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On Almost Geodesic Mappings of the Second Type Between Manifolds with Non-symmetric Linear Connection

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Abstract. We derive two mixed systems of Cauchy type in covariant derivatives of the first and second kind that ensures the existence of almost geodesic mappings of the second type between manifolds with non-symmetric linear connection. Also, we consider a particular class of these mappings determined by the condition $\nabla F = 0$, where ∇ is the symmetric part of non-symmetric linear connection $\underset{1}{\nabla}$ and F is the affinor structure. The same special class of almost geodesic mappings of the second type between generalized Riemannian spaces was recently considered in the paper $\big($M.Z. Petrović, *Special almost geodesic mappings of the second type between generalized Riemannian spaces*, Bull. Malays. Math. Sci. Soc. (2), DOI :10.1007/s40840-017-0509-5$\big)$.

1. Introduction and Preliminaries

Geodesic and almost geodesic lines play an important role in geometry and physics. A diffeomorphism $f : M \to \overline{M}$ of manifolds M and $\overline{M}$ endowed with linear connections ∇ and $\overline{\nabla}$ is said to be an *almost geodesic mapping* if maps every geodesic line of the space (M, ∇) into an almost geodesic line of the space $(\overline{M}, \overline{\nabla})$. This concept was introduced by Sinyukov [22] for the mappings between affine connected spaces without torsion. Mikeš [1–4, 6–11, 31] gave some of the significant contributions to the study of geodesic and almost geodesic mappings of affine connected, Riemannian and Einstein spaces. Sobchuk, Mikeš and Pokorná [23] studied special almost geodesic mappings of the second type between semi-symmetric Riemannian spaces. In the same manner special almost geodesic mappings of the second type of the first and second kind are considered between generalized Riemannian spaces in [19]. The aim of this paper is to consider these mappings between manifolds with non-symmetric linear connection. The non-symmetric linear connections and non-symmetric metrics are specially important in the Relativistic theory [5]. The almost geodesic mappings between generalized Riemannian spaces as well as between manifolds with non-symmetric linear connection are studied in [18–20, 24–27, 30]. The authors of the paper [16] defined generalized Kählerian spaces and the authors of the papers [33] and [34] also defined generalized Kählerian spaces in a certain manner. Generalized hyperbolic Kähler spaces are defined and equitorsion

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holomorphically projective mappings of these spaces are studied in [17]. Invariant geometric objects of equitorsion holomorphically projective mappings of generalized Kählerian spaces are found in [34]. The same approach was also used for finding invariant geometric objects of equitorsion geodesic mappings of general affine connection spaces in [32]. Geodesic mappings of general affine connection spaces and holomorphically projective mappings of both generalized elliptic and hyperbolic Kählerian spaces are just particular cases of almost geodesic mappings of the second type of manifolds with non-symmetric linear connection that will be considered in this paper.

Let M be an n-dimensional differentiable manifold with non-symmetric linear connection $\underset{1}{\nabla}$. It is well known that another non-symmetric linear connection $\underset{2}{\nabla}$ on the manifold M is defined by [14]

$$\underset{2}{\nabla}_X Y = \underset{1}{\nabla}_Y X + [X, Y], \ \ X, Y \in \mathcal{X}(M),$$

where as usual $\mathcal{X}(M)$ denotes a set of smooth vector fields on M and $[\cdot, \cdot]$ denotes the Lie bracket.

Let $\big(U, u\big)$, $u = (u^1, \dots, u^n)$ be a local chart at the point $p \in M$. The set of vectors at p is the vector space with basis

$$\frac{\partial}{\partial u^1}, \dots, \frac{\partial}{\partial u^n}.$$

Let us denote

$$X = \frac{\partial}{\partial u^i}, \ \ Y = \frac{\partial}{\partial u^j}, \ \ Z = \frac{\partial}{\partial u^k},$$

and abbreviate $\frac{\partial}{\partial u^i}$ by ∂_i, then we have

$$\underset{1}{\nabla}_{\partial_i} \partial_j = L^h_{ij} \partial_h \quad \text{and} \quad \underset{2}{\nabla}_{\partial_i} \partial_j = L^h_{ji} \partial_h,$$

where the functions L^h_{ij} are called linear connection coefficients or components of non-symmetric linear connection.

In this section we consider basic equations of almost geodesic mappings of the second type between manifolds with non-symmetric linear connections $\underset{1}{\nabla}$ and $\underset{2}{\nabla}$. The non-symmetric linear connections $\underset{\theta}{\nabla}$ ($\theta \in \{1, 2\}$) induce covariant derivatives of tensor fields [13, 15]

$$\underset{1}{\nabla}_m a^i_j \equiv a^i_{j\underset{1}{|}m} = a^i_{j,m} + L^i_{pm} a^p_j - L^p_{jm} a^i_p,$$
$$\underset{2}{\nabla}_m a^i_j \equiv a^i_{j\underset{2}{|}m} = a^i_{j,m} + L^i_{mp} a^p_j - L^p_{mj} a^i_p,$$

where $a^i_{j,m}$ denotes the partial derivative of a tensor a^i_j with respect to x^m and $\underset{\theta}{|}$ denotes the covariant derivative with respect to the connection $\underset{\theta}{\nabla}$, $\theta \in \{1, 2\}$.

Let $c : I \to M$ be a curve on a manifold M with non-symmetric linear connection satisfying the regularity condition $c'(t) \neq 0$, and let $\xi(t) = (c(t), c'(t))$ be the tangent vector field along c. The curve c is called an *almost geodesic of the kind* θ ($\theta \in \{1, 2\}$) if there exist vector fields X_1 and X_2 satisfying $\underset{\theta}{\nabla}_\xi X_i = a^j_i X_j$, for some differentiable functions $a^j_i : I \to \mathbb{R}$ and differentiable real functions $b^i(t)$ along c such that $\xi = b^1 X_1 + b^2 X_2$ holds.

In the present paper we deal with almost geodesic mappings of the second type between manifolds with non-symmetric linear connection [25]. By using two kinds of covariant differentiation we examine two mixed systems of Cauchy type in covariant derivatives for the existence of almost geodesic mappings of the second type between manifolds with non-symmetric linear connection. Some relations between independent curvature tensors of non-symmetric linear connection with respect to special almost geodesic mappings of the second type between manifolds with non-symmetric linear connection are examined. Also, some generalizations of the Weyl (projective) curvature tensor are obtained.

An outline of this paper is as follows. Mixed systems of Cauchy type in covariant derivatives for the existence of almost geodesic mappings of the second type between manifolds with non-symmetric linear connection are given in Section 2. Relations between five independent curvature tensors of non-symmetric linear connection with respect to special almost geodesic mappings of the second type between manifolds with non-symmetric linear connection are given in Section 3. Section 4 is devoted to geometric objects that are invariant with respect to special almost geodesic mappings of the second type between manifolds with non-symmetric linear connection.

2. Almost Geodesic Mappings of the Second Type of Manifolds with Non-symmetric Linear Connection

In [24] the basic facts on almost geodesic lines and almost geodesic mappings of manifolds with non-symmetric linear connection (in the index notation) are given. Let M and $\overline{M}$ be two n-dimensional manifolds ($n > 2$) with non-symmetric linear connections $\underset{1}{\nabla}$ and $\underset{1}{\overline{\nabla}}$, respectively. We can consider these manifolds in the *common coordinate system with respect to the diffeomorphism* $f : M \to \overline{M}$. In this coordinate system the corresponding points $p \in M$ and $f(p) \in \overline{M}$ have the same coordinates. Therefore we can suppose $M \equiv \overline{M}$ and for $\theta \in \{1, 2\}$ we can put

$$\underset{\theta}{P} = \underset{\theta}{\overline{\nabla}} - \underset{\theta}{\nabla},$$

where $\underset{\theta}{P}$ is a tensor field of type $(1, 2)$, called the *deformation tensor field of linear connections* $\underset{\theta}{\overline{\nabla}}$ *and* $\underset{\theta}{\nabla}$ *with respect to the mapping* f. In local coordinates, we have

$$\underset{1}{P}{}^h_{ij} = \overline{L}^h_{ij} - L^h_{ij}, \quad \underset{2}{P}{}^h_{ij} = \overline{L}^h_{ji} - L^h_{ji}, \tag{1}$$

where $L^h_{ij}(L^h_{ji})$ and $\overline{L}^h_{ij}(\overline{L}^h_{ji})$ are components of the non-symmetric linear connections $\underset{1}{\nabla}(\underset{2}{\nabla})$ and $\underset{1}{\overline{\nabla}}(\underset{2}{\overline{\nabla}})$, respectively. Analogously as in [4] let us introduce tensor fields $\underset{1}{P}$ and $\underset{2}{P}$ of type $(1, 3)$ given by

$$\underset{1}{P}(X, Y, Z) = \sum_{CS(X,Y,Z)} \underset{1}{\nabla}_Z \underset{1}{P}(X, Y) + \underset{1}{P}(\underset{1}{P}(X, Y), Z), \ \ X, Y, Z \in \mathcal{X}(M) \tag{2}$$

and

$$\underset{2}{P}(X, Y, Z) = \sum_{CS(X,Y,Z)} \underset{2}{\nabla}_Z \underset{2}{P}(X, Y) + \underset{2}{P}(Z, \underset{2}{P}(X, Y)), \ \ X, Y, Z \in \mathcal{X}(M), \tag{3}$$

where $\sum_{CS(\cdot,\cdot,\cdot)}$ denotes the cyclic sum on arguments in brackets.

A diffeomorphism f of a manifold M with non-symmetric linear connection $\underset{\theta}{\nabla}$ onto the manifold $\overline{M}$ with non-symmetric linear connection $\underset{\theta}{\overline{\nabla}}$ is an *almost geodesic mapping of the kind* θ ($\theta \in \{1, 2\}$) if and only if the following condition is satisfied

$$\underset{\theta}{P}(X_1, X_2, X_3) \wedge \underset{\theta}{P}(X_4, X_5) \wedge X_6 = 0, \ \ X_i \in \mathcal{X}(M), \ i = 1, \dots, 6,$$

where $\underset{\theta}{P}$ is the deformation tensor field of the non-symmetric linear connections $\underset{\theta}{\overline{\nabla}}$ and $\underset{\theta}{\nabla}$, with respect to the diffeomorphism f, whereas $\underset{1}{P}(X, Y, Z)$ and $\underset{2}{P}(X, Y, Z)$ are respectively defined by (2) and (3).

Basic facts on almost geodesic mappings of the second type between non-symmetric affine connection spaces are given in [25]. We have two kinds of almost geodesic mappings of the second type $\underset{\theta}{\pi_2}$, $\theta \in \{1, 2\}$.

The basic equations of these mappings are given by [25] M.S. Stanković,*On a canonic almost geodesic mappings of the second type of affine spaces*, Filomat 13, (1999), 105–114.

$$\begin{aligned} &\underset{1}{P}(X,Y) = \sum_{CS(X,Y)} \Big(\psi(X)\cdot Y + \sigma(X)\cdot FY\Big) + K(X,Y),\\ &\underset{2}{P}(X,Y) = \underset{1}{P}(Y,X), \end{aligned} \tag{4}$$

where $X, Y \in \mathcal{X}(M)$, ψ and σ are 1-forms, K is an anti-symmetric tensor field of type $(1,2)$ and F is a tensor field of type $(1,1)$ satisfying

$$\begin{aligned} &\sum_{CS(X,Y)} \Big(\underset{\theta}{\nabla}_Y FX + F^2X\cdot\sigma(Y) + K(FY,X)\Big) =\\ &\sum_{CS(X,Y)} \Big(\mu(X)\cdot FY + \nu(X)\cdot Y\Big),\ \ X,Y\in\mathcal{X}(M),\ \theta\in\{1,2\}, \end{aligned} \tag{5}$$

for some 1-forms μ and ν.

Let us introduce a tensor field $\underset{1}{Q}$ of type $(1,2)$ defined by

$$\underset{1}{Q}(X,Y) = \sum_{CS(X,Y)} \Big(-F^2X\cdot\sigma(Y) - K(FY,X) + \mu(X)\cdot FY + \nu(X)\cdot Y\Big).$$

Then for $\theta = 1$ the condition (5) can be represented in the following way

$$\underset{1}{\nabla}_Y F(X) + \underset{1}{\nabla}_X F(Y) = \underset{1}{Q}(X,Y), \tag{6}$$

which further implies

$$\underset{1}{\nabla}_Z\underset{1}{\nabla}_Y FX + \underset{1}{\nabla}_Z\underset{1}{\nabla}_X FY = \underset{1}{\nabla}_Z\underset{1}{Q}(X,Y).$$

Now, let us consider the following expression

$$\begin{aligned} &\underset{1}{\nabla}_Y\underset{1}{Q}(X,Z) - \underset{1}{\nabla}_Z\underset{1}{Q}(X,Y) + \underset{1}{\nabla}_X\underset{1}{Q}(Y,Z) =\\ &\quad \underset{1}{\nabla}_Y\underset{1}{\nabla}_Z FX + \underset{1}{\nabla}_Y\underset{1}{\nabla}_X FZ - \underset{1}{\nabla}_Z\underset{1}{\nabla}_Y FX\\ &\quad -\underset{1}{\nabla}_Z\underset{1}{\nabla}_X FY + \underset{1}{\nabla}_X\underset{1}{\nabla}_Z FY + \underset{1}{\nabla}_X\underset{1}{\nabla}_Y FZ =\\ &\quad \underset{1}{\nabla}_Y\underset{1}{\nabla}_Z FX - \underset{1}{\nabla}_Z\underset{1}{\nabla}_Y FX + \underset{1}{\nabla}_X\underset{1}{\nabla}_Z FY - \underset{1}{\nabla}_Z\underset{1}{\nabla}_X FY\\ &\quad +\underset{1}{\nabla}_X\underset{1}{\nabla}_Y FZ - \underset{1}{\nabla}_Y\underset{1}{\nabla}_X FZ + 2\underset{1}{\nabla}_Y\underset{1}{\nabla}_X FZ =\\ &\quad \underset{1}{R}(Y,Z)FX - F(\underset{1}{R}(Y,Z)X) - \underset{1}{\nabla}_{T(Z,Y)}F(X)\\ &\quad + \underset{1}{R}(X,Z)FY - F(\underset{1}{R}(X,Z)Y) - \underset{1}{\nabla}_{T(Z,X)}F(Y)\\ &\quad + \underset{1}{R}(X,Y)FZ - F(\underset{1}{R}(X,Y)Z) - \underset{1}{\nabla}_{T(Y,X)}F(Z) + 2\underset{1}{\nabla}_Y\underset{1}{\nabla}_X FZ, \end{aligned} \tag{7}$$

where we used the first Ricci type identity [13, 15]

$$\underset{1}{\nabla}_X\underset{1}{\nabla}_Y FZ - \underset{1}{\nabla}_Y\underset{1}{\nabla}_X FZ = \underset{1}{R}(X,Y)FZ - F(\underset{1}{R}(X,Y)Z) - \underset{1}{\nabla}_{T(Y,X)}FZ.$$

From (7) by applying the property $\underset{1}{R}(X,Y)Z = -\underset{1}{R}(Y,X)Z$ of the curvature tensor $\underset{1}{R}$ we can easily obtain the following system of Cauchy type

$$\begin{aligned}
\underset{1}{V}(X,Z) =& \underset{1}{\nabla}_Z F(X),\\
\underset{1}{\nabla}_Y \underset{1}{V}(X,Z) =& -2F(\underset{1}{R}(Z,X)Y) - \underset{1}{R}(Y,Z)FX\\
&+ \underset{1}{R}(Y,X)FZ + \underset{1}{R}(Z,X)FY\\
&+ F\Big(\underset{1}{R}(X,Y)Z + \underset{1}{R}(Y,Z)X - \underset{1}{R}(Z,X)Y\Big)\\
&+ \underset{1}{V}(Z,T(X,Y)) + \underset{1}{V}(Y,T(X,Z)) + \underset{1}{V}(X,T(Y,Z))\\
&+ \underset{1}{\nabla}_Y \underset{1}{Q}(Z,X) - \underset{1}{\nabla}_X \underset{1}{Q}(Z,Y) + \underset{1}{\nabla}_Z \underset{1}{Q}(Y,X)
\end{aligned} \tag{8}$$

with respect to the tensor fields F and $\underset{1}{V}$.According to the first equation in (8) equation (6) takes form

$$\underset{1}{V}(X,Y) + \underset{1}{V}(Y,X) = \underset{1}{Q}(X,Y),$$

algebraic character with respect to $\underset{1}{V}$ and $\underset{1}{Q}$. Together with (8) give a mixed system of Cauchy type in covariant derivatives.

Analogously, we can consider almost geodesic mappings of type $\underset{2}{\pi}_2$. By using the second Ricci type identity [13, 15]

$$\underset{2}{\nabla}_X \underset{2}{\nabla}_Y FZ - \underset{2}{\nabla}_Y \underset{2}{\nabla}_X FZ = \underset{2}{R}(X,Y)FZ - F(\underset{2}{R}(X,Y)Z) + \underset{2}{\nabla}_{T(Y,X)} FZ,$$

and the property $\underset{2}{R}(X,Y)Z = -\underset{2}{R}(Y,X)Z$ of the curvature tensor $\underset{2}{R}$. From (7) one can easily obtain the following mixed Cauchy type system in covariant derivatives

$$\begin{aligned}
\underset{2}{V}(X,Z) =& \underset{2}{\nabla}_Z F(X),\\
\underset{2}{\nabla}_Y \underset{2}{V}(X,Z) =& -2F(\underset{2}{R}(Z,X)Y) - \underset{2}{R}(Y,Z)FX\\
&+ \underset{2}{R}(Y,X)FZ + \underset{2}{R}(Z,X)FY\\
&+ F\Big(\underset{2}{R}(X,Y)Z + \underset{2}{R}(Y,Z)X - \underset{2}{R}(Z,X)Y\Big)\\
&+ \underset{2}{V}(Z,T(X,Y)) + \underset{2}{V}(Y,T(X,Z)) + \underset{2}{V}(X,T(Y,Z))\\
&+ \underset{2}{\nabla}_Y \underset{2}{Q}(Z,X) - \underset{2}{\nabla}_X \underset{2}{Q}(Z,Y) + \underset{2}{\nabla}_Z \underset{2}{Q}(Y,X),
\end{aligned} \tag{9}$$

where $\underset{2}{Q}(X,Y) = \underset{2}{V}(X,Y) + \underset{2}{V}(Y,X)$.

The systems of equations (8) and (9) are nonlinear. Hence if we consider almost geodesic mappings of type $\underset{\theta}{\pi}_2$ ($\theta = 1,2$) we encounter difficulties and do not come up with some interesting results, therefore we will consider some subclasses of these mappings. Almost geodesic mappings of type $\underset{\theta}{\pi}_2(e)$ $e = \pm 1$, $\theta = 1,2$ between manifolds with non-symmetric linear connection shall be of particular importance.

3. Special Almost Geodesic Mappings of the Second Type and Relations Between Curvature Tensors

We consider special almost geodesic mappings of type $\underset{\theta}{\pi}_2(e)$, $e = \pm 1$ ($\theta = 1,2$) between manifolds with non-symmetric linear connection, we denote then by $\underset{\theta}{\pi}_2(e, \nabla F)$. The mappings of this type have been already studied in the case of generalized Riemannian spaces [19]. *Basic equations of mappings* $\underset{\theta}{\pi}_2(e,\nabla F)$, $e=\pm 1$, $\theta\in\{1,2\}$ are given by [19]

$$\begin{aligned}&\underset{\theta}{P}(X,Y)=(-1)^{(\theta-1)}\Big(\sum_{CS(X,Y)}\big(\psi(X)\cdot Y+\sigma(X)\cdot FY\big)+K(X,Y)\Big),\\ &\sum_{CS(X,Y)}\big(\underset{\theta}{\nabla}_Y FX-(-1)^{\theta}K(FY,X)\big)=\sum_{CS(X,Y)}\big(\mu(X)\cdot FY-\mu(FX)\cdot Y\big),\end{aligned}\tag{1}$$

where $X,Y\in\mathcal{X}(M)$, ψ, σ are 1-forms, K is an anti-symmetric tensor field of type $(1,2)$ and F is a tensor field of type $(1,1)$ satisfying

$$F^2=eI,\quad \nabla F=0.\tag{2}$$

In (2) and in what follows ∇ denotes the symmetric part of non-symmetric linear connection $\underset{1}{\nabla}$, i.e.,

$$\nabla_X Y=\frac{1}{2}\big(\underset{1}{\nabla}_X Y+\underset{1}{\nabla}_Y X\big).$$

A non-symmetric linear connection $\underset{1}{\nabla}$ is determined by its symmetric part ∇ and the torsion tensor field T by

$$\underset{1}{\nabla}_X Y=\nabla_X Y+\frac{1}{2}T(X,Y).$$

The deformation tensor field P of symmetric linear connection ∇ is given by

$$P(X,Y)=\frac{1}{2}\big(\underset{\theta}{P}(X,Y)+\underset{\theta}{P}(Y,X)\big),\ \ \theta\in\{1,2\}.\tag{3}$$

Example 3.1. (See Example 3.1 in [19]) *A generalized Kähler space* (see [16, 28, 29]) *is a generalized Riemannian space equipped with a non-symmetric Riemannian metric g and an affinor structure F that satisfy*

$$\begin{aligned}F^2&=-I,\\ \underline{g}(FX,FY)&=-\underline{g}(X,Y),\\ \underset{1}{\nabla}F=0\ \ &and\ \ \underset{2}{\nabla}F=0,\end{aligned}$$

where $\underline{g}$ is the symmetric part of metric g.

A generalized Kähler space is a manifold with non-symmetric linear connection. The non-symmetric linear connection $\underset{1}{\nabla}$ is explicitly defined by

$$\underline{g}(\underset{1}{\nabla}_X Y,Z)=\frac{1}{2}(Xg(Y,Z)+Yg(Z,X)-Zg(Y,X)),$$

whereas the non-symmetric linear connection $\underset{2}{\nabla}$ is given by

$$\underset{2}{\nabla}_X Y=\underset{1}{\nabla}_Y X.$$

Let (M,g,F) and $(\overline{M},\overline{g},\overline{F})$ be two generalized Kähler spaces of dimension $2n\geq 4$. An equitorsion holomorphically projective mapping $f:M\to\overline{M}$ (see [28, 29]) *is an equitorsion almost geodesic mapping of type $\underset{\theta}{\pi}_2(-1,\nabla F)$, $\theta\in\{1,2\}$.*

On manifolds with non-symmetric linear connection one can define five independent curvature tensors [12, 14, 21]:

$$\begin{aligned}
\underset{\theta}{R}(X,Y)Z =& \underset{\theta}{\nabla}_X \underset{\theta}{\nabla}_Y Z - \underset{\theta}{\nabla}_Y \underset{\theta}{\nabla}_X Z, \quad \theta = 1,2;\\
\underset{3}{R}(X,Y)Z =& \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{1}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{2}{\nabla}_X Y} Z;\\
\underset{4}{R}(X,Y)Z =& \underset{2}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z + \underset{2}{\nabla}_{\underset{2}{\nabla}_Y X} Z - \underset{1}{\nabla}_{\underset{1}{\nabla}_X Y} Z;\\
\underset{5}{R}(X,Y)Z =& \frac{1}{2}\Big(\underset{1}{\nabla}_X \underset{1}{\nabla}_Y Z - \underset{2}{\nabla}_Y \underset{1}{\nabla}_X Z + \underset{2}{\nabla}_X \underset{2}{\nabla}_Y Z - \underset{1}{\nabla}_Y \underset{2}{\nabla}_X Z\Big).
\end{aligned} \tag{4}$$

Curvature tensors R and $\overline{R}$ of symmetric linear connections ∇ and $\overline{\nabla}$, respectively, satisfy the following relation [22], p. 170

$$\begin{aligned}
\overline{R}(X,Y)Z =& R(X,Y)Z + (\nabla_X P)(Z,Y) - (\nabla_Y P)(Z,X) + P(P(Z,Y),X)\\
&- P(P(Z,X),Y).
\end{aligned} \tag{5}$$

Relations between the curvature tensors $\underset{\theta}{R}$ $(\theta = 1,\dots,5)$ and the curvature tensor R are examined in [12]:

$$\underset{1}{R}(X,Y)Z = R(X,Y)Z + \frac{1}{2}(\nabla_X T)(Z,Y) - \frac{1}{2}(\nabla_Y T)(Z,X) + \frac{1}{4}T(T(Z,Y),X) - \frac{1}{4}T(T(Z,X),Y); \tag{6}$$

$$\underset{2}{R}(X,Y)Z = R(X,Y)Z - \frac{1}{2}(\nabla_X T)(Z,Y) + \frac{1}{2}(\nabla_Y T)(Z,X) + \frac{1}{4}T(T(Z,Y),X) - \frac{1}{4}T(T(Z,X),Y);$$

$$\underset{3}{R}(X,Y)Z = R(X,Y)Z + \frac{1}{2}(\nabla_X T)(Z,Y) + \frac{1}{2}(\nabla_Y T)(Z,X) - \frac{1}{4}T(T(Z,Y),X) + \frac{1}{4}T(T(Z,X),Y) + \frac{1}{2}T(T(Y,X),Z);$$

$$\underset{4}{R}(X,Y)Z = R(X,Y)Z + \frac{1}{2}(\nabla_Z T)(Z,Y) + \frac{1}{2}(\nabla_Y T)(Z,X) - \frac{1}{4}T(T(Z,Y),X) + \frac{1}{4}T(T(Z,X),Y) + \frac{1}{2}T(T(Y,X),Z);$$

$$\underset{5}{R}(X,Y)Z = R(X,Y)Z + \frac{1}{4}T(T(Z,Y),X) + \frac{1}{4}T(T(Z,X),Y).$$

By $\sum_{CA(\cdot,\cdot,\cdot)}$ we denote [19]

$$\sum_{CA(Y,Z)} A(X,Y,Z) = A(X,Y,Z) - A(X,Z,Y),$$

where A is an arbitrary tensor field.

Relations between the curvature tensors $\underset{\theta}{R}$ and $\underset{\theta}{\overline{R}}$ $(\theta = 1,\dots,5)$ of generalized Riemannian (M,g) and $(\overline{M},\overline{g})$ with respect to special almost geodesic mappings $\underset{\theta}{\pi_2}(e,\nabla F)$, $e = \pm 1$ $(\theta = 1,2)$ are derived in [19]. These relations are given in Proposition 3.2.

Proposition 3.2. *Let $f : M \to \overline{M}$ be an almost geodesic mapping of type $\underset{\theta}{\pi_2}(e,\nabla F)$, $e = \pm 1$, $\theta \in \{1,2\}$ and let $\underset{\theta}{R}$, $\underset{\theta}{\overline{R}}$ are θ-kind $(\theta \in \{1,\dots,5\})$ curvature tensors of manifolds M and $\overline{M}$ with non-symmetric linear connections $\underset{1}{\nabla}$ and $\underset{1}{\overline{\nabla}}$, respectively. Then the following relations are valid*

$$\begin{aligned}
\underset{1}{\overline{R}}(X,Y)Z = \underset{1}{R}(X,Y)Z - \sum_{CA(X,Y)} \Big(&\lambda(Z,Y)\cdot X + \eta(Z,Y)\cdot FX - (\nabla_X K)(Z,Y)\\
&- \frac{1}{4}\overline{T}(\overline{T}(Z,Y),X) + \frac{1}{4}T(T(Z,Y),X)\Big),
\end{aligned} \tag{7}$$

$$\overline{\underset{2}{R}}(X,Y)Z = \underset{2}{R}(X,Y)Z - \sum_{CA(X,Y)} \Big(\lambda(Z,Y)\cdot X + \eta(Z,Y)\cdot FX + (\nabla_X K)(Z,Y) - \frac{1}{4}\overline{T}(\overline{T}(Z,Y),X) + \frac{1}{4}T(T(Z,Y),X)\Big),$$

$$\overline{\underset{3}{R}}(X,Y)Z = \underset{3}{R}(X,Y)Z - \sum_{CA(X,Y)} \Big(\lambda(Z,Y)\cdot X + \eta(Z,Y)\cdot FX + \frac{1}{4}\overline{T}(\overline{T}(Z,Y),X) - \frac{1}{4}T(T(Z,Y),X) + \frac{1}{2}\overline{T}(\overline{T}(Y,X),Z) - \frac{1}{2}T(T(Y,X),Z)\Big) + \sum_{CS(X,Y)} (\nabla_X K)(Z,Y),$$

$$\overline{\underset{4}{R}}(X,Y)Z = \underset{4}{R}(X,Y)Z - \sum_{CA(X,Y)} \Big(\lambda(Z,Y)\cdot X + \eta(Z,Y)\cdot FX + \frac{1}{4}\overline{T}(\overline{T}(Z,Y),X) - \frac{1}{4}T(T(Z,Y),X) - \frac{1}{2}\overline{T}(\overline{T}(Y,X),Z) + \frac{1}{2}T(T(Y,X),Z)\Big) + \sum_{CS(X,Y)} (\nabla_Z K)(X,Y),$$

$$\overline{\underset{5}{R}}(X,Y)Z = \underset{5}{R}(X,Y)Z - \sum_{CA(X,Y)} \Big(\lambda(Z,Y)\cdot X + \eta(Z,Y)\cdot FX\Big) + \sum_{CS(X,Y)} \Big(\frac{1}{4}\overline{T}(\overline{T}(Z,Y),X) - \frac{1}{4}T(T(Z,Y),X)\Big),$$

where λ and η are symmetric bilinear forms given by

$$\lambda(X,Y) = (\nabla_Y\psi)(X) + e\sigma(X)\cdot\sigma(Y) - \psi(X)\cdot\psi(Y) - \sum_{CS(X,Y)} \sigma(X)\cdot\psi(FY),$$

$$\eta(X,Y) = (\nabla_Y\sigma)(X) - \sum_{CS(X,Y)} \sigma(X)\cdot\sigma(FY).$$

4. Invariants of Special Equitorsion Almost Geodesic Mappings of the Second Type

An almost geodesic mapping $f : M \to \overline{M}$ of the second type has the *property of reciprocity* (see [22, 25]) if its inverse mapping $f^{-1} : \overline{M} \to M$ is an almost geodesic mapping of the second type and corresponds to the same affinor structure F. Since the deformation tensor fields $\underset{1}{P}$ and $\overline{\underset{1}{P}}$ of linear connections $\underset{1}{\nabla}$ and $\overline{\underset{1}{\nabla}}$ with respect to the mappings f and f^{-1} satisfy the relation

$$\overline{\underset{1}{P}}(X,Y) = -\underset{1}{P}(X,Y),$$

without loss of generality we can suppose

$$\overline{\psi} = -\psi, \quad \overline{\sigma} = -\sigma, \quad \overline{F} = F, \quad \overline{K} = -K.$$

Proposition 4.1. *Let M and $\overline{M}$ be two n-dimensional differentiable manifolds with non-symmetric linear connection. A necessary and sufficient condition for an almost geodesic mapping $f : M \to \overline{M}$ of type $\underset{1}{\pi}_2$ to have the property of reciprocity is expressed by the following relation*

$$F^2 = \alpha I + \beta F,$$

where F is the affinor structure corresponding to the mapping f, and α, β are invariants (scalar functions).

It is a simple matter to verify that almost geodesic mappings of type $\underset{\theta}{\pi}_2(e,\nabla F)$, $e = \pm 1$, $\theta \in \{1,2\}$ have the property of reciprocity [19].

Definition 4.2. [18, 19] *Let M and $\overline{M}$ be two n-dimensional affine connected manifolds with torsion tensor fields T and $\overline{T}$, respectively. An almost geodesic mapping $f : M \to \overline{M}$ is an* equitorsion almost geodesic mapping *if the torsion tensor is preserved, i.e.,*

$$T(X, Y) = \overline{T}(X, Y). \tag{8}$$

Let M and $\overline{M}$ be two n-dimensional manifolds with non-symmetric linear connection and let $f : M \to \overline{M}$ be an equitorsion almost geodesic mapping of type $\underset{\theta}{\pi}_2(e, \nabla F)$, $\theta \in \{1, 2\}$, $e = \pm 1$. Let us construct the geometric objects $\underset{\theta}{\widetilde{W}}$, $\theta = 1, \ldots, 5$ in the same way as in [19]

$$\underset{1}{\widetilde{W}}(X, Y)Z = \underset{1}{R}(X, Y)Z + \sum_{CA(X,Y)} \Big(\underset{1}{\omega}(Z, X)Y - \frac{1}{2}\nabla_X\sigma(Z)FY - \frac{1}{4}T(T(Z, Y), X)\Big), \tag{9}$$

where

$$\underset{1}{\omega}(Z, X) = \frac{1}{N-1}\Big(\underset{1}{\mathrm{Ric}}(Z, X) - \frac{1}{2}(\nabla_{FX}\sigma)(Z) - \frac{1}{4}\sum_{CA(X,Y)} \mathrm{Tr}\Big(Y \to T(T(Z, X), Y)\Big)\Big);$$

$$\underset{2}{\widetilde{W}}(X, Y)Z = \underset{2}{R}(X, Y)Z + \sum_{CA(X,Y)} \Big(\underset{2}{\omega}(Z, X) \cdot Y - \frac{1}{2}(\nabla_X\sigma)(Z) \cdot FY - \frac{1}{4}T(T(Z, Y), X)\Big), \tag{10}$$

where

$$\underset{2}{\omega}(Z, X) = \frac{1}{N-1}\Big(\underset{2}{\mathrm{Ric}}(Z, X) - \frac{1}{2}(\nabla_{FX}\sigma)(Z) - \frac{1}{4}\sum_{CA(X,Y)} \mathrm{Tr}\Big(Y \to T(T(Z, X), Y)\Big)\Big);$$

$$\begin{aligned}\underset{3}{\widetilde{W}}(X, Y)Z = \underset{3}{R}(X, Y)Z + \sum_{CA(X,Y)} \Big(&\underset{3}{\omega}(Z, X) \cdot Y - \frac{1}{2}(\nabla_X\sigma)(Z) \cdot FY + \frac{1}{4}T(T(Z, Y), X)\\ &+ \frac{1}{2}T(T(Y, X), Z)\Big),\end{aligned} \tag{11}$$

where

$$\begin{aligned}\underset{3}{\omega}(Z, X) = \frac{1}{N-1}\Big(&\underset{3}{\mathrm{Ric}}(Z, X) - \frac{1}{2}(\nabla_{FX}\sigma)(Z) + \frac{1}{4}\sum_{CA(X,Y)} \mathrm{Tr}\Big(Y \to T(T(Z, X), Y)\Big)\\ &+ \frac{1}{2}\mathrm{Tr}\Big(Y \to T(T(X, Y), Z)\Big)\Big);\end{aligned}$$

$$\begin{aligned}\underset{4}{\widetilde{W}}(X, Y)Z = \underset{4}{R}(X, Y)Z + \sum_{CA(X,Y)} \Big(&\underset{4}{\omega}(Z, X) \cdot Y - \frac{1}{2}(\nabla_X\sigma)(Z)FY + \frac{1}{4}T(T(Z, Y), X)\\ &- \frac{1}{2}T(T(Y, X), Z)\Big),\end{aligned} \tag{12}$$

where

$$\begin{aligned}\underset{4}{\omega}(Z, X) = \frac{1}{N-1}\Big(&\underset{4}{\mathrm{Ric}}(Z, X) - \frac{1}{2}(\nabla_{FX}\sigma)(Z) + \frac{1}{4}\sum_{CA(X,Y)} \mathrm{Tr}\Big(Y \to T(T(Z, X), Y)\Big)\\ &- \frac{1}{2}\mathrm{Tr}\Big(Y \to T(T(X, Y), Z)\Big)\Big);\end{aligned}$$

$$\underset{5}{\widetilde{W}}(X, Y)Z = \underset{5}{R}(X, Y)Z + \sum_{CA(X,Y)} \Big(\underset{5}{\omega}(Z, X) \cdot Y - \frac{1}{2}(\nabla_X\sigma)(Z)FY\Big) - \sum_{CS(Y,X)} \frac{1}{4}T(T(Z, Y), X), \tag{13}$$

where

$$\underset{5}{\omega}(Z,X)=\frac{1}{N-1}\Big(\underset{5}{\mathrm{Ric}}(Z,X)-\frac{1}{2}(\nabla_{FX}\sigma)(Z)-\frac{1}{4}\sum_{CS(Y,X)}\mathrm{Tr}\Big(Y\to T(T(Z,X),Y)\Big)\Big).$$

We consider special equitorsion almost geodesic mappings of type $\underset{\theta}{\pi}_2(e,\nabla F)$, $\theta\in\{1,2\}$, $e=\pm1$ between manifolds with non-symmetric linear connection. By applying the same technique as in [19] we obtained the invariant geometric objects with respect to these mappings that are algebraically identical with those in the case of generalized Riemannian spaces derived in [19].

Proposition 4.3. *Let M and $\overline{M}$ be two n-dimensional manifolds with non-symmetric linear connection and let $f:M\to\overline{M}$ be an equitorsion almost geodesic mapping of type $\underset{\theta}{\pi}_2(e,\nabla F)$, $\theta\in\{1,2\}$, $e=\pm1$. The geometric objects $\underset{\theta}{\widetilde{W}}$, $\theta=1,\dots,5$ defined by* (9)–(13) *are invariant with respect to the mapping f.*

5. Conclusion

First, for almost geodesic mappings of the second type between manifolds with non-symmetric linear connection mixed systems of Cauchy type in covariant derivatives of the first and second kind are examined. These are analogous to the related system of Cauchy type for the existence of almost geodesic mappings of the second type between affine connected manifolds without torsion which could be found in the famous book by Sinyukov [22]. In that manner we filled the gap in the theory of almost geodesic mappings of the second type between manifolds with non-symmetric linear connection.

Second, we considered special almost geodesic mappings of the second type between manifolds with non-symmetric linear connection. The same type of almost geodesic mappings between semi-symmetric Riemannian spaces was under consideration in the paper [23] which had motivated us to consider such mappings in the more general situation and to find some invariant geometric objects of these mappings. The special almost geodesic mapping $f:M\to\overline{M}$ of the second type are considered in the particular cases, when the manifolds M and $\overline{M}$ were generalized Riemannian or Kähler manifolds [19], and in the present paper these mappings were considered between manifolds with non-symmetric linear connection.

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