{[0]}b_{[0][0]}$$
$$\overset{(3)}{=} f(S^{-1}_{\alpha^{-1}}(b_{([1],\beta^{-1})2i}{}^{l_{\alpha^{-1}}})S^{-1}_{\alpha^{-1}}(c_{([1],(\alpha\beta)^{-1}2\alpha^{-1})}a_{([1],(\alpha\beta)^{-1}2\alpha^{-1})}))g(S^{-1}_{\beta^{-1}}(c_{([1],(\alpha\beta)^{-1}1\beta^{-1})}a_{([1],(\alpha\beta)^{-1})1\beta^{-1}}b_{([1],\beta^{-1})1\beta^{-1}}))c_{[0]}a_{[0]}b_{[0]}$$
$$\overset{(14)[11]}{=} f(S^{-1}_{\alpha^{-1}}(1_{2\alpha^{-1}})S^{-1}_{\alpha^{-1}}(c_{([1],(\alpha\beta)^{-1}2\alpha^{-1})}a_{([1],(\alpha\beta)^{-1}2\alpha^{-1})}))g(S^{-1}_{\beta^{-1}}(c_{([1],(\alpha\beta)^{-1}1\beta^{-1})}a_{([1],(\alpha\beta)^{-1})1\beta^{-1}}1_{1\beta^{-1}}b_{([1],\beta^{-1})}))c_{[0]}a_{[0]}b_{[0]}$$

$= f(S^{-1}_{\alpha^{-1}}(c_{([1],(\alpha\beta)^{-1}2\alpha^{-1})}a_{([1],(\alpha\beta)^{-1}2\alpha^{-1})}))g(S^{-1}_{\beta^{-1}}(c_{([1],(\alpha\beta)^{-1}1\beta^{-1})}a_{([1],(\alpha\beta)^{-1})1\beta^{-1}}b_{([1],\beta^{-1})}))c_{[0]}a_{[0]}b_{[0]}$
$= (c_{[0]}a_{[0]}) \triangleleft (b\sharp g)f(S^{-1}_{\alpha^{-1}}(c_{([1],\alpha^{-1})}a_{([1],\alpha^{-1})})) = (c \triangleleft (a\sharp f)) \triangleleft (b\sharp g);$

$c \triangleleft (1\sharp\varepsilon) = \varepsilon(S_i^{-1}(c_{([1],i)}))c_{[0]} = \varepsilon(c_{([1],i)})c_{[0]} = c.$

To prove the statement *(2)* we only need to show that A is a bimodule. In fact, for any $a, b \in A,\ f \in H_\alpha{}^*,\ x \in A^{coH}$, we have

$$(xa) \triangleleft (b\sharp f) = f(S^{-1}_{\alpha^{-1}}(x_{([1],\alpha^{-1})}a_{([1],\alpha^{-1})}b_{([1],\alpha^{-1})}))x_{[0]}a_{[0]}b_{[0]} \overset{(Th.1.8)^{[11]}}{=} f(S^{-1}_{\alpha^{-1}}(S^{-1}_{\alpha}(x_{([1],i)}{}^{r_\alpha})a_{([1],\alpha^{-1})}b_{([1],\alpha^{-1})}))x_{[0]}a_{[0]}b_{[0]}$$
$$\overset{(15)}{=} f(S^{-1}_{\alpha^{-1}}(1_{([1],\alpha^{-1})}a_{([1],\alpha^{-1})}b_{([1],\alpha^{-1})}))x1_{[0]}a_{[0]}b_{[0]} = f(S^{-1}_{\alpha^{-1}}(a_{([1],\alpha^{-1})}b_{([1],\alpha^{-1})}))xa_{[0]}b_{[0]} = x(a \triangleleft (b\sharp f));$$

$$(b\sharp f) \triangleright (ax) = ba_{[0]}x_{[0]}f(a_{([1],\alpha)}x_{([1],\alpha)}) = ba_{[0]}x_{[0]}f(a_{([1],\alpha)}x_{([1],i)}{}^{l_\alpha}) \overset{(15)}{=} ba_{[0]}1_{[0]}xf(a_{([1],\alpha)}1_{([1],\alpha)})$$
$$= ba_{[0]}xf(a_{([1],\alpha)}) = ((b\sharp f) \triangleright a)x.$$

Thus we complete the proof. □

Lemma 1.16. *Let H be a weak Hopf π-coalgebra with S-fixed left integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ and A a right weak H-π-comdulelike algebra and A a right weak H-π-comdulelike algebra. Then for any $a \in A,\ f \in H_\beta{}^*$,*

$$\lambda_\alpha af = \lambda_\alpha(S^{*-1}_{\beta^{-1}}(f)\bullet_{\beta^{-1}}a). \tag{34}$$

Proof. Firstly by (26) and $S^*_\alpha(\lambda_\alpha) = \lambda_{\alpha^{-1}}$, we have

$$S^*_{\alpha^{-1}\beta}(\lambda_{\alpha^{-1}\beta2\alpha^{-1}\beta}) \otimes fS^*_{\alpha^{-1}\beta}(\lambda_{\alpha^{-1}\beta1\alpha^{-1}\beta}) = S^*_\beta(f)S^*_{\alpha^{-1}}(\lambda_{\alpha^{-1}2\alpha^{-1}}) \otimes S^*_{\alpha^{-1}}(\lambda_{\alpha^{-1}1\alpha^{-1}}),$$

applying $S^{*-1}_{\alpha^{-1}\beta} \otimes S^{*-1}_{\alpha^{-1}}$ to both sides of the above equality we get

$$\lambda_{\alpha^{-1}\beta2\alpha^{-1}\beta} \otimes \lambda_{\alpha^{-1}\beta1\alpha^{-1}\beta}S^{*-1}_\beta(f) = \lambda_{\alpha^{-1}2\alpha^{-1}}f \otimes \lambda_{\alpha^{-1}1\alpha^{-1}}. \tag{35}$$

For any $a \in A,\ f \in H_\beta{}^*$, we compute

$$\lambda_\alpha af \overset{(33)}{=} \lambda_\alpha f_{2\beta}(S^{*-1}_{\beta^{-1}}(f_{1\beta})\bullet_\beta a) \overset{(31)}{=} (\lambda_{\alpha1\alpha}f_{2\beta}S^{*-1}_{\beta^{-1}}(f_{1\beta})\bullet_\alpha a)\lambda_{\alpha2\alpha}f_{3\beta} \overset{(24)}{=} (\lambda_{\alpha1\alpha}S^{*-1}_{\beta^{-1}}(f_{1\beta})\bullet_{\alpha\beta^{-1}}a)\lambda_{\alpha2\alpha}f_{2\beta}{}^{r_\beta}$$
$$\overset{(35)}{=} (\lambda_{\alpha1\alpha}S_i^{*-1}(f_{2\beta}{}^{r_\beta})S^{*-1}_{\beta^{-1}}(f_{1\beta})\bullet_{\alpha\beta^{-1}}a)\lambda_{\alpha2\alpha} \overset{(23)}{=} (\lambda_{\alpha1\alpha}S^{*-1}_{\beta^{-1}}(f_{2\beta})^{l_{\beta^{-1}}}S^{*-1}_{\beta^{-1}}(f_{1\beta})\bullet_{\alpha\beta^{-1}}a)\lambda_{\alpha2\alpha}$$
$$\overset{(19)}{=} ((\lambda_\alpha S^{*-1}_{\beta^{-1}}(f_{2\beta})^{l_{\beta^{-1}}})_{1\alpha}S^{*-1}_{\beta^{-1}}(f_{1\beta})\bullet_{\alpha\beta^{-1}}a)(\lambda_\alpha S^{*-1}_{\beta^{-1}}(f_{2\beta})^{l_{\beta^{-1}}})_{2\alpha} \overset{(31)}{=} \lambda_\alpha S^{*-1}_{\beta^{-1}}(f_{2\beta})^{l_{\beta^{-1}}}(S^{*-1}_{\beta^{-1}}(f_{1\beta})\bullet_{\beta^{-1}}a) = \lambda_\alpha(S^{*-1}_{\beta^{-1}}(f)\bullet_{\beta^{-1}}a),$$

Thus we complete the proof. □

After these preparations we can give the main result.

Theorem 1.17. *Let H be a weak Hopf π-coalgebra with S-fixed left integral $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ and A a right weak H-π-comdulelike algebra. Then $[A^{coH}, {}_{A^{coH}}A_{A\sharp H^*}, {}_{A\sharp H^*}A_{A^{coH}}, \tau, \chi]$ forms a Morita context, where*

$$\tau : A\otimes_{A^{coH}}A \to A\sharp H^*,\ \tau(a \otimes b) = (a\lambda_\alpha b)_{\alpha\in\pi},$$
$$\chi : A\otimes_{A\sharp H^*}A \to A^{coH},\ \chi(a \otimes b) = (\lambda_\alpha\bullet_\alpha(ab))_{\alpha\in\pi}.$$

Proof. From Lemma1.15 we know that A is both an A^{coH}-$A\sharp H^*$-bimodules and an $A\sharp H^*$-A^{coH}-bimodules. To verify the conditions for a Morita context we have to show that τ is an $A\sharp H^*$-bimodule map which is middle A^{coH}-linear, χ is an A^{coH}-bimodule map which is middle $A\sharp H^*$-linear and associativity holds.

Firstly we claim that τ is an $A\sharp H^*$-bimodule map which is middle A^{coH}-linear. In fact, for any $a, b, c \in A,\ x \in A^{coH}$ and $f \in H_\beta{}^*$ we have

$$(c\sharp f)\tau(a \otimes b) = cfa\lambda_\alpha b \overset{(31)}{=} c(f_{1\beta}\bullet_\beta a)f_{2\beta}\lambda_\alpha b \overset{(17)}{=} c(f_{1\beta}\bullet_\beta a)f_{2\beta}{}^{l_\beta}\lambda_\alpha b \overset{(30)}{=} c(f\bullet_\beta a)\lambda_\alpha b = \tau((c\sharp f) \triangleright a \otimes b);$$

$$\tau(a \otimes b \triangleleft (c\sharp f)) = \tau(a \otimes S^{*-1}_{\beta^{-1}}(f)\bullet_{\beta^{-1}}(bc)) = a\lambda_\alpha(S^{*-1}_{\beta^{-1}}(f)\bullet_{\beta^{-1}}(bc)) \overset{(34)}{=} a\lambda_\alpha bcf = \tau(a \otimes b)(c\sharp f);$$

$$\tau(ax\otimes b)=ax\lambda_\alpha b\overset{(32)}{=}a\lambda_\alpha xb=\tau(a\otimes xb).$$

Secondly we prove that χ is an A^{coH}-bimodule map which is middle $A\sharp H^*$-linear. In fact, for any $a,b,c\in A,\ x\in A^{coH}$ and $f\in H_\beta{}^*$ we have

$$\chi(xa\otimes b)=\lambda_\alpha\bullet_\alpha(xab)\overset{(28)}{=}(\lambda_{\alpha 1\alpha}\bullet_\alpha x)(\lambda_{\alpha 2\alpha}\bullet_\alpha(ab))=(\lambda_{\alpha 1\alpha}{}^{l_\alpha}\bullet_i x)(\lambda_{\alpha 2\alpha}\bullet_\alpha(ab))\overset{(21)}{=}(S_i^*(\varepsilon_{1i})\bullet_i x)(\varepsilon_{2i}\lambda_\alpha\bullet_\alpha(ab))$$
$$=(\varepsilon_{1i}{}^{l_i}\bullet_i x)(\varepsilon_{2i}\lambda_\alpha\bullet_\alpha(ab))=(\varepsilon_{1i}\bullet_i x)(\varepsilon_{2i}\lambda_\alpha\bullet_\alpha(ab))\overset{(28)}{=}\varepsilon\bullet_i(x(\lambda_\alpha\bullet_\alpha(ab)))=x(\lambda_\alpha\bullet_\alpha(ab))=x\chi(a\otimes b);$$

$$\chi(a\otimes bx)=\lambda_\alpha\bullet_\alpha(abx)\overset{(28)}{=}(\lambda_{\alpha 1\alpha}\bullet_\alpha(ab))(\lambda_{\alpha 2\alpha}\bullet_\alpha x)=(\lambda_{\alpha 1\alpha}\bullet_\alpha(ab))(\lambda_{\alpha 2\alpha}{}^{l_\alpha}\bullet_i x)\overset{(21)}{=}(\varepsilon_{1i}\lambda_\alpha\bullet_\alpha(ab))(\varepsilon_{2i}\bullet_i x)$$
$$\overset{(28)}{=}\varepsilon\bullet_i((\lambda_\alpha\bullet_\alpha(ab))x)=(\lambda_\alpha\bullet_\alpha(ab))x=\chi(a\otimes b)x;$$

$$\chi(a\lhd(b\sharp f)\otimes c)=\chi(S^{*-1}_{\beta^{-1}}(f)\bullet_{\beta^{-1}}(ab)\otimes c)=\lambda_\alpha\bullet_\alpha((S^{*-1}_{\beta^{-1}}(f)\bullet_{\beta^{-1}}(ab))c)\overset{(28)}{=}(\lambda_{\alpha 1\alpha}S^{*-1}_{\beta^{-1}}(f)\bullet_{\alpha\beta^{-1}}(ab))(\lambda_{\alpha 2\alpha}\bullet_\alpha c)$$
$$\overset{(35)}{=}(\lambda_{\alpha 1\alpha}\bullet_\alpha(ab))(\lambda_{\alpha 2\alpha}f\bullet_{\alpha\beta}c)\overset{(28)}{=}\lambda_\alpha\bullet_\alpha(ab(f\bullet_\beta c))=\chi(a\otimes b(f\bullet_\beta c))=\chi(a\otimes(b\sharp f)\rhd c).$$

Finally we claim that the associativity holds, that is, $\chi(a\otimes b)c=a\lhd\tau(b\otimes c)$ and $\tau(a\otimes b)\rhd c=a\chi(b\otimes c)$ for any $a,b,c\in A$. Indeed, we have

$$a\lhd\tau(b\otimes c)=a\lhd(b\lambda_\alpha c)\overset{(31)}{=}S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 2\alpha})\bullet_{\alpha^{-1}}(ab(\lambda_{\alpha 1\alpha}\bullet_\alpha c))\overset{(28)}{=}(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 4\alpha})\bullet_{\alpha^{-1}}a)(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 3\alpha})\bullet_{\alpha^{-1}}b)(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 2\alpha})\lambda_{\alpha 1\alpha}\bullet_i c)$$

$$=(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 3\alpha})\bullet_{\alpha^{-1}}a)(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 2\alpha})\bullet_{\alpha^{-1}}b)(S_i^{*-1}(\lambda_{\alpha 2\alpha}{}^{r_{\alpha^{-1}}})\bullet_i c)\overset{(21)}{=}(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 2\alpha})\bullet_{\alpha^{-1}}a)(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 1\alpha}\varepsilon_{2i})\bullet_{\alpha^{-1}}b)(S_i^{*-1}(\varepsilon_{1i})\bullet_i c)$$
$$=(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 2\alpha})\bullet_{\alpha^{-1}}a)(\varepsilon_{1i}S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 1\alpha})\bullet_{\alpha^{-1}}b)(\varepsilon_{2i}\bullet_i c)\overset{(28)}{=}(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 2\alpha})\bullet_{\alpha^{-1}}a)(S^{*-1}_{\alpha^{-1}}(\lambda_{\alpha 1\alpha})\bullet_{\alpha^{-1}}b)c\overset{(28)}{=}(S^{*-1}_{\alpha^{-1}}(\lambda_\alpha)\bullet_{\alpha^{-1}}(ab))c$$
$$=(\lambda_{\alpha^{-1}}\bullet_{\alpha^{-1}}(ab))c=\chi(a\otimes b)c;$$

$$\tau(a\otimes b)\rhd c=(a\lambda_\alpha b)\rhd c=a(\lambda_{\alpha 1\alpha}\bullet_\alpha b)(\lambda_{\alpha 2\alpha}\bullet_\alpha c)\overset{(28)}{=}a(\lambda_\alpha\bullet_\alpha(bc))=a\chi(b\otimes c).$$

This concludes the proof. □

Let H be a weak Hopf π-coalgebra, then there exists a weak Hopf π-coalgebra $(H_i{}^{l_\alpha}H_i{}^{r_\alpha},\mu_\alpha,\eta_\alpha,\Delta_{\alpha,\beta},\varepsilon,S_\alpha)_{\alpha,\beta\in\pi}$ denoted by H^lH^r. In fact, for any $a,b,c,d\in H_i$ and $\alpha,\beta\in\pi$, we have

$$(a^{l_\alpha}b^{r_\alpha})(c^{l_\alpha}d^{r_\alpha})\overset{(13)}{=}(a^{l_\alpha}c^{l_\alpha})(b^{r_\alpha}d^{r_\alpha})\overset{(11)}{=}(a^{l_i}c)^{l_\alpha}(bd^{r_i})^{r_\alpha}\in H_i{}^{l_\alpha}H_i{}^{r_\alpha};$$

$$\Delta_{\alpha,\beta}(a^{l_{\alpha\beta}}b^{r_{\alpha\beta}})=a^{l_{\alpha\beta}}{}_{1\alpha}b^{r_{\alpha\beta}}{}_{1\alpha}\otimes a^{l_{\alpha\beta}}{}_{2\beta}b^{r_{\alpha\beta}}{}_{2\beta}\overset{(16)^{[11]}}{=}1_{1\alpha}1'_{1\alpha}a^{l_\alpha}\otimes 1_{2\beta}1'_{2\beta}b^{r_\beta}\overset{(13)}{=}a^{l_\alpha}1_{1\alpha}\otimes 1_{2\beta}b^{r_\beta}$$
$$\overset{(10)^{[11]}}{=}a^{l_\alpha}1_{1i}{}^{r_\alpha}\otimes 1_{2i}{}^{l_\beta}b^{r_\beta}\in H_i{}^{l_\alpha}H_i{}^{r_\alpha}\otimes H_i{}^{l_\beta}H_i{}^{r_\beta};$$

$$S_\alpha(a^{l_{\alpha^{-1}}}b^{r_{\alpha^{-1}}})=S_\alpha(b^{r_{\alpha^{-1}}})S_\alpha(a^{l_{\alpha^{-1}}})\overset{Th.1.8^{[11]}}{=}S_i(b)^{l_\alpha}S_i(a)^{r_\alpha}\in H_i{}^{l_\alpha}H_i{}^{r_\alpha}.$$

For any $\alpha\in\pi,\ B=H_i{}^{r_\alpha}$ is a weak right H^lH^r- π-comodulelike algebra via the coaction $\rho=\{\rho_\beta=\Delta_{\alpha,\beta}\}_{\beta\in\pi}$. Directly from the equity (11) B is an algebra and from Eq. (16) in [11] it is also a weak right H^lH^r-π-comodulelike algebra.

Example 1.18. *Let H ba a weak Hopf π-coalgebra, then there exists S-fixed left integral $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ for $U=H^lH^R$, hence we have a Morita context $[B^{coU},{}_{B^{coU}}B_{B\sharp U^*},{}_{B\sharp U^*}B_{B^{coU}},\tau,\chi]$.*

We define $\lambda_\alpha:U_\alpha=H_i{}^{l_\alpha}H_i{}^{r_\alpha}\to k,\ \lambda_\alpha(a^{l_\alpha}b^{r_\alpha})=\varepsilon(a)\varepsilon(b)$, then we claim that $\lambda=(\lambda_\alpha)_{\alpha\in\pi}$ is a S-fixed left integral. Indeed, for any $f\in U_\alpha{}^*,\ a,b\in H_i,\ (S^*_\alpha(\lambda_\alpha))(a^{l_{\alpha^{-1}}}b^{r_{\alpha^{-1}}})=\varepsilon(a)\varepsilon(b)=\lambda_{\alpha^{-1}}(a^{l_{\alpha^{-1}}}b^{r_{\alpha^{-1}}})$, it means that $S^*_\alpha(\lambda_\alpha)=\lambda_{\alpha^{-1}}$. Also we compute

$$f((a^{l_{\alpha\beta}}b^{r_{\alpha\beta}})_{1i}{}^{l_\alpha})\lambda_{\alpha\beta}((a^{l_{\alpha\beta}}b^{r_{\alpha\beta}})_{2\alpha\beta})\overset{(16)^{[11]}}{=}f((1_{1i}1'_{1i}a^{l_i})^{l_\alpha})\lambda_{\alpha\beta}(1_{2\alpha\beta}1'_{2\alpha\beta}b^{r_{\alpha\beta}})=f((1_{1i}a^{l_\alpha})^{l_\alpha})\lambda_{\alpha\beta}(1_{2i}{}^{l_{\alpha\beta}}b^{r_{\alpha\beta}})$$
$$=f((1_{1i}a^{l_\alpha})^{l_\alpha})\varepsilon(1_{2i}{}^{l_{\alpha\beta}})\varepsilon(b^{r_{\alpha\beta}})=f(a^{l_\alpha})\varepsilon(b)=f((a^{l_{\alpha\beta}}b^{r_{\alpha\beta}})_{1\alpha})\lambda_\beta((a^{l_{\alpha\beta}}b^{r_{\alpha\beta}})_{2\beta}),\ \ \text{so}\ \ \lambda=(\lambda_\alpha)_{\alpha\in\pi}\ \ \text{is a left integral.}$$

2. The Morita Contexts for Weak π-Galois Extensions

In this section we will construct a Morita context for a weak π-Galois extension which extends the results in [6].

Definition 2.1. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. A right A-module M is called a weak right (H,A)-π-Hopf module* if M is also a weak right H-π-comodulelike object such that the following compatibility condition holds for any $\alpha \in \pi$, $a \in A$ and $m \in M$,

$$(m \cdot a)_{[0]} \otimes (m \cdot a)_{([1],\alpha)} = m_{[0]} \cdot a_{[0]} \otimes m_{([1],\alpha)} a_{([1],\alpha)}. \tag{36}$$

Example 2.2. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra via $\rho = \{\rho_\alpha\}_{\alpha\in\pi}$. Then*

(1) $A \otimes_{A^{coH}} A$ is a weak right (H,A)-π-Hopf module via the right action and coaction given by $(a \otimes b) \circ c = a \otimes bc$ and $(a \otimes b)_{[0]} \otimes (a \otimes b)_{([1],\alpha)} = a \otimes b_{[0]} \otimes b_{([1],\alpha)}$ for any $a, b, c \in A$ respectively. Obviously $A \otimes_{A^{coH}} A$ is a right A-module , in what follows we only show that the coaction is reasonable, the other proof is tirvial. Indeed, for any $a, b, c \in A$ and $x \in A^{coH}$, we have

$$\begin{aligned}
&(a \otimes xb)_{[0]} \otimes (a \otimes xb)_{([1],\alpha)} = a \otimes x_{[0]} b_{[0]} \otimes x_{([1],\alpha)} b_{([1],\alpha)} = a \otimes x_{[0]} b_{[0]} \otimes {x_{([1],i)}}^{l_\alpha} b_{([1],\alpha)} \\
&\overset{Th.1.8^{[11]}}{=} a \otimes x_{[0]} b_{[0]} \otimes S^{-1}_{\alpha^{-1}}({x_{([1],i)}}^{r_{\alpha^{-1}}}) b_{([1],\alpha)} \overset{(16)}{=} a \otimes x 1_{[0]} b_{[0]} \otimes 1_{([1],\alpha)} b_{([1],\alpha)} = ax \otimes b_{[0]} \otimes b_{([1],\alpha)} \\
&= (ax \otimes b)_{[0]} \otimes (ax \otimes b)_{([1],\alpha)}.
\end{aligned}$$

(2) $A\sharp\overline{H} = ((A \otimes H_\alpha)/ker\theta_\alpha)_{\alpha\in\pi}$ is a weak right (H,A)-π-Hopf module where $\overline{H} = \oplus_{\alpha\in\pi} H_\alpha$, $\theta_\alpha : A \otimes H_\alpha \to A \otimes H_\alpha$, $a \otimes h \mapsto a1_{[0]} \otimes h1_{([1],\alpha)}$, and the right action and coaction are defined by $(a\sharp h) \cdot b = a_{[0]}\sharp hb_{([1],\alpha)}$ and $(a\sharp g)_{[0]} \otimes (a\sharp g)_{([1],\alpha)} = (a\sharp g_{1\alpha}) \otimes g_{2\beta}$, $a, b \in A$, $h \in H_\alpha$, $g \in H_{\alpha\beta}$ respectively. Obviously $\theta_\alpha^2 = \theta_\alpha$.

Here we only show that the right action and coaction on $A\sharp\overline{H}$ are well-defined, the other proof is easy and obvious. In fact, for any $a, b \in A$, $h \in H_\alpha$, $g \in H_{\alpha\beta}$ we have

$$(a1_{[0]}\sharp h1_{([1],\alpha)}) \cdot b = a1_{[0]} b_{[0]}\sharp h1_{([1],\alpha)} b_{([1],\alpha)} = ab_{[0]}\sharp hb_{([1],\alpha)} = (a\sharp h) \cdot b;$$

$$\begin{aligned}
&(a1_{[0]}\sharp g1_{([1],\alpha\beta)})_{[0]} \otimes (a1_{[0]}\sharp g1_{([1],\alpha\beta)})_{([1],\beta)} = a1_{[0]}\sharp g_{1\alpha} 1_{([1],\alpha\beta)1\alpha} g_{2\beta} 1_{([1],\alpha\beta)2\beta} \overset{(15)}{=} a1_{[0]}\sharp g_{1\alpha} 1'_{1\alpha} 1_{([1],\alpha)} \otimes g_{2\beta} 1'_{2\beta} \\
&= a1_{[0]}\sharp g_{1\alpha} 1_{([1],\alpha)} \otimes g_{2\beta} = a\sharp g_{1\alpha} \otimes g_{2\beta} = (a\sharp g)_{[0]} \otimes (a\sharp g)_{([1],\beta)}.
\end{aligned}$$

Definition 2.3. *([8]) Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. We say that $A^{coH} \hookrightarrow A$ is a weak right π- Galois extension (or weak right group Galois extension)* if $can = \oplus_{\alpha\in\pi} can_\alpha$ is bijective as a left A-linear and weak right H-π-colinear map, where $can_\alpha : A \otimes_{A^{coH}} A \to A\sharp H_\alpha$, $a \otimes b \mapsto ab_{[0]} \otimes b_{([1],\alpha)}$, $a, b \in A$.

Let H be a weak Hopf π-coalgebra and A be a weak right H-π-comodulelike algebra. Set $\widehat{Hom}(H,A) = \oplus_{\alpha\in\pi} \widehat{Hom}(H_\alpha, A)$, where $\widehat{Hom}(H_\alpha, A) = Hom(H_\alpha, A)/ker\omega_\alpha$, $\omega_\alpha : Hom(H_\alpha, A) \to Hom(H_\alpha, A)$,

$$\omega_\alpha(f)(h) = \widehat{f}(h) = f(h1_{([1],\alpha)})1_{[0]},\ f \in Hom(H_\alpha, A),\ h \in H_\alpha.$$

Clearly $\omega_\alpha^2 = \omega_\alpha$.

Lemma 2.4. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. Then $\widehat{Hom}(H,A)$ is a unital π-graded algebra, where the multiplication is defined by for any $f \in \widehat{Hom}(H_\alpha, A)$, $g \in \widehat{Hom}(H_\beta, A)$, $x \in H_{\beta\alpha}$, $(fg)(x) = f(x_{2\alpha})_{[0]} g(x_{1\beta} f(x_{2\alpha})_{([1],\beta)})$ and the unit is given by $\widehat{\varepsilon 1}_A : H_i \to A$, $(\widehat{\varepsilon 1}_A)(h) = \varepsilon(h1_{([1],i)})1_{[0]}$.*

Proof. First we show that the multiplication is reasonable. In fact, for any $f \in \widehat{Hom}(H_\alpha, A)$, $g \in \widehat{Hom}(H_\beta, A)$, $x \in H_{\beta\alpha}$, we have

$$\begin{aligned}
&(\omega_\alpha(f)g)(x) = 1_{[0][0]} f(x_{2\alpha} 1_{([1],\alpha)})_{[0]} g(x_{1\beta} 1_{[0]([1],\beta)} f(x_{2\alpha} 1_{([1],\alpha)})_{([1],\beta)}) \overset{(15)}{=} 1_{[0]} f(x_{2\alpha} 1'_{2\alpha})_{[0]} g(x_{1\beta} 1'_{1\beta} 1_{([1],\beta)} f(x_{2\alpha} 1'_{2\alpha})_{([1],\beta)}) \\
&\overset{(14)}{=} f(x_{2\alpha})_{[0]} g(x_{1\beta} f(x_{2\alpha} 1'_{2\alpha})_{([1],\beta)}) = (fg)(x);
\end{aligned}$$

$$(f\omega_\beta(g))(x) = f(x_{2\alpha})_{[0]} 1_{[0]} g(x_{1\beta} f(x_{2\alpha})_{([1],\beta)} 1_{([1],\beta)}) \overset{(14)}{=} f(x_{2\alpha})_{[0]} g(x_{1\beta} f(x_{2\alpha})_{([1],\beta)}) = (fg)(x).$$

Next we prove that $\widehat{Hom}(H,A)$ is a unital π-graded algebra. In fact, for any $f \in \widehat{Hom}(H_\alpha, A)$, $g \in \widehat{Hom}(H_\beta, A)$, $l \in \widehat{Hom}(H_\gamma, A)$, $y \in H_{\gamma\beta\alpha}$, $x \in H_\alpha$, we have

$$\begin{aligned}
((fg)l)(y) &= f(y_{2\beta\alpha 2\alpha})_{[0][0]} g(y_{2\beta\alpha 1\beta} f(y_{2\beta\alpha 2\alpha})_{([1],\beta)})_{[0]} l(y_{1\gamma} f(y_{2\beta\alpha 2\alpha})_{[0]([1],\gamma)} g(y_{2\beta\alpha 1\beta} f(y_{2\beta\alpha 2\alpha})_{([1],\beta)})_{([1],\gamma)}) \\
&\overset{(3)}{=} f(y_{2\beta\alpha 2\alpha})_{[0]} g(y_{2\beta\alpha 1\beta} f(y_{2\beta\alpha 2\alpha})_{([1],\gamma\beta)2\beta})_{[0]} l(y_{1\gamma} f(y_{2\beta\alpha 2\alpha})_{([1],\gamma\beta)1\gamma} g(y_{2\beta\alpha 1\beta} f(y_{2\beta\alpha 2\alpha})_{([1],\gamma\beta)2\beta})_{([1],\gamma)}) \\
&\overset{(1)}{=} f(y_{2\alpha})_{[0]} g(y_{1\gamma\beta 2\beta} f(y_{2\alpha})_{([1],\gamma\beta)2\beta})_{[0]} l(y_{1\gamma\beta 1\gamma} f(y_{2\alpha})_{([1],\gamma\beta)1\gamma} g(y_{1\gamma\beta 2\beta} f(y_{2\alpha})_{([1],\gamma\beta)2\beta})_{([1],\gamma)}) \\
&= f(y_{2\alpha})_{[0]} (gl)(y_{1\gamma\beta} f(y_{2\alpha})_{([1],\gamma\beta)}) = (f(gl))(y);
\end{aligned}$$

$$\begin{aligned}
(f\widehat{\varepsilon 1_A})(x) &= f(x_{2\alpha})_{[0]} 1_{[0]} \varepsilon(x_{1i} f(x_{2\alpha})_{([1],i)} 1_{([1],i)}) = f(x_{2\alpha})_{[0]} \varepsilon(x_{1i} f(x_{2\alpha})_{([1],i)}{}^{l_i}) \overset{(15)}{=} 1_{[0]} f(x_{2\alpha}) \varepsilon(x_{1i} 1_{([1],i)}) \\
&\overset{(8)[11]}{=} 1_{[0]} f(x 1_{([1],i)}{}^{l_i}) \overset{(16)}{=} 1_{[0]} f(x 1_{([1],i)}) = f(x);
\end{aligned}$$

$$(\widehat{\varepsilon 1_A} f)(x) = \varepsilon(x_{2i} 1_{([1],i)}) 1_{[0][0]} f(x_{1\alpha} 1_{[0]([1],\alpha)}) \overset{(15)}{=} \varepsilon(x_{2i} 1'_{2i}) 1_{[0]} f(x_{1\alpha} 1'_{1\alpha} 1_{([1],\alpha)}) = 1_{[0]} f(x 1_{([1],\alpha)}) = f(x).$$

Therefore we complete the proof. □

Lemma 2.5. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. Then A is an A^{coH}-$\widehat{Hom}(H,A)$-bimodule via the left multiplication and the right $\widehat{Hom}(H,A)$-module given by $a \trianglelefteq f = a_{[0]} f(a_{([1],\alpha)})$, $a \in A$, $f \in \widehat{Hom}(H_\alpha, A)$.*

Proof. Clearly A is a left A^{coH}-module. Firstly we show the right $\widehat{Hom}(H,A)$-action is reasonable. Indeed, for any $a \in A$, $f \in \widehat{Hom}(H_\alpha, A)$,

$$a \trianglelefteq \widehat{f} = a_{[0]} \widehat{f}(a_{([1],\alpha)}) = a_{[0]} 1_{[0]} f(a_{([1],\alpha)} 1_{([1],\alpha)}) = a_{[0]} f(a_{([1],\alpha)}) = a \trianglelefteq f.$$

Secondly we claim that A is not only a right $\widehat{Hom}(H,A)$-module, but also an A^{coH}-$\widehat{Hom}(H,A)$-bimodule. In fact, for any $a \in A$, $b \in A^{coH}$, $f \in \widehat{Hom}(H_\alpha, A)$, $g \in \widehat{Hom}(H_\beta, A)$ we have

$$(a \trianglelefteq f) \trianglelefteq g = a_{[0][0]} f(a_{([1],\alpha)[0]} g(a_{[0]([1],\beta)} f(a_{([1],\alpha)([1],\beta)}) \overset{(3)}{=} a_{[0]} f(a_{([1],\beta\alpha 2\alpha)})_{[0]} g(a_{([1],\beta\alpha 1\beta)} f(a_{([1],\beta\alpha 2\alpha)})_{([1],\beta)}) = a \trianglelefteq (fg);$$

$$a \trianglelefteq (\widehat{\varepsilon 1_A}) = a_{[0]} 1_{[0]} \varepsilon(a_{([1],i)} 1_{([1],i)}) = a_{[0]} \varepsilon(a_{([1],i)}) = a;$$

$$\begin{aligned}
(ba) \trianglelefteq f &= b_{[0]} a_{[0]} f(b_{([1],\alpha)} a_{([1],\alpha)}) \overset{Th1.8[11]}{=} b_{[0]} a_{[0]} f(S^{-1}_{\alpha^{-1}}(b_{([1],i)}{}^{r_{\alpha^{-1}}}) a_{([1],\alpha)}) \overset{(16)}{=} b 1_{[0]} a_{[0]} f(1_{([1],\alpha)} a_{([1],\alpha)}) \\
&= b a_{[0]} f(a_{([1],\alpha)}) = b(a \trianglelefteq f).
\end{aligned}$$

Thus we complete the proof. □

Let Q be the subset of $\widehat{Hom}(H,A)$ consisting of maps $f = (f_\alpha)_{\alpha\in\pi} \in \widehat{Hom}(H,A)$ satisfying $f_\alpha(h_{2\alpha})_{[0]} \otimes h_{1\beta} f_\alpha(h_{2\alpha})_{([1],\beta)} = f_{\beta\alpha}(h) 1_{[0]} \otimes 1_{([1],\beta)}$ for any $\alpha, \beta \in \pi$. The α-th component of Q is denoted by Q_α.

Lemma 2.6. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. Then Q is a $\widehat{Hom}(H,A)$-A^{coH}-bimodule via the left multiplication and the right A^{coH}-module given by $q * a = qj(a)$, where $j : A \to \widehat{Hom}(H_i, A)$, $j(b)(h) = 1_{[0]} b \varepsilon(h 1_{([1],i)})$, $a \in A^{coH}$, $b \in A$, $h \in H_i$, $q \in Q_\alpha$.*

Proof. Firstly we claim that the left and right actions are well-defined. Indeed, for any $q_\beta \in Q_\beta$, $f_\alpha \in \widehat{Hom}(H_\alpha, A)$, $x \in H_{\gamma\beta\alpha}$, $y \in H_{\gamma\beta}$, $a \in A^{coH}$ we have

$$\begin{aligned}
(f_\alpha q_\beta)(x_{2\alpha})_{[0]} \otimes x_{1\gamma} (f_\alpha q_\beta)(x_{2\alpha})_{([1],\gamma)} &= f_\alpha(x_{3\alpha})_{[0][0]} q_\beta(x_{2\beta} f_\alpha(x_{3\alpha})_{([1],\beta)})_{[0]} \otimes x_{1\gamma} f_\alpha(x_{3\alpha})_{[0]([1],\gamma)} q_\beta(x_{2\beta} f(x_{3\alpha})_{([1],\beta)})_{([1],\gamma)} \\
&\overset{(3)}{=} f_\alpha(x_{3\alpha})_{[0]} q_\beta(x_{2\beta} f_\alpha(x_{3\alpha})_{([1],\gamma\beta 2\beta)})_{[0]} \otimes x_{1\gamma} f_\alpha(x_{3\alpha})_{([1],\gamma\beta)1\gamma} q_\beta(x_{2\beta} f_\alpha(x_{3\alpha})_{([1],\gamma\beta)2\beta})_{([1],\gamma)} \\
&= f_\alpha(x_{2\alpha})_{[0]} q_{\gamma\beta}(x_{1\gamma\beta} f_\alpha(x_{2\alpha})_{([1],\gamma\beta)}) 1_{[0]} \otimes 1_{([1],\gamma)} = (f_\alpha q_{\gamma\beta})(x) 1_{[0]} \otimes 1_{([1],\gamma)};
\end{aligned}$$

$$\begin{aligned}
&(q_\beta * a)(y_{2\beta})_{[0]} \otimes y_{1\gamma} (q_\gamma * a)(y_{2\beta})_{([1],\gamma)} \\
&= q_\beta(y_{3\beta})_{[0]} 1_{[0][0]} a_{[0]} \varepsilon(y_{2i} q_\beta(y_{3\beta})_{([1],\gamma)2i} 1_{([1],i)}) \otimes y_{1\gamma} q_\beta(y_{3\beta})_{([1],\gamma)1\gamma} 1_{[0]([1],\gamma)} a_{([1],\gamma)} \\
&\overset{(15)}{=} q_\beta(y_{2\beta})_{[0]} a_{[0]} \otimes y_{1\gamma} q_\beta(y_{2\beta})_{([1],\gamma)1\gamma} a_{([1],\gamma)} = q_{\gamma\beta}(y) 1_{[0]} a_{[0]} \otimes 1_{([1],\gamma)} a_{([1],\gamma)} = q_{\gamma\beta}(y) a_{[0]} \otimes a_{([1],\gamma)}
\end{aligned}$$

$$\overset{Th.1.8^{[11]}}{=} q_{\gamma\beta}(y)a_{[0]}\otimes S^{-1}_{\gamma^{-1}}(a_{([1],i)}{}^{r_{\gamma^{-1}}}) \overset{(16)}{=} q_{\gamma\beta}(y)a1_{[0]}\otimes 1_{([1],\gamma)} = q_{\gamma\beta}(y_{2\gamma\beta})_{[0]}\varepsilon(y_{1i}q_{\gamma\beta}(y_{2\gamma\beta})_{([1],i)})a1_{[0]}\otimes 1_{([1],\gamma)}$$
$$= q_{\gamma\beta}(y_{2\gamma\beta})_{[0]}1'_{[0]}\varepsilon(y_{1i}q_{\gamma\beta}(y_{2\gamma\beta})_{([1],i)}1'_{([1],i)})a1_{[0]}\otimes 1_{([1],\gamma)} = (q_{\gamma\beta}*a)(y)1_{[0]}\otimes 1_{([1],\gamma)}.$$

Hence the above actions are well-defined.

Next we prove that Q is an $\widehat{Hom}(H,A)$-A^{coH}-bimodule. In fact, for any $q_\beta\in Q_\beta,\ f_\alpha\in\widehat{Hom}(H_\alpha,A),\ u\in H_{\beta\alpha},\ z\in H_\beta,\ a,b\in A^{coH}$,

$$((q_\beta*a)*b)(z) = q_\beta(z_{3\beta})_{[0]}1_{[0][0]}a_{[0]}1'_{[0]}b\varepsilon(z_{2i}q_\beta(z_{3\beta})_{([1],i)2i}1_{([1],i)})\varepsilon(z_{1i}q_\beta(z_{3\beta})_{([1],i)1i}1_{[0]([1],i)}a_{([1],i)}1'_{([1],i)})$$
$$\overset{(3)}{=} q_\beta(z_{2\beta})_{[0]}a_{[0]}b\varepsilon(z_{1i}q_\beta(z_{2\beta})_{([1],i)}a_{([1],i)}) \overset{(16)}{=} q_\beta(z_{2\beta})_{[0]}1_{[0]}ab\varepsilon(z_{1i}q_\beta(z_{2\beta})_{([1],i)}1_{([1],i)}) = (q_\beta*(ab))(z);$$
$$(q_\beta*1)(z) = q_\beta(z_{2\beta})_{[0]}1_{[0]}\varepsilon(z_{1i}q_\beta(z_{2\beta})_{([1],i)}1_{([1],i)}) = q_\beta(z_{2\beta})_{[0]}\varepsilon(z_{1i}q_\beta(z_{2\beta})_{([1],i)}) = q_\beta(z)1_{[0]}\varepsilon(1_{([1],i)}) = q_\beta(z),$$

so Q is a right A^{coH}-module. Also we compute

$$((f_\alpha q_\beta)*a)(u) = f_\alpha(u_{3\alpha})_{[0][0]}q_\beta(u_{2\beta}f_\alpha(u_{3\alpha})_{([1],\beta)})_{[0]}1_{[0]}a\varepsilon(u_{1i}f_\alpha(u_{3\alpha})_{[0]([1],i)}q_\beta(u_{2\beta}f_\alpha(u_{3\alpha})_{([1],\beta)})_{([1],i)}1_{([1],i)})$$
$$\overset{(3)}{=} f_\alpha(u_{3\alpha})_{[0]}q_\beta(u_{2\beta}f_\alpha(u_{3\alpha})_{([1],\beta)2\beta})_{[0]}a\varepsilon(u_{1i}f_\alpha(u_{3\alpha})_{([1],\beta)1i}q_\beta(u_{2\beta}f_\alpha(u_{3\alpha})_{([1],\beta)2\beta})_{([1],i)})$$
$$= f_\alpha(u_{2\alpha})_{[0]}q_\beta(u_{1\beta}f_\alpha(u_{2\alpha})_{([1],\beta)})1_{[0]}a\varepsilon(1_{([1],i)}) = f_\alpha(u_{2\alpha})_{[0]}q_\beta(u_{1\beta}f_\alpha(u_{2\alpha})_{([1],\beta)})a$$
$$= f_\alpha(u_{3\alpha})_{[0]}q_\beta(u_{2\beta}f_\alpha(u_{3\alpha})_{([1],\beta)})_{[0]}\varepsilon(u_{1i}q_\beta(u_{2\beta}f_\alpha(u_{3\alpha})_{([1],\beta)})_{([1],i)})a = (f_\alpha(q_\beta*a))(u),$$

Therefore Q is a $\widehat{Hom}(H,A)$-A^{coH}-bimodule. □

In what follows we will compute a Morita context associated to a weak Hopf group Galois extension.

Theorem 2.7. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. Then*

$$(A^{coH},\widehat{Hom}(H,A),A,Q,\tau,\mu)$$

is a Morita context, where $\tau: A\otimes_{\widehat{Hom}(H,A)}Q\to A^{coH}$ and $\mu: Q\otimes_{A^{coH}}A\to\widehat{Hom}(H,A)$ are given by the formulas $\tau(a\otimes q_\alpha)=a_{[0]}q_\alpha(a_{([1],\alpha)})$; $\mu(q_\alpha\otimes a)=q_\alpha j(a),\ a\in A,\ q_\alpha\in Q_\alpha$.

Proof. From Lemma 2.5 and 2.6 we only need to show the following assertions hold.

(1) τ is an A^{coH}-bimodule map which is middle $\widehat{Hom}(H,A)$-linear.

Firstly we claim that τ is well-defined. Indeed, for any $a\in A,\ q_\alpha\in Q_\alpha,\ \alpha,\beta\in\pi$, we compute

$$(a_{[0]}q_\alpha(a_{([1],\alpha)}))_{[0]}\otimes(a_{[0]}q_\alpha(a_{[\alpha]}))_{([1],\beta)} = a_{[0][0]}q_\alpha(a_{([1],\alpha)})_{[0]}\otimes a_{[0]([1],\beta)}q_\alpha(a_{([1],\alpha)})_{([1],\beta)}$$
$$\overset{(3)}{=} a_{[0]}q_\alpha(a_{([1],\beta\alpha)2\alpha})_{[0]}\otimes a_{([1],\beta\alpha)1\beta}q_\alpha(a_{([1],\alpha)2\alpha})_{([1],\beta)} = a_{[0]}q_{\beta\alpha}(a_{([1],\beta\alpha)})1_{[0]}\otimes 1_{([1],\beta)}$$
$$\overset{(16)}{=} a_{[0]}q_{\beta\alpha}(a_{([1],\beta\alpha)})1_{[0]}\otimes 1_{([1],i)}{}^{l_\beta}\in A^{coH}.$$

Next we prove that τ is an A^{coH}-bimodule map. In fact, for any $a\in A,\ q_\alpha\in Q_\alpha,\ b\in A^{coH}$, we have

$$\tau(ba\otimes q_\alpha) = b_{[0]}a_{[0]}q_\alpha(b_{([1],\alpha)}a_{([1],\alpha)}) \overset{Th.1.8^{[11]}}{=} b_{[0]}a_{[0]}q_\alpha(S^{-1}_{\alpha^{-1}}(b_{([1],i)}{}^{r_{\alpha^{-1}}})a_{([1],\alpha)})$$
$$\overset{(16)}{=} b1_{[0]}a_{[0]}q_\alpha(1_{([1],\alpha)}a_{([1],\alpha)}) = ba_{[0]}q_\alpha(a_{([1],\alpha)}) = b\tau(a\otimes q_\alpha);$$
$$\tau(a\otimes q_\alpha*b) = a_{[0][0]}q_\alpha(a_{([1],\alpha)})_{[0]}\varepsilon(a_{[0]([1],i)}q_\alpha(a_{([1],\alpha)})_{([1],i)})b$$
$$\overset{(3)}{=} a_{[0]}q_\alpha(a_{([1],\alpha)2\alpha})_{[0]}\varepsilon(a_{([1],\alpha)1i}q_\alpha(a_{([1],\alpha)2\alpha([1],i)}))b = a_{[0]}q_\alpha(a_{([1],\alpha)})1_{[0]}\varepsilon(1_{([1],i)})b = a_{[0]}q_\alpha(a_{([1],\alpha)})b = \tau(a\otimes q_\alpha)b.$$

Finally we show that τ is middle $\widehat{Hom}(H,A)$-linear. For any $a\in A,\ q_\alpha\in Q_\alpha,\ f\in\widehat{Hom}(H_\beta,A)$, we have

$$\tau(a\trianglelefteq f\otimes q_\alpha) = a_{[0][0]}f(a_{([1],\beta)})_{[0]}q_\alpha(a_{[0]([1],\alpha)}f(a_{([1],\beta)})_{([1],\alpha)}) \overset{(3)}{=} a_{[0]}f(a_{([1],\alpha\beta)2\beta})_{[0]}q_\alpha(a_{([1],\alpha\beta)1\alpha}f(a_{([1],\alpha\beta)2\beta})_{([1],\alpha)})$$
$$= a_{[0]}(fq_\alpha)(a_{([1],\alpha\beta)}) = \tau(a\otimes fq_\alpha).$$

(2) μ is an $\widehat{Hom}(H,A)$-bimodule map which is middle A^{coH}-linear.

Firstly we claim that μ is a $\widehat{Hom}(H,A)$-bimodule map. Indeed, for any $a\in A,\ q_\alpha\in Q_\alpha,\ f\in\widehat{Hom}(H_\beta,A),\ h\in H_{\alpha\beta},\ x\in H_{\beta\alpha}$, we compute

$$
\begin{aligned}
&\mu(fq_\alpha\otimes a)(h)=f(h_{3\beta})_{[0][0]}q_\alpha(h_{2\alpha}f(h_{3\beta})_{([1],\alpha)})_{[0]}1_{[0]}a\varepsilon(h_{1i}f(h_{3\beta})_{[0]([1],i)}q_\alpha(h_{2\alpha}f(h_{3\beta})_{([1],\alpha)})_{([1],i)}1_{([1],i)})\\
&\overset{(3)}{=}f(h_{3\beta})_{[0]}q_\alpha(h_{2\alpha}f(h_{3\beta})_{([1],\alpha)2\alpha})_{[0]}1_{[0]}a\varepsilon(h_{1i}f(h_{3\beta})_{([1],\alpha)1i}q_\alpha(h_{2\alpha}f(h_{3\beta})_{([1],\alpha)2\alpha})_{([1],i)}1_{([1],i)})\\
&=f(h_{2\beta})_{[0]}(q_\alpha j(a))(h_{1\alpha}f(h_{2\beta})_{([1],\alpha)})=(f\mu(q_\alpha\otimes a))(h);
\end{aligned}
$$

$$
\begin{aligned}
&\mu(q_{\beta\alpha}\otimes a\trianglelefteq f)(x)=q_{\beta\alpha}(x_{2\beta\alpha})_{[0]}1_{[0]}a_{[0]}f(a_{([1],\beta)})\varepsilon(x_{1i}q_{\beta\alpha}(x_{2\beta\alpha})_{([1],i)}1_{([1],i)})=q_{\beta\alpha}(x_{\beta\alpha})1_{[0]}a_{[0]}f(a_{([1],\beta)})\varepsilon(1_{([1],i)})\\
&=q_{\beta\alpha}(x_{\beta\alpha})a_{[0]}f(a_{([1],\beta)})=q_{\beta\alpha}(x_{2\beta\alpha})_{[0]}a_{[0]}f(x_{1\beta}q_{\beta\alpha}(x_{2\beta\alpha})_{([1],\beta)}a_{([1],\beta)})\\
&\overset{(15)}{=}q_\beta(x_{3\beta})_{[0]}1_{[0][}a_{[0]}\varepsilon(x_{2i}q_\beta(x_{3\beta})_{([1],\beta)2i}1_{([1],\beta)2i})f(x_{1\beta}q_\beta(x_{3\beta})_{([1],\beta)1\beta}1_{([1],\beta)1\beta}a_{([1],\beta)})\\
&=(q_\beta j(a)(x_{2\alpha}))_{[0]}f(x_{1\beta}(q_\beta j(a)(x_{2\alpha}))_{([1],\beta)})=(\mu(q_\beta\otimes a)f)(x).
\end{aligned}
$$

Next we show that μ is middle A^{coH}-linear. In fact, for any $a\in A,\ q_\alpha\in Q_\alpha,\ b\in A^{coH},\ y\in H_\alpha$ we have

$$
\begin{aligned}
&\mu(q_\alpha*b\otimes a)(y)=q_\alpha(y_{3\alpha})_{[0]}1_{[0][0]}b_{[0]}1'_{[0]}a\varepsilon(y_{2i}q_\alpha(y_{3\alpha})_{([1],i)2i}1_{([1],i)})\varepsilon(y_{1i}q_\alpha(y_{3\alpha})_{([1],i)1i}1_{[0]([1],i)}b_{([1],i)}1'_{([1],i)})\\
&\overset{(15)}{=}q_\alpha(y_{2\alpha})_{[0]}b_{[0]}a\varepsilon(y_{1i}q_\alpha(y_{2\alpha})_{([1],i)}b_{([1],i)})=q_\alpha(y)1_{[0]}b_{[0]}a\varepsilon(1_{([1],i)}b_{([1],i)})\\
&=q_\alpha(y)ba=q_\alpha(y_{2\alpha})_{[0]}1_{[0]}ba\varepsilon(y_{1i}q_\alpha(y_{2\alpha})_{([1],i)}1_{([1],i)})=(q_\alpha j(ba))(y)=\mu(q_\alpha\otimes ba)(y).
\end{aligned}
$$

(3) We prove that the associativity holds, that is, for any $a,b\in A,\ q=(q_\beta)_{\beta\in\pi}\in Q,\ p=(p_\alpha)_{\alpha\in\pi}\in Q,\ \tau(a\otimes q_\beta)b=a\trianglelefteq\mu(q_\beta\otimes b)$ and $\mu(q_\beta\otimes a)p_\alpha=q_{\alpha\beta}*\tau(a\otimes p_\alpha)$. In fact, for any $h\in H_{\alpha\beta}$ we have

$$
\begin{aligned}
&a\trianglelefteq\mu(q_\beta\otimes b)=a_{[0][0]}q_\beta(a_{([1],\beta)})_{[0]}1_{[0]}b\varepsilon(a_{[0]([1],i)}q_\beta(a_{([1],\beta)})_{([1],i)}1_{([1],i)})\overset{(3)}{=}a_{[0]}q_\beta(a_{([1],\beta)2\beta})_{[0]}b\varepsilon(a_{[0]([1],i)}q_\beta(a_{([1],\beta)})_{([1],\beta)1i})\\
&=a_{[0]}q_\beta(a_{([1],\beta)})1_{[0]}b\varepsilon(1_{([1],i)})=a_{[0]}q_\beta(a_{([1],\beta)})b=\tau(a\otimes q_\beta)b;
\end{aligned}
$$

$$
\begin{aligned}
&(\mu(q_\beta\otimes a)p_\alpha)(h)=q_\beta(h_{3\beta})_{[0]}1_{[0][0]}a_{[0]}\varepsilon(h_{2i}q_\beta(h_{3\beta})_{([1],\alpha)2i}1_{([1],i)})p_\alpha(h_{1\alpha}q(h_{3\beta})_{([1],\alpha)1\alpha}1_{[0]([1],\alpha)}a_{([1],\alpha)})\\
&\overset{(15)}{=}q_\beta(h_{2\beta})_{[0]}a_{[0]}p_\alpha(h_{1\alpha}q_\beta(h_{2\beta})_{([1],\alpha)}a_{([1],\alpha)})=q_{\alpha\beta}(h)a_{[0]}p_\alpha(a_{([1],\alpha)})\\
&=q_{\alpha\beta}(h_{2\alpha\beta})_{[0]}a_{[0]}\varepsilon(h_{1i}q_{\alpha\beta}(h_{2\alpha\beta})_{([1],i)})p_\alpha(a_{([1],\alpha)})=(q_{\alpha\beta}j(a_{[0]}p_\alpha(a_{([1],\alpha)})))(h)=(q_{\alpha\beta}*\tau(a\otimes p_\alpha))(h),
\end{aligned}
$$

This concludes the proof. □

Lemma 2.8. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. Then the left dual of $A\sharp\overline{H}$, ${}_AHom(A\sharp\overline{H},A)=\oplus_{\alpha\in\pi A}Hom(A\sharp H_\alpha,A)$ denoted by ${}^*(A\sharp\overline{H})$ is isomorphic to $\widehat{Hom}(H,A)$ as an algebra under the multiplication given by for any $h\in H_{\beta\alpha},\ f\in_A Hom(A\sharp H_\alpha,A),\ g\in_A Hom(A\sharp H_\beta,A)$,*

$$(fg)(a\sharp h)=af((1\sharp h_{2\alpha})_{[0]})g(1\sharp h_{1\beta}(1\sharp h_{2\alpha})_{([1],\beta)}).$$

Proof. We define the two maps $\kappa=(\kappa_\alpha)_{\alpha\in\pi}:{}^*(A\sharp\overline{H})\to\widehat{Hom}(H,A)$ and $\xi=(\xi_\alpha)_{\alpha\in\pi}:\widehat{Hom}(H,A)\to{}^*(A\sharp\overline{H})$, where for any $h\in H_\alpha,\ a\in A,\ f\in_A Hom(A\sharp H_\alpha,A),\ l\in\widehat{Hom}(H_\alpha,A)$,

$$\kappa_\alpha(f)(h)=f(1_{[0]}\sharp h1_{([1],\alpha)}),\ \xi_\alpha(l)(a\sharp h)=al(h).$$

Now we have to show that for any $\alpha\in\pi,\ \xi_\alpha$ is well-defined. Indeed, for any $a\in A,\ l\in\widehat{Hom}(H_\alpha,A),\ h\in H_\alpha,\ \xi_\alpha(l)(a1_{[0]}\sharp h1_{([1],\alpha)})=a1_{[0]}l(h1_{([1],\alpha)})=al(h)=\xi_\alpha(l)(a\sharp h)$.

Next we claim that κ is the inverse of ξ. In fact, for any $\alpha\in\pi,\ h\in H_\alpha,\ a\in A,\ f\in_A Hom(A\sharp H_\alpha,A),\ l\in\widehat{Hom}(H_\alpha,A)$, we have

$$
\begin{aligned}
(\xi_\alpha\kappa_\alpha)(f)(a\sharp h)&=a\kappa_\alpha(f)(h)=af(1_{[0]}\sharp h1_{([1],\alpha)})=f(a\sharp h);\\
(\kappa_\alpha\xi_\alpha)(l)(h)&=\xi_\alpha(l)(1_{[0]}\sharp h1_{([1],\alpha)})=1_{[0]}l(h1_{([1],\alpha)})=l(h).
\end{aligned}
$$

And we prove that κ is an algebra morphism. In fact, for any $\alpha \in \pi$, $h \in H_{\beta\alpha}$, $f \in_A Hom(A\sharp H_\alpha, A)$, $g \in_A Hom(A\sharp H_\beta, A)$, we compute

$$\kappa_{\beta\alpha}(fg)(h) = (fg)(1_{[0]}\sharp h 1_{([1],\beta\alpha)}) = f(1\sharp h_{2\alpha})_{[0]} g(1\sharp h_{1\beta} f(1\sharp h_{2\alpha})_{([1],\beta)})$$
$$= f(1_{[0]}\sharp h_{2\alpha}1_{([1],\alpha)})_{[0]} g(1'_{[0]}\sharp h_{1\beta} f(1_{[0]}\sharp h_{2\alpha}1_{([1],\alpha)})_{([1],\beta)}1'_{([1],\beta)}) = \kappa_\alpha(f)(h_{2\alpha})_{[0]}\kappa_\beta(g)(h_{1\beta}\kappa_\alpha(f)(h_{2\alpha})_{([1],\beta)})$$
$$= (\kappa_\alpha(f)\kappa_\beta(g))(h).$$

Thus we complete the proof. □

The dual of the canonical map ${}^*can_\alpha :{}^* (A\sharp H_\alpha) \to_A Hom(A\otimes_{A^{coH}}A, A)$ is given by ${}^*can_\alpha(f)(a\otimes b) = f(can_\alpha(a\otimes b)) = f(ab_{[0]}\otimes b_{([1],\alpha)})$. It follows from Theorem 2.7 that τ is surjective if and only if there exists $q = (q_\alpha)_{\alpha\in\pi} \in Q$ such that $1_{[0]}q_\alpha(1_{([1],\alpha)}) = 1$. With the above notations, we immediately obtain the following result.

Corollary 2.9. *Let H be a weak Hopf π-coalgebra and A a weak right H-π-comodulelike algebra. Then the following assertions are equivalent.*

(1) $A^{coH} \hookrightarrow A$ is Galois;

(2) ${}^*can$ is an isomorphism;

(3) The Morita context $(A^{coH}, \widehat{Hom}(H, A), A, Q, \tau, \mu)$ is strict.

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