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Chaotic Behavior Analysis and Control of a Toxin Producing Phytoplankton and Zooplankton System Based on Linear Feedback

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Abstract. In this paper, we study the dynamic behavior and control of the fractional-order nutrient-phytoplankton-zooplankton system. First, we analyze the stability of the fractional-order nutrient-plankton system and get the critical stable value of fractional orders. Then, by applying the linear feedback control and Routh-Hurwitz criterion, we yield the sufficient conditions to stabilize the system to its equilibrium points. Finally, Under a modified fractional-order Adams-Bashforth-Monlton algorithm, we simulate the results respectively.

1. Introduction

Marine ecosystem has attracted scientists a lot due to its strong ability of adjusting global climate. In recent researches, scientists found that ocean carbon uptake had increased a lot during few decades in [10], which meant that the balance of marine ecology has changed. Authors in [36] pointed out that plankton, which is a general term used to describe freely-floating and weekly-swimming marine or freshwater organisms, played an important role in keeping the balance of marine ecology. Therefore, the principle of nutrient-plankton cycle in water or marine system is significant and the control of the system should also be concerned when imbalance occurs.

Plankton are divided into two groups generally, including phytoplankton and zooplankton. Phytoplankton are consumed by zooplankton, the animals, which in turn are eaten by larger organisms. Besides their role in basic food chain, phytoplankton have important impacts on the global carbon cycle [1]. Direct measurement of plankton is quite hard to realize, so qualitative researches about the cycle of plankton came up naturally [13, 16, 33, 34]. Steele [33] first proposed a 3-D differential system containing nutrient, phytoplankton and zooplankton (NPZ). The model described the relation between each variable and analyzed the system in a qualitative method. Results from simple models could help researchers deal with complicated system where patterns are quite crucial in determining the output. Ruan [31] further discussed the persistence and co-existence of nutrient-plankton interaction. Edwards [13, 14] investigated

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the dynamic behavior of plankton population with each term of the model like zooplankton mortality or higher predation. He also extended the NPZ model into a 4-D model with the component of detritus, which was named NPZD model [15]. Franks [17] coupled the model into physics and used a NPZ-Physics model to simulate the ecological system.

Researches on the dynamic nutrient-phytoplankton-zooplankton system were more complicated than before when it comes to the additional effects of toxicity. Toxin producing phytoplankton is a group of phytoplankton that have the capability of producing some toxic chemicals. The role of toxin and nutrient on the plankton system was discussed in [6, 8, 21]. What's more, time delayed system were being discussed in the next decade.

As we all know, time delay is a common phenomenon in biological system. The analysis of the system is rather complex and sometimes it will become unstable when time delay happens [23]. Ruan [30] and Das [9] considered the interaction of NPZ system with time delay due to gestation and nutrient recycling and Sharma [32] concluded the properties of Toxin-NPZ system interaction with different functional response and time delay. Moreover, with the development above, fractional-order ecology systems grew up gradually.

Fractional-order calculus is the extension and generalization of integer-order calculus, which has more than 300 years of history [4]. It is known that fractional differential equations are more effective and valuable in science and engineering due to its naturally relation to system with memory which is a common feature of many phenomena. There are numerous application of fractional differential equations in many subjects like polymer rheology, regular variation in thermodynamics, biophysics, blood flow phenomena, electrical circuits, energy demand-supply system, ecology system, love triangle system etc. [2, 3, 7, 22, 25, 28]. Then, fractional-oder NPZ models were studied gradually, including stabilities of the system, properties with different functional response and time delay in system [20, 29].

In [20], Javidi proposed a fractional-order NPZ system based on [21] as follows:

$$\begin{cases} \dfrac{\mathrm{d}^{\alpha_1} N}{\mathrm{d}t^{\alpha_1}} = N_0 - AN - BNP + CP, \\ \dfrac{\mathrm{d}^{\alpha_2} P}{\mathrm{d}t^{\alpha_2}} = A_1 NP - B_1 P - \dfrac{WPZ}{D+P}, \\ \dfrac{\mathrm{d}^{\alpha_3} Z}{\mathrm{d}t^{\alpha_3}} = \dfrac{W_1 PZ}{D+P} - A_2 Z - C_1 PZ, \end{cases} \tag{1}$$

where $0 < \alpha_i < 1$, $N_0, A, B, C, A_1, B_1, W, W_1, A_2, C_1$ are all positive parameters. $N(t)$, $P(t)$, $Z(t)$ denote the concentration of nutrient, the biomass of phytoplankton and zooplankton, respectively; N_0 is the constant supply rate of nutrient to the system and AN is the loss of nutrient due to leaching; We make the assumption that the growth rate of phytoplankton biomass is A, W is the rate of predation phytoplankton by zooplankton and the corresponding conversion rate of zooplankton is W_1; The Holling type-II funtional response here with D as half-saturation constant is used to explain the interaction; We take BN as the specific rate of nutrient uptaken per unit biomass of phytoplankton in unit time and depletion of zooplankton biomass due to toxin producing phytoplankton is given C_1PZ; What's more, the phytoplankton and zooplankton biomass depletion due to natural mortality at the rate of B_1 and A_2, respectively. Much work need to be done of system (1) and Javidi investigated the stable condition of system (1) at each equilibrium by changing parameters.

During past decades, chaos control of different chaotic fractional-order system was also a hot issue due to its importance in stabilizing the system in life. There were various approaches to control different chaotic fractional-order systems. For example, Li and Chen [24] applied linear feedback control in a chaotic fractional Chen system, Zhang et al. [38] investigated an adaptive single driving variable controller in a chaotic fractional-order Lü-Lü system and Yin et al. [37] analyzed adaptive sliding mode control methods. A proper controller could help managers to do better in dealing with imbalance phenomena in nature.

As far as we know, control of chaotic fractional-order NPZ system has not been studied in any literature. Motivated by the researches above, we investigate the problem of the control of fractional-order NPZ system in this article. First, we look back on the stability at each equilibrium point of the system (1) and analyze the dynamic behavior in system. Moreover, in order to stabilize the system, we design a linear feedback control and give the sufficient condition with corresponding proof about feedback gains of each equilibrium point, respectively. Finally, we do some numerical simulation and verify the results obtained before.

The paper is arranged as follows. Section 2 gives the preliminary of the fractional calculus and analyzes the stability of the fractional-order nutrient-phytoplankton-zooplankton system. Section 3 does the linear feedback control in fractional-order NPZ system and gives the sufficient condition about feedback of each equilibrium point, respectively. Section 4 shows the numerical simulations to verify our results in Section 2 and 3.

2. The Stability of the Fractional-order NPZ System

2.1. Basics of fractional-order calculus

The Caputo definition of fractional derivative [5] is given as

$$D^{\alpha} f(t) = J^{l-\alpha} f^{(l)}(t), \alpha > 0,$$

where the operator D^{α} is referred as the "α-order Caputo differential operator", $f^{(l)}$ represents the l-order derivative of $f(t)$, $l = \lfloor \alpha \rfloor$ is the smallest integer which is not less than α and J^{θ} is the θ-order Riemann-Liouville integral operator which can be described as

$$J^{\theta} u(t) = \frac{1}{\Gamma(\theta)} \int_0^t (t-\tau)^{(\theta-1)} u(\tau) d(\tau), \theta > 0,$$

where $\Gamma(\theta)$ is the Euler's Gamma function.

In [19, 26, 27], stability conditions and their applications to systems of fraction-order differential equations were reported. We consider the following nonlinear autonomous fractional-order system

$$D^{\alpha} X(t) = F(X(t)), X(0) = X_0, \tag{2}$$

where $X(t) = (x_1, x_2, x_3)^T \in R^3$, $F : R^3 \to R^3$ is a nonlinear vector function in terms of X. The Jacobian matrix evaluated at the equilibrium point $X^* = (x_1^*, x_2^*, x_3^*)$ is

$$J(X^*) = \left(\frac{\partial F_i}{\partial x_j} \right)_{ij} \Big|_{x=x^*}. \tag{3}$$

The local stability of the equilibrium points of a linearized fractional-order system can be obtained from the following lemma [27]:

Lemma 2.1. *If all the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of the equilibrium point X^* of Jacobian matrix* (3)*, satisfy the Matignon's conditions [26] which can be described as*

$$\left| \arg \lambda_i \right| > \alpha\pi/2, (i = 1, 2, 3), \tag{4}$$

where $|\arg \lambda_i|(i = 1, 2, 3)$ denotes the argument value of the eigenvalue λ_i. Then X^* is locally asymptotically stable.

2.2. Stability in equilibrium points

In this section, we look back on the stability of system (1). Let

$$\frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} N = 0, \qquad \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} P = 0, \qquad \frac{\mathrm{d}^{\alpha}}{\mathrm{d}t^{\alpha}} Z = 0. \tag{5}$$

There exist four equilibrium points as follows:

(i) Boundary equilibrium point $E_1 = \left(N_1^*, 0, 0\right)$, where $N_1^* = \frac{N_0}{A}$.

(ii) Zooplankton-free point $E_2 = \left(N_2^*, P_2^*, 0\right)$ if $(A_1N_0 - AB_1)(BB_1 - A_1C_1) > 0$, where $N_2^* = \frac{B_1}{A_1}$, $P_2^* = \frac{A_1N_0-AB_1}{BB_1-A_1C_1}, 0$.

(iii) Interior equilibrium points $E_3 = \left(N_3^*, P_3^*, Z_3^*\right)$ and $E_4 = \left(N_4^*, P_4^*, Z_4^*\right)$ exist if $S < 0$, $\frac{A_1\left(N_0+CP_3^*\right)}{A+BP_3^*} > B_1$, $\frac{A_1\left(N_0+CP_3^*\right)}{A+BP_3^*} > B_1$, and $S^2 > 4C_1DA_2$, where

$$
\begin{aligned}
P_3^* &= \frac{-S - \sqrt{S^2 - 4C_1DA_2}}{2C_1}, & P_4^* &= \frac{-S + \sqrt{S^2 - 4C_1DA_2}}{2C_1},\\
N_3^* &= \frac{N_0 + CP_3^*}{A + BP_3^*}, & N_4^* &= \frac{N_0 + CP_4^*}{A + BP_4^*},\\
Z_3^* &= \frac{\left(D + P_3^*\right)\left(A_1N_3^*\right) - B_1}{W}, & Z_4^* &= \frac{\left(D + P_4^*\right)\left(A_1N_4^*\right) - B_1}{W},
\end{aligned}
$$

and $S = A_2 + DC_1 - W_1$.

The Jacobian matrix of system (1) at equilibrium points is

$$
J(E^*) = \left(J_{ij}\right)\big|_{E^*}, \tag{6}
$$

where

$$
\begin{aligned}
&J_{11} = -A - BP^*, \qquad J_{12} = -BN^* + C, \qquad J_{13} = 0,\\
&J_{21} = A_1P^*, \qquad J_{22} = A_1N^* - B_1 - \frac{WDZ^*}{(D + P^*)^2}, \qquad J_{23} = -\frac{WP^*}{D + P^*},\\
&J_{31} = 0, \qquad J_{32} = \frac{W_1DZ^*}{(D + P^*)^2} - C_1Z^*, \qquad J_{33} = \frac{W_1P^*}{D + P^*} - A_2 - C_1P^*,
\end{aligned}
$$

and the corresponding characteristic equation of matrix (6) is given by

$$
\lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3 = 0, \tag{7}
$$

where

$$
\begin{aligned}
\sigma_1 &= -\left(J_{11} + J_{22} + J_{33}\right),\\
\sigma_2 &= J_{11}J_{22} + J_{11}J_{33} + J_{22}J_{33} - J_{23}J_{32} - J_{12}J_{21},\\
\sigma_3 &= J_{12}J_{21}J_{33} + J_{11}J_{23}J_{32} - J_{11}J_{22}J_{33}.
\end{aligned}
$$

Table 1: Stability of each equilibrium of system (1)

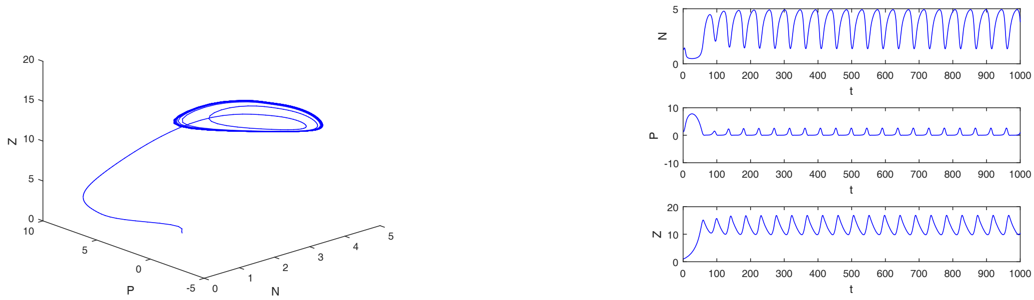
	E_1^*	E_2^*	E_3^*	E_4^*
λ_1	-0.2	-4.1158	$0.1019 + 0.2790i$	-6.0422
λ_2	2.6	0.0158	$0.1019 - 0.2790i$	0.4555
λ_3	-0.05	0.0614	-0.0981	0.4412
Stability	unstable saddle point	unstable saddle point	conditionally stable*	unstable saddle point
Index	1	2	–	2

*System (1) is stable if $\alpha_i < \min \frac{2}{\pi}\left|\arg\lambda_i\right| = \frac{2}{\pi}|\arctan(0.2790/0.1019)| = 0.7771$ according to Lemma 2.1.

Due to the complexity of getting the analytical solutions of Eq. (7) if parameters are continuous functions, we take $N_0 = 1.1, A = 0.2, B = 0.3, C = 0.1, A_1 = 0.5, B_1 = 0.15, W = 0.2, W_1 = 0.26, A_2 = 0.05, C_1 = 0.01, D = 1$

from [20] to show this stability. Substituting the parameters and $E_i^*(i = 1, 2, 3, 4)$ into Eq. (7), one obtains the roots of Eq. (7) and stabilities as shown in Table 1 according to Lemma 2.1.

From Table 1, there exists only one stable equilibrium point when the maximum fractional order among α_1, α_2 and α_3 is less than 0.7771. When $\alpha_1 = \alpha_2 = \alpha_3 = 0.7771$, system (1) will display instability and there is no stable equilibrium if $\alpha_1, \alpha_2, \alpha_3$ are all greater than 0.7771. Thus, system (1) is oscillatory which is displayed in Figure 1(a) and Figure 1(b) when $\alpha_1 = \alpha_2 = \alpha_3 = 0.95$.



(a) The phase orbit

(b) The evolution over time

Figure 1: The phase orbit and the evolution over time of fractional-order nutrient-phytoplankton-zooplankton system

To be more precise, we calculate the Lyapunov exponents of the system by methods in [35] when $\alpha_i = 0.95(i = 1, 2, 3)$. As shown in Figure 2, three Lyapunov exponents are 0, -0.1347, and -1.4335 respectively, which means that the trend of system (1) is a limit cycle.

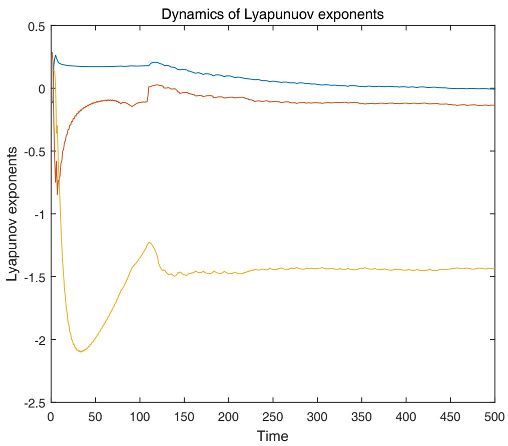


Figure 2: Dynamics of Lyapunov exponents of the fractional-order NPZ system

3. Control of the System

Since the balance of nutrient and plankton plays an important role in the preservation of the ecological environment, this chaotic fractional-order nutrient-plankton system needed to be controlled for a healthy ecology.

In this section, linear state feedback controller is designed to control fractional-order chaotic NPZ system in its equilibrium points as introduced in [24, 27]. Linear feedback controllers have advantages of simple structure and intuitive. To be convenient, we set α_1=α_2=α_3=α, the controlled fractional-order NPZ system is given by

$$\begin{cases} \dfrac{\mathrm{d}^\alpha N}{\mathrm{d}t^\alpha} = N_0 - AN - BNP + CP + k_1\left(N - N^*\right), \\ \dfrac{\mathrm{d}^\alpha P}{\mathrm{d}t^\alpha} = A_1 NP - B_1 P - \dfrac{WPZ}{D+P} + k_2\left(P - P^*\right), \\ \dfrac{\mathrm{d}^\alpha Z}{\mathrm{d}t^\alpha} = \dfrac{W_1 PZ}{D+P} - A_2 Z - C_1 PZ + k_3\left(Z - Z^*\right), \end{cases} \tag{8}$$

where k_1, k_2 and k_3 are feedback gains and (N^*, P^*, Z^*) is the desired equilibrium point to be controlled.

The Jacobian matrix of system (8) at (N^*, P^*, Z^*) is

$$J'\left(E^*\right) = \left(J'_{ij}\right)\Big|_{E^*}, \tag{9}$$

where

$$\begin{aligned} &J'_{11} = -A - BP^* + k_1, \qquad J'_{12} = -BN^* + C, \qquad J'_{13} = 0, \\ &J'_{21} = A_1 P^*, \qquad J'_{22} = A_1 N^* - B_1 - \frac{WDZ^*}{\left(D+P^*\right)^2} + k_2, \qquad J'_{23} = -\frac{WP^*}{D+P^*}, \\ &J'_{31} = 0, \qquad J'_{32} = \frac{W_1 DZ^*}{\left(D+P^*\right)^2} - C_1 Z^*, \qquad J'_{33} = \frac{W_1 P^*}{D+P^*} - A_2 - C_1 P^* + k_3, \end{aligned}$$

and the corresponding characteristic equation of matrix (9) is given by

$$\lambda^3 + \sigma'_1 \lambda^2 + \sigma'_2 \lambda + \sigma'_3 = 0, \tag{10}$$

where

$$\begin{aligned} \sigma'_1 =& -\left(J'_{11} + J'_{22} + J'_{33}\right) - \left(k_1 + k_2 + k_3\right), \\ \sigma'_2 =& J'_{11}J'_{22} + J'_{11}J'_{33} + J'_{22}J'_{33} - J'_{23}J'_{32} - J'_{12}J'_{21} + k_1\left(J'_{22} + J'_{33}\right) + k_2\left(J'_{11} + J'_{33}\right) + k_3\left(J'_{11} + J'_{22}\right) \\ &+ k_1k_2 + k_2k_3 + k_1k_3 - J'_{23}J'_{32} - J'_{12}J'_{21}, \\ \sigma'_3 =& J'_{12}J'_{21}J'_{33} + J'_{11}J'_{23}J'_{32} - J'_{11}J'_{22}J'_{33} + k_1\left(J'_{23}J'_{32} - J'_{22}J'_{33}\right) + k_3\left(J'_{12}J'_{21} - J'_{11}J'_{22}\right) - k_2J'_{11}J'_{33} - k_1k_2k_3. \end{aligned}$$

Our goal is to find suitable feedback gains such that system (8) can be controlled in its equilibrium points. We have the following theorems to describe the control the system:

Theorem 3.1. *The equilibrium* $E_1^* = \left(N_1^*, 0, 0\right)$ *is locally asymptotically stable if* k_1, k_2, k_3 *satisfy*

$$k_1 < A_1, \qquad k_2 < B_1 - \frac{A_1 N_0}{A}, \qquad k_3 < A_2.$$

Proof. Substituting $E_1^* = \left(\frac{N_0}{A}, 0, 0\right)$ into Eq. (10), we have

$$\left[\lambda - \left(k_1 - A_1\right)\right]\left[\lambda - \left(k_2 - B_1 + \frac{A_1 N_0}{A}\right)\right]\left[\left(\lambda - \left(k_3 - A_2\right)\right] = 0. \tag{11}$$

The roots of Eq. (11) are $\lambda_1 = k_1 - A_1$, $\lambda_2 = k_2 - B_1 + \frac{A_1 N_0}{A}$, $\lambda_3 = k_3 - A_2$. Notice that $k_1 < A_1, k_2 < B_1 - \frac{A_1 N_0}{A}$, $k_3 < A_2$, one obtains that all roots are real and negative, which means that $\left|\arg\left(\lambda_i\right)\right| = \pi > \frac{\alpha\pi}{2}$ $(i = 1, 2, 3)$. Then the proof is completed. □

In the same way as in Theorem 3.1, we have the following theorem:

Theorem 3.2. *The equilibrium $E_2^* = \left(N_2^*, P_2^*, 0\right)$ is locally asymptotically stable if feedback gains k_1, k_2, k_3 satisfy one of the following conditions:*

$$\begin{aligned}&(1)\ E < 0,\ F > 0,\ \Delta > 0;\\ &(2)\ E < 0,\ F > 0,\ \Delta < 0 \textit{ for all } \alpha < \frac{2}{\pi}\arctan\left(\frac{\sqrt{-\Delta}}{F}\right),\end{aligned} \tag{12}$$

where

$$\begin{aligned}E &:= \frac{W_1P_2^*}{D+P_2^*} - A_2 - C_1P_2^* + k_3,\\ F &:= \left(-A - BP_2^* + k_1\right) + +k_2,\\ G &:= \left(-A - BP_2^* + k_1\right)k_2 + A_1\left(N_0 - AN_2^*\right),\\ \Delta &:= F^2 - 4G.\end{aligned}$$

Proof. Substituting $E_2^* = \left(N_2^*, P_2^*, 0\right)$ into Eq. (10), one obtains

$$(\lambda - E)\left(\lambda^2 - F\lambda + G\right) = 0.$$

Then we have

$$\lambda_1 = E, \qquad \lambda_{2,3} = \frac{1}{2}\left(-F \pm \sqrt{\Delta}\right).$$

If $(12)_1$ holds, then $\lambda_1 < 0$, $\lambda_{2,3} < 0$. Thus, all the eigenvalues with control corresponding to the equilibrium E_2 are real and negative.

If $(12)_2$ holds, then $\lambda_1 < 0$ and $\lambda_{2,3}$ are complex as follows:

$$\lambda_{2,3} = \frac{-F \pm i\sqrt{-\Delta}}{2},$$

where $i = \sqrt{-1}$. Thus from Lemma 2.1, if $\left|\arg(\lambda_{2,3})\right| = \arctan\left(\frac{\sqrt{-\Delta}}{F}\right) > \frac{\alpha\pi}{2}$ hold, then the equilibrium E_2^* is asymptotically stable. Then the proof is completed. □

Now, we recall a theorem by fractional-order Routh-Hurwitz stability criterion to determine the local stability of interior equilibrium points E_3^* and E_4^*.

Theorem 3.3. [27] *If the discriminant of the eigenvalues of matrix* (9) *is given as*

$$D(P) = 18\sigma_1'\sigma_2'\sigma_3' + (\sigma_1'\sigma_2')^2 - 4\sigma_3'(\sigma_1')^3 - 4(\sigma_2')^3 - 27(\sigma_3')^2,$$

then equilibrium points E_3^* and E_4^* are locally asymptotically stable if they satisfy the following fractional Routh-Hurwitz conditions:

(i) *If $D(P) > 0$, then the necessary and sufficient condition for equilibrium point $E_3^*(E_4^*)$, to be locally asymptotically stable, is $\sigma_1' > 0$, $\sigma_3' > 0$, $\sigma_1'\sigma_2' - \sigma_3' > 0$.*

(ii) *If $D(P) < 0$, $\sigma_1' \geq 0$, $\sigma_2' \geq 0$, $\sigma_3' > 0$, then $E_3^*(E_4^*)$ is locally asymptotically stable for $\alpha < \frac{2}{3}$. However, if $D(P) < 0$, $\sigma_1' < 0$, $\sigma_2' < 0$, $\alpha > \frac{2}{3}$, then all roots of equation* (10) *satisfy the condition $|arg(\lambda)| < \alpha\pi/2$.*

(iii) *If $D(P) < 0$, $\sigma_1' > 0$, $\sigma_2' > 0$, $\sigma_1'\sigma_2' - \sigma_3' = 0$, then $E_3^*(E_4^*)$ is locally asymptotically stable for all $\alpha \in (0,1)$.*

(iv) *The necessary condition for equilibrium point $E_3^*(E_4^*)$, to be locally asymptotically stable, is $\sigma_3' > 0$.*

4. Numerical Simulations

In this section, to verify theoretical results obtained before, we will do corresponding numerical simulations by applying a modified Adams-Bashforth-Moulton algorithm proposed by Diethelm et al. [11, 12].

4.1. *Dynamic behaviors in different fractional orders*

The parameters of system (8) are chosen as $N_0 = 1.1, A = 0.2, B = 0.3, C = 0.1, A_1 = 0.5, B_1 = 0.15, W = 0.2, W_1 = 0.26, A_2 = 0.05, C_1 = 0.01, D = 1$. Based on the algorithm above, we can simulate our results of the fractional-order NPZ system. According to what have been discussed in Section 2, we know that the system exhibit instabilities when $\alpha \geq 0.7771$ as shown in Figure 3(a) and 3(b). And when $\alpha < 0.7771$, we find the cases that the system (1) does not show chaotic behavior in Figure 3(c) and 3(d).

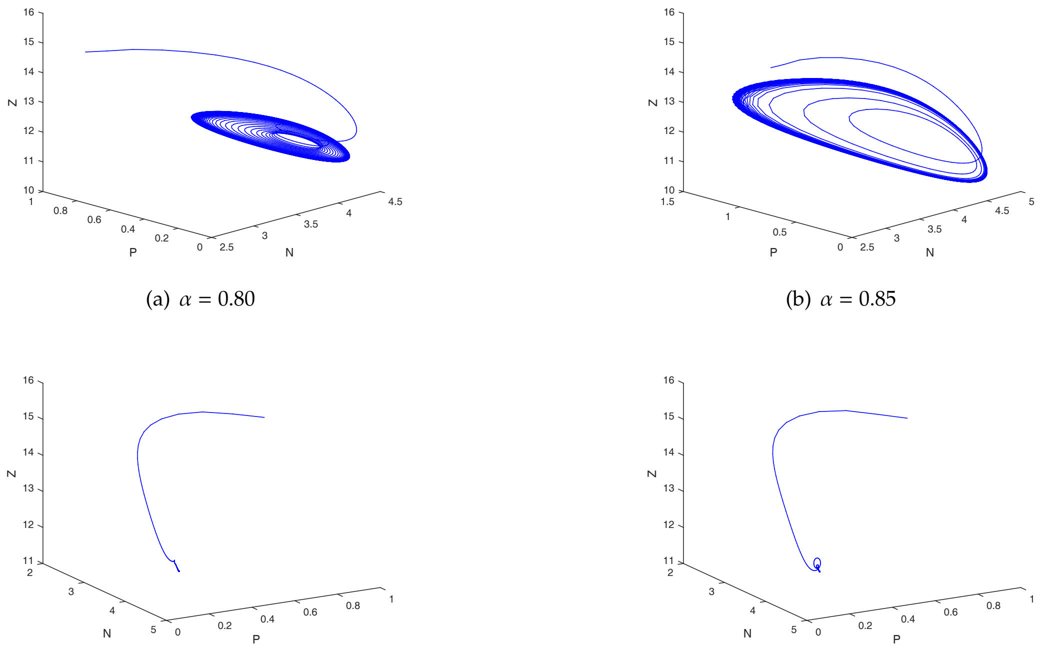


Figure 3: Phase diagrams for system using the fixed values and fractional orders are $\alpha = 0.80$, $\alpha = 0.85$, $\alpha = 0.70$ and $\alpha = 0.75$.

4.2. *Control of the system*

Let us now chose $\alpha = 0.95$ and do the following work to verify whether system (1) can be controlled by our results or not.

4.2.1. *Case 1: Plankton-free point* $E_1^*(5.5, 0, 0)$

Based on Theorem 3.1, let us chose the feedback gain $(k_1, k_2, k_3) = (-1, -3, -1)$ and $\alpha = 0.95$. By calculation, eigenvalues are $\lambda_1 = -1.2$, $\lambda_2 = -0.4$, $\lambda_3 = -1.05$, and the maximum lyapunov exponent is -0.4128, which means that system (8) is asymptotically stable at E_1^* as shown in Figure 4(a) and Figure 5(a).

4.2.2. *Case 2: Zooplankton-free point* $E_2^*(0.3, 13, 0)$

Based on Theorem 3.2, let us chose the feedback gain $(k_1, k_2, k_3) = (4, -2, -2)$ and $\alpha = 0.95$. By calculation, eigenvalues are $\lambda_1 = -0.0664$, $\lambda_2 = -2.0336$, $\lambda_3 = -1.9386$, and the maximum lyapunov exponent is -1.2715, which means that system (8) is asymptotically stable at E_2^* as shown in Figure 4(b) and Figure 5(b).

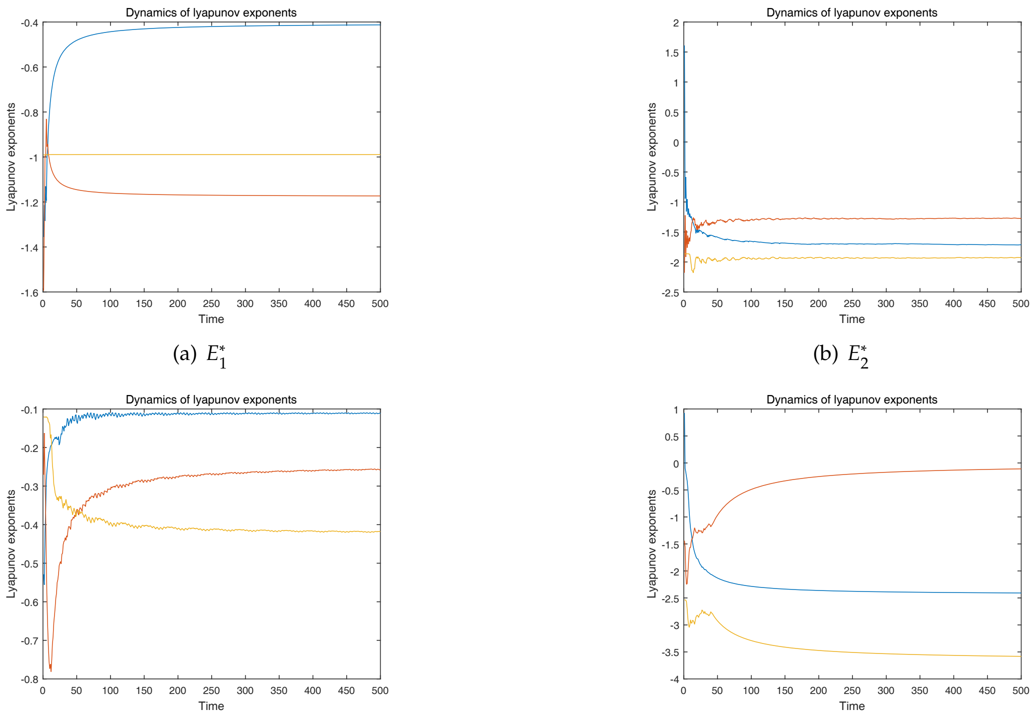
4.2.3. Case 3: Interior point E_3^* (4.0778, 0.2532, 11.8359)

Let us chose the feedback gain $(k_1, k_2, k_3) = (0, -0.5, 0)$ and $\alpha = 0.95$. From Theorem 3.3 (iii), we have $D(P) = -0.0276 < 0$, $\sigma_1' = 0.3943 > 0$, $\sigma_2' = 0.2607 > 0$ and $\sigma_1'\sigma_2' - \sigma_3' \approx 0$, and the maximum lyapunov exponent is -0.1116, which means that system (8) is asymptotically stable at E_3^* as shown in Figure 4(c) and Figure 5(c).

4.2.4. Case 4: Interior point E_4^* (0.5021, 19.7468, 10.4806)

Let us chose the feedback gain $(k_1, k_2, k_3) = (1.5, -0.5, -3)$ and $\alpha = 0.95$. From Theorem 3.3 (iii), we have $D(P) = -21866 < 0$, $\sigma_1' = 8.0279 > 0$, $\sigma_2' = 18.3206 > 0$ and $\sigma_1'\sigma_2' - \sigma_3' \approx 0$, and the maximum lyapunov exponent is -0.1094, which means that system (8) is asymptotically stable at E_4^* as shown in Figure 4(d) and Figure 5(d).

From cases above, we know that the control of fractional-order NPZ system is feasible. Therefore, linear control of the system can be applied when the water ecology is unstable. For example, as shown in case 3, when the system is unstable, which means that biomass of phytoplankton may increase due to eutrophication caused by sewage disposal, decision makers can decrease the phytoplankton growth rate by 0.5 in some approach to make sure the system can return to a dynamical stable state in several days.



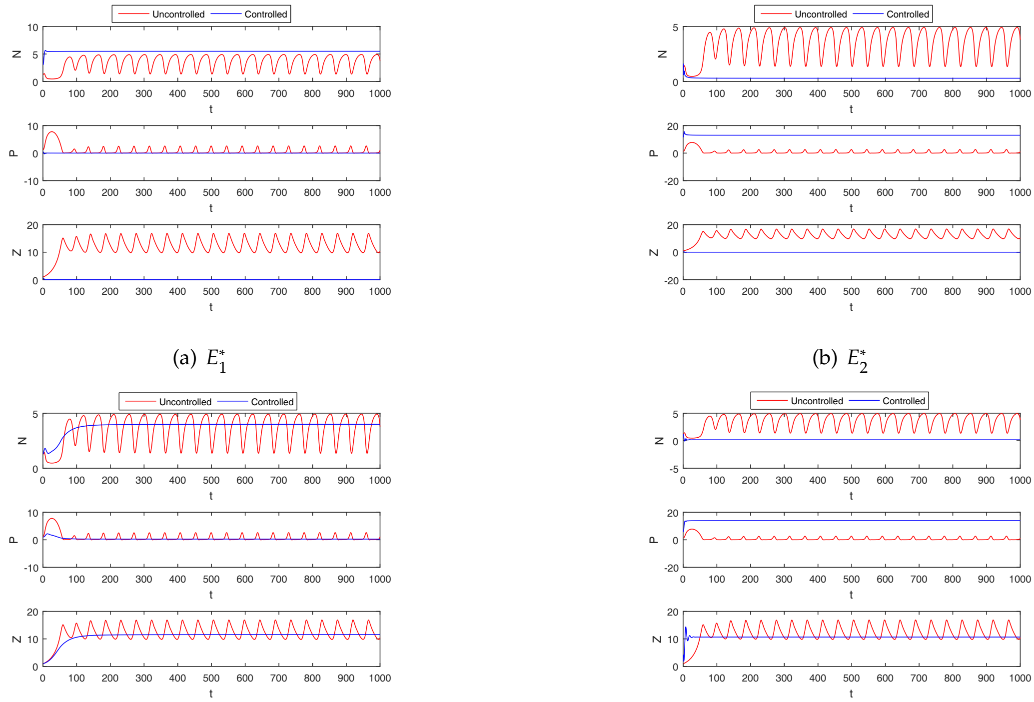
(a) E_1^* (b) E_2^*

(c) E_3^* (d) E_4^*

Figure 4: Dynamics of lyapunov exponents for the controlled system.

5. Conclusion

Overall, our study has revealed the stability in equilibrium points of the fractional-order nutrient-phytoplankton-zooplankton system and applied a simple approach, which is linear feedback controller, to control the instability. Through the work above, we found that fractional order $\alpha = 0.7771$ is the critical value of stability which means that every equilibrium point of system (1) is an unstable saddle point if $\alpha > 0.7771$. What's more, we yielded the sufficient condition of local stability in the equilibrium points of the controlled fractional-order system by using Lemma 2.1 and fractional Routh-Hurwitz criterion. Finally, numerical simulation verified that results obtained above are credible. Thus, linear feedback control is a feasible way to control the fractional NPZ system in water or marine ecology.



(a) E_1^* (b) E_2^*

(c) E_3^* (d) E_4^*

Figure 5: The time response of the state for stabilizing system (8) to each equilibrium point.

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