Filomat 32:11 (2018), 3853–3865
https://doi.org/10.2298/FIL1811853S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Tauberian Conditions with Controlled Oscillatory Behavior for Statistical Convergence

Sefa Anıl Sezer[a], Rahmet Savaş[a], İbrahim Çanak[b]

[a]*İstanbul Medeniyet University, Department of Mathematics, İstanbul, Turkey*
[b]*Ege University, Department of Mathematics, İzmir, Turkey*

Abstract. We present new Tauberian conditions in terms of the general logarithmic control modulo of the oscillatory behavior of a real sequence (s_n) to obtain

$$\lim_{n\to\infty} s_n = \xi \;\text{ from } st - \lim_{n\to\infty} s_n = \xi,$$

where ξ is a finite number. We also introduce the statistical (ℓ, m) summability method and extend some Tauberian theorems to this method. The main results improve some well-known Tauberian theorems obtained for the statistical convergence.

1. Introduction and Background

Let $\mathbb{N}$ denote the set of all natural numbers. The natural (or asymptotic) density of $E \subseteq \mathbb{N}$ is defined by

$$\delta(E) = \lim_{N\to\infty} \frac{1}{N+1} |\{n \le N : n \in E\}|$$

if the limit exists. Note that the vertical bars indicate the number of elements in the enclosed set.

The idea of the statistical convergence, which is closely related to the concept of natural density, was introduced by Fast [1].

A real sequence (s_n) is called statistically convergent to ξ if for every $\epsilon > 0$, the set $E_\epsilon = \{n \le N : |s_n - \xi| \ge \epsilon\}$ has natural density zero, i.e.

$$\lim_{N\to\infty} \frac{1}{N+1} |\{n \le N : |s_n - \xi| \ge \epsilon\}| = 0.$$

In symbol, we write $st - \lim s_n = \xi$. Obviously, ξ is uniquely determined.

Although the term "statistical convergence" first appeared in Fast [1], it was first used by Zygmund who gave a relation between this concept and strong summability in ([15], page 181) where he used the term "almost convergence" in place of the statistical convergence.

2010 *Mathematics Subject Classification*. Primary 40E05; Secondary 40A35, 40G15, 40C05

Keywords. (Tauberian theorems, statistical convergence, logarithmic summability method, one-sided Tauberian conditions, slowly decreasing sequence)

Received: 18 October 2017; Accepted: 05 January 2018

Communicated by Ivana Djolović

This work was supported by Research Fund of the İstanbul Medeniyet University. Project Number: F-GAP-2017-995

Email addresses: sefaanil.sezer@medeniyet.edu.tr (Sefa Anıl Sezer), rahmet.savas@medeniyet.edu.tr (Rahmet Savaş), ibrahim.canak@ege.edu.tr (İbrahim Çanak)

Note that ordinary convergence implies the statistical convergence to the same limit, so statistical convergence may be considered as a regular summability method. Hovewer, the converse is not necessarily true. For example,

$$s_n = \begin{cases} 1, & n = m^2,\ m = 0, 1, ... \\ 0, & n \neq m^2,\ m = 0, 1, ... \end{cases}$$

is statistically convergent to 0. However it is not convergent in the ordinary sense. Additionally, notice that a statistically convergent sequence may not be bounded. Consider the sequence

$$s_n = \begin{cases} \sqrt{n}, & n = m^2,\ m = 0, 1, ... \\ 1, & n \neq m^2,\ m = 0, 1, ... \,. \end{cases}$$

Then, $st - \lim s_n = 1$, but (s_n) is not bounded.

In the present paper we use the common notation for matrix summability methods: Let $\mathcal{A} = [a_{nk}]$ be an infinite real matrix, then the matrix transformation $(\mathcal{A}s)_n$ of (s_n) is given by

$$(\mathcal{A}s)_n = \sum_{k=0}^{\infty} a_{nk} s_k, \quad n = 0, 1, 2, ... \tag{1}$$

Thus, "(s_n) is $\mathcal{A}$-summable to ξ" means that $\lim(\mathcal{A}s)_n = \xi$. $\mathcal{A}$ is called a "regular" summability method if it transforms convergent sequences into other convergent sequences and preserves limits.

In the matrix summability method defined in (1), if we choose

$$a_{nk} = \begin{cases} \dfrac{1}{n+1}, & k \leq n \\ 0, & \text{otherwise}, \end{cases}$$

we get a well-known regular summability method called $(C, 1)$ summability. Given a sequence (s_n), the transformation defined by

$$\sigma_n^{(1)}(s) = \frac{1}{n+1} \sum_{k=0}^{n} s_k$$

is said to be the arithmetic mean of (s_n). A sequence (s_n) is called $(C, 1)$ summable to ξ and written $\lim s_n = \xi(C, 1)$ if

$$\lim_{n\to\infty} \sigma_n^{(1)}(s) = \xi.$$

In ([9], Lemma 4), Schoenberg obtained that a bounded and statistically convergent sequence is summable $(C, 1)$. Then, the question arises whether or not the $(C, 1)$ summability includes the statistical convergence regardless of boundedness. Fridy [2] gave a negative answer to this question and proved that statistical convergence can not be included by any matrix method.

Later, Fridy and Miller [3] established a connection between statistical convergence and a certain class of matrix summability methods and generalized the result of Schoenberg.

Lemma 1.1. *Let $\mathcal{T}$ be a collection of lower triangular non-negative summability matrices T which are regular. The bounded sequence (s_n) is statistically convergent to ξ if and only if it is T summable to ξ for all $T \in \mathcal{T}$.*

It is obvious that $(C, 1) \in \mathcal{T}$. As a different example of a matrix method in $\mathcal{T}$ we may give $(\ell, 1)$ summability which have the matrix representation

$$a_{nk} = \begin{cases} \dfrac{1}{(k+1)\ell_n}, & k \leq n \\ 0, & k > n \end{cases}$$

where

$$\ell_n = \sum_{k=0}^{n} \frac{1}{k+1} \sim \log n.$$

The transformation of (s_n) defined by

$$t_n^{(1)}(s) = \frac{1}{\ell_n} \sum_{k=0}^{n} \frac{s_k}{k+1}$$

is said to be the logarithmic mean of (s_n). A sequence (s_n) is called $(\ell, 1)$ summable to ξ and written $\lim s_n = \xi(\ell, 1)$ if

$$\lim_{n\to\infty} t_n^{(1)}(s) = \xi.$$

Besides, a given sequence (s_n) may not be summable $(\ell, 1)$, but the sequence $(t_n^{(1)})$ may be summable $(\ell, 1)$, in other saying, the repetition of the $(\ell, 1)$ method may generate a convergent sequence. Hence, m-fold application of the $(\ell, 1)$ method is defined by

$$t_n^{(m)}(s) = \begin{cases} \dfrac{1}{\ell_n} \displaystyle\sum_{k=0}^{n} \frac{t_k^{(m-1)}(s)}{k+1} & , m \geq 1 \\ s_n & , m = 0. \end{cases}$$

If

$$\lim_{n\to\infty} t_n^{(m)}(s) = \xi,$$

we say that (s_n) is summable to ξ by the (ℓ, m) method. Trivially, if (s_n) is (ℓ, m) summable, then it is $(\ell, m+1)$ summable to the same number. However, the converse is not valid, in general, provided by the example (see [12])

$$s_n = \left((-1)^n \left(n \log n + (n+1)\log(n+1)\right)\right).$$

Here, (s_n) is $(\ell, 2)$ summable to 0. Nevertheless, (s_n) is neither convergent nor $(\ell, 1)$ summable.

On the other hand, if

$$st - \lim_{n\to\infty} t_n^{(m)}(s) = \xi,$$

we say that (s_n) is statistically (ℓ, m) summable to ξ.

Taking Lemma 1.1 into account together with the fact that ordinary convergence implies statistical convergence, we get the following result.

Lemma 1.2. *Let (s_n) be a bounded sequence. If*

$$st - \lim_{n\to\infty} s_n = \xi, \tag{2}$$

then for every $m \geq 1$

$$st - \lim_{n\to\infty} t_n^{(m)}(s) = \xi. \tag{3}$$

Consider the sequence

$$s_n = \left(2(-1)^n n + (-1)^n\right).$$

The sequence (s_n) is statistically $(\ell, 1)$ summable to 0, but not statistically convergent. More precisely, the limit (3) may not imply (2).

If a sequence is convergent, then it is summable to the same limit by a regular method. The converse case is not always true. However, it may be true under certain supplementary conditions. Such condition is said to be a Tauberian condition with respect to the summability method in question and the resulting theorem is said to be a Tauberian theorem, honoring Austrian mathematician Alfred Tauber, who first obtained a converse theorem for the Abel method. One may consult Korevaar's book "*Tauberian Theory: A Century of Developments*" [5] for further results on Tauberian type theorems.

In this study, we deal with Tauberian theorems for the statistical convergence and the logarithmic (ℓ, m) summability.

2. Auxilary Results

In this section, we introduce some fundamental identities and lemmas which will be needed in the sequel.

In this work, H represents a positive constant, possibly different at every occurrence and notations $s_n = O(1)$ and $s_n = o(1)$ refer that (s_n) is bounded for sufficiently large n and $\lim_{n\to\infty} s_n = 0$, respectively.

The classical logarithmic control modulo of the oscillatory behavior of (s_n) is given by

$$\omega_n^{(0)}(s) = \alpha_n \Delta s_n \sim n \log n \Delta s_n, \tag{4}$$

where

$$\alpha_n = (n+1)\ell_{n-1} \text{ and } \Delta s_n = \begin{cases} s_n - s_{n-1} & , n \geq 1 \\ s_0 & , n = 0 \end{cases} .$$

(4) has a significant role when determining Tauberian conditions (see [4] and [11] for numerical sequences, [12] for improper integrals, [10] and [14] for sequences of fuzzy numbers).

A sequence (s_n) is called slowly decreasing in the $(\ell, 1)$ sense if

$$\lim_{\lambda\to 1^+} \liminf_{n\to\infty} \min_{n<k\leq[n^\lambda]} (s_k - s_n) \geq 0 \tag{5}$$

or equivalently

$$\lim_{\lambda\to 1^-} \liminf_{n\to\infty} \min_{[n^\lambda]\leq k<n} (s_n - s_k) \geq 0, \tag{6}$$

where [.] denotes the integer part. This definition was presented by Móricz [8]. Actually, it was Kwee [6] who first used slowly decreasing sequences while proving the following Tauberian type result.

Theorem 2.1. *If (s_n) is $(\ell, 1)$ summable to ξ and*

$$\liminf(s_m - s_n) \geq 0 \textit{ whenever } m > n \to \infty \textit{ and } \frac{\log m}{\log n} \to 1, \tag{7}$$

then $\lim s_n = \xi$.

Notice that (7) is equivalent to (5). Besides, if $\omega_n^{(0)}(s) \geq -H$, then slow decrease condition (5) is satisfied. Eventually, we attain the next result as a corollary of the last theorem.

Theorem 2.2. *Let (s_n) be $(\ell, 1)$ summable to ξ and*

$$\omega_n^{(0)}(s) \geq -H,$$

then $\lim s_n = \xi$.

Later, Móricz [8] established the statistical analogues of Theorem 2.1 and Theorem 2.2 as follows.

Theorem 2.3. *Let (s_n) be statistically convergent to ξ. If (s_n) is slowly decreasing in the $(\ell, 1)$ sense, then* $\lim s_n = \xi$.

Theorem 2.4. *Let (s_n) be statistically convergent to ξ. If*

$$\omega_n^{(0)}(s) \geq -H,$$

then $\lim s_n = \xi$.

The difference of a sequence and its logarithmic mean is represented by

$$s_n - t_n^{(1)}(s) = v_n^{(0)}(\Delta s) \tag{8}$$

where

$$v_n^{(0)}(\Delta s) = \frac{1}{\ell_n}\sum_{k=1}^{n} \ell_{k-1}\Delta s_k.$$

The identity (8) is called the Kronecker identity in the $(\ell, 1)$ sense and it will be used in the several steps of proofs.

Kwee [7] sets a restriction on the sequence $(v_n^{(0)}(\Delta s))$ and get the following Tauberian type result.

Theorem 2.5. *Let (s_n) be $(\ell, 1)$ summable to ξ. If*

$$v_n^{(0)}(\Delta s) = o(1),$$

then $\lim s_n = \xi$.

The next theorem is the statistical version of Theorem 2.5.

Theorem 2.6. *Let (s_n) be statistically convergent to ξ. If*

$$v_n^{(0)}(\Delta s) = o(1),$$

then $\lim s_n = \xi$.

Proof. Suppose $\lim v_n^{(0)}(\Delta s) = 0$, then $st - \lim v_n^{(0)}(\Delta s) = 0$. Hence, via the logarithmic Kronecker identity

$$s_n - t_n^{(1)}(s) = v_n^{(0)}(\Delta s),$$

we get $st - \lim t_n^{(1)}(s) = \xi$. Also, from the hypothesis

$$v_n^{(0)}(\Delta s) = \alpha_n \Delta t_n^{(1)}(s) \geq -H.$$

Now, by applying Theorem 2.4 to $(t_n^{(1)}(s))$, we obtain

$$\lim_{n\to\infty} s_n = \xi(\ell, 1).$$

Therefore, $\lim s_n = \xi$ follows from Theorem 2.5. □

For every integer $m > 0$, we introduce m-th order iterated logarithmic means of $v_n^{(0)}(\Delta s)$ by

$$v_n^{(m)}(\Delta s) = \begin{cases} \dfrac{1}{\ell_n}\displaystyle\sum_{k=0}^{n} \dfrac{v_k^{(m-1)}(\Delta s)}{k+1} & , m \geq 1 \\ v_n^{(0)}(\Delta s) & , m = 0. \end{cases}$$

Lemma 2.7. ([11]) *For every integer $m \geq 1$,*

(i)$\alpha_n \Delta v_n^{(m)}(\Delta s) = v_n^{(m-1)}(\Delta s) - v_n^{(m)}(\Delta s)$,

(ii)$\alpha_n \Delta t_n^{(m)}(s) = v_n^{(m-1)}(\Delta s)$.

For each integers $m \geq 0$ and $r \geq 0$ we have

$$\left(\alpha_n\Delta\right)_r s_n = \left(\alpha_n\Delta\right)_{r-1}\left(\alpha_n \Delta s_n\right) = \alpha_n\Delta\left(\left(\alpha_n\Delta\right)_{r-1} s_n\right),$$

where $\left(\alpha_n\Delta\right)_0 s_n = s_n$ and $\left(\alpha_n\Delta\right)_1 s_n = \alpha_n \Delta s_n$.

The general logarithmic control modulo of integer order $m \geq 1$ of (s_n) is recursively defined in [11] by

$$\omega_n^{(m)}(s) = \omega_n^{(m-1)}(s) - t_n^{(1)}(\omega^{(m-1)}(s)). \tag{9}$$

The next lemmas show two different representations of $(\omega_n^{(m)}(s))$.

Lemma 2.8. ([11]) *For every integer $m \geq 1$,*

$$\omega_n^{(m)}(s) = \left(\alpha_n \Delta\right)_m v_n^{(m-1)}(\Delta s).$$

Lemma 2.9. *For every integer $m \geq 1$,*

$$\omega_n^{(m)}(s) = \omega_n^{(0)}(s) + \sum_{j=1}^{m} (-1)^j \binom{m}{j} v_n^{(j-1)}(\Delta s), \tag{10}$$

where $\binom{m}{j} = \dfrac{m(m-1)...(m-j+1)}{j!}$.

Proof. We will prove with induction. If $m = 1$, the assertion is

$$\begin{aligned}
\omega_n^{(1)}(s) &= \omega_n^{(0)}(s) - t_n^{(1)}(\omega^{(0)}(s)) \\
&= \omega_n^{(0)}(s) - v_n^{(0)}(\Delta s) \\
&= \omega_n^{(0)}(s) + \sum_{j=1}^{1} (-1)^j \binom{1}{j} v_n^{(j-1)}(\Delta s),
\end{aligned}$$

which is obviously valid. Let $k \in \mathbb{N}$ be given and suppose (10) is true for $m = k$. Namely,

$$\omega_n^{(k)}(s) = \omega_n^{(0)}(s) + \sum_{j=1}^{k} (-1)^j \binom{k}{j} v_n^{(j-1)}(\Delta s). \tag{11}$$

We should now demonstrate that the lemma is valid for $m = k + 1$. More precisely,

$$\omega_n^{(k+1)}(s) = \omega_n^{(0)}(s) + \sum_{j=1}^{k+1} (-1)^j \binom{k+1}{j} v_n^{(j-1)}(\Delta s).$$

Then, considering (11) we obtain

$$\begin{aligned}
\omega_n^{(k+1)}(s) &= \omega_n^{(k)}(s) - t_n^{(1)}(\omega^{(k)}(s)) \\
&= \omega_n^{(0)}(s) + \sum_{j=1}^{k} (-1)^j \binom{k}{j} v_n^{(j-1)}(\Delta s) - \left(v_n^{(0)}(\Delta s) + \sum_{j=1}^{k} (-1)^j \binom{k}{j} v_n^{(j)}(\Delta s) \right) \\
&= \omega_n^{(0)}(s) + \sum_{j=1}^{k} (-1)^j \binom{k}{j} v_n^{(j-1)}(\Delta s) + \sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} v_n^{(j)}(\Delta s) \\
&= \omega_n^{(0)}(s) + \sum_{j=1}^{k} (-1)^j \binom{k}{j} v_n^{(j-1)}(\Delta s) + \sum_{j=1}^{k+1} (-1)^j \binom{k}{j-1} v_n^{(j-1)}(\Delta s) \\
&= \omega_n^{(0)}(s) + \sum_{j=1}^{k} (-1)^j \left[\binom{k}{j} + \binom{k}{j-1} \right] v_n^{(j-1)}(\Delta s) + (-1)^{k+1} \binom{k}{k} v_n^{(k)}(\Delta s).
\end{aligned}$$

Since $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$, the last identity may be written as

$$\begin{aligned}
\omega_n^{(k+1)}(s) &= \omega_n^{(0)}(s) + \sum_{j=1}^{k} (-1)^j \binom{k+1}{j} v_n^{(j-1)}(\Delta s) + (-1)^{k+1} \binom{k}{k} v_n^{(k)}(\Delta s) \\
&= \omega_n^{(0)}(s) + \sum_{j=1}^{k+1} (-1)^j \binom{k+1}{j} v_n^{(j-1)}(\Delta s).
\end{aligned}$$

The lemma therefore is valid for all $m \in \mathbb{N}$. $\square$

Also, the following lemmas are quite important and repeatedly used in the proofs.

Lemma 2.10. ([8]) *Let (s_n) be slowly decreasing in the $(\ell, 1)$ sense, then so is $(t_n^{(1)}(s))$.*

Lemma 2.11. ([8]) *Let (s_n) be slowly decreasing in the $(\ell, 1)$ sense, then*

$$v_n^{(0)}(\Delta s) \geq -H.$$

Lemma 2.12. ([11]) *For a real sequence (s_n)*
(i) *If $\lambda > 1$,*

$$s_n - t_n^{(1)}(s) = \frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]} - \ell_n}\left(t_{[n^\lambda]}^{(1)}(s) - t_n^{(1)}(s)\right) - \frac{1}{\ell_{[n^\lambda]} - \ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{s_k - s_n}{k+1}.$$

(ii) *If $0 < \lambda < 1$,*

$$s_n - t_n^{(1)}(s) = \frac{\ell_{[n^\lambda]}}{\ell_n - \ell_{[n^\lambda]}}\left(t_n^{(1)}(s) - t_{[n^\lambda]}^{(1)}(s)\right) + \frac{1}{\ell_n - \ell_{[n^\lambda]}}\sum_{k=[n^\lambda]+1}^{n}\frac{s_n - s_k}{k+1}.$$

Here, $[n^\lambda]$ denotes the integer part of n^λ.

3. Tauberian Theorems for Statistical Convergence

In this section we recover ordinary convergence of (s_n) from its statistical convergence by imposing certain restrictions on the sequence $(\omega_n^{(r)}(s))$.

Theorem 3.1. *Let (s_n) be a bounded sequence which is statistically convergent to ξ. If for any nonnegative integer r*

$$\omega_n^{(r)}(s) \geq -H, \tag{12}$$

then (s_n) converges to ξ.

Proof. Since $st - \lim s_n = \xi$ and (s_n) is bounded, we have $st - \lim t_n^{(1)}(s) = \xi$. Then, by (8), for every integer $m \geq 0$,

$$st - \lim_{n\to\infty} v_n^{(m)}(\Delta s) = 0. \tag{13}$$

Taking the logarithmic mean of both sides of the identity (10) gives

$$t_n^{(1)}(\omega^{(m)}(s)) = \sum_{j=0}^{m}(-1)^j\binom{m}{j}v_n^{(j)}(\Delta s). \tag{14}$$

Combining (13) and (14), we easily get

$$st - \lim_{n\to\infty} t_n^{(1)}(\omega^{(m)}(s)) = 0 \tag{15}$$

for all integer $m \geq 0$. On the other hand, by the assumption

$$\omega_n^{(r)}(s) = \alpha_n \Delta t_n^{(1)}(\omega^{(r-1)}(s)) \geq -H. \tag{16}$$

Taking (15) into account for $m = r - 1$, (16) and Theorem 2.4, it follows

$$\lim_{n\to\infty} t_n^{(1)}(\omega^{(r-1)}(s)) = 0. \tag{17}$$

Hence, using (16) and (17), we obtain via

$$\omega_n^{(r)}(s) = \omega_n^{(r-1)}(s) - t_n^{(1)}(\omega^{(r-1)}(s))$$

that

$$\omega_n^{(r-1)}(s) = \alpha_n \Delta t_n^{(1)}(\omega^{(r-2)}(s)) \geq -H. \tag{18}$$

Considering (15) for $m = r - 2$ together with (18) and Theorem 2.4 yields

$$\lim_{n\to\infty} t_n^{(1)}(\omega^{(r-2)}(s)) = 0. \tag{19}$$

Now, by using (18) and (19), we obtain from the identity

$$\omega_n^{(r-1)}(s) = \omega_n^{(r-2)}(s) - t_n^{(1)}(\omega^{(r-2)}(s))$$

that

$$\omega_n^{(r-2)}(s) = \alpha_n \Delta t_n^{(1)}(\omega^{(r-3)}(s)) \geq -H \tag{20}$$

In the light of (12), (18) and (20), if we continue in the same fashion, then we find

$$\omega_n^{(0)}(s) \geq -H.$$

Consequently, the proof follows from Theorem 2.4. □

Corollary 3.2. *Let (s_n) be a bounded sequence which is statistically convergent to ξ. If for any nonnegative integer r*

$$\omega_n^{(r)}(s) = O(1),$$

then (s_n) converges to ξ.

Corollary 3.3. *Let (s_n) be a bounded sequence which is statistically convergent to ξ. If for any nonnegative integer r*

$$\omega_n^{(r)}(s) = o(1),$$

then (s_n) converges to ξ.

Theorem 3.4. *Let (s_n) be a bounded sequence which is statistically convergent to ξ. If for any nonnegative integer r*

$$(t_n^{(1)}(\omega^{(r)}(s))) \text{ is slowly decreasing in the } (\ell, 1) \text{ sense,} \tag{21}$$

then (s_n) converges to ξ.

Proof. Let (s_n) be bounded and statistically convergent to ξ, then $(t_n^{(1)}(s))$ is also statistically convergent to the same limit. So, by (8), $(v_n^{(0)}(\Delta s))$ is statistically convergent to zero. If we replace (s_n) by $(v_n^{(0)}(\Delta s))$ in (8), we may write

$$v_n^{(0)}(\Delta s) - v_n^{(1)}(\Delta s) = \alpha_n \Delta v_n^{(1)}(\Delta s) = t_n^{(1)}(\omega^{(1)}(s)). \tag{22}$$

It follows from the identity (22) that

$$st-\lim_{n\to\infty} t_n^{(1)}(\omega^{(1)}(s))=0.$$

Now, applying (8) to $(\alpha_n\Delta v_n^{(1)}(\Delta s))$, we have

$$\alpha_n\Delta v_n^{(1)}(\Delta s)-\alpha_n\Delta v_n^{(2)}(\Delta s)=(\alpha_n\Delta)_2 v_n^{(2)}(\Delta s)=t_n^{(1)}(\omega^{(2)}(s)). \tag{23}$$

Hence, by (23)

$$st-\lim_{n\to\infty} t_n^{(1)}(\omega^{(2)}(s))=0.$$

If we continue in the same fashion, then for each integer $r\geq 0$

$$st-\lim_{n\to\infty} t_n^{(1)}(\omega^{(r)}(s))=0. \tag{24}$$

So, Lemma 1.2 implies that for each integer $r\geq 0$

$$st-\lim_{n\to\infty} t_n^{(2)}(\omega^{(r)}(s))=0. \tag{25}$$

Taking $(t_n^{(1)}(\omega^{(r)}(s)))$ instead of (s_n) in (8), we may write the following identity

$$t_n^{(1)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s))=v_n^{(0)}(\Delta t^{(1)}(\omega^{(r)}(s)))=\alpha_n\Delta t_n^{(2)}(\omega^{(r)}(s)). \tag{26}$$

We obtain from (21), (26) and Lemma 2.11 that

$$\alpha_n\Delta t_n^{(2)}(\omega^{(r)}(s))\geq -H. \tag{27}$$

In that case, considering (25) and (27) and applying Theorem 2.4 to $(t_n^{(2)}(\omega^{(r)}(s)))$ yields

$$\lim_{n\to\infty} t_n^{(2)}(\omega^{(r)}(s))=0. \tag{28}$$

Now, handling the Lemma 2.12 (i) in terms of $(t_n^{(1)}(\omega^{(r)}(s)))$, we have

$$t_n^{(1)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s))=\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\left(t_{[n^\lambda]}^{(2)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s))\right)-\frac{1}{\ell_{[n^\lambda]}-\ell_n}\sum_{k=n+1}^{[n^\lambda]}\frac{t_k^{(1)}(\omega^{(r)}(s))-t_n^{(1)}(\omega^{(r)}(s))}{k+1}. \tag{29}$$

If $\lambda>1$,

$$\frac{\lambda}{2(\lambda-1)}\leq\frac{\ell_{[n^\lambda]}}{\ell_{[n^\lambda]}-\ell_n}\leq\frac{3\lambda}{2(\lambda-1)}. \tag{30}$$

So, from (29) and (30)

$$t_n^{(1)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s))\leq\frac{3\lambda}{2(\lambda-1)}\left(t_{[n^\lambda]}^{(2)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s))\right)-\min_{n<k\leq[n^\lambda]}\left(t_k^{(1)}(\omega^{(r)}(s))-t_n^{(1)}(\omega^{(r)}(s))\right). \tag{31}$$

Taking the supremum limit as $n\to\infty$ and letting $\lambda\to 1^+$, respectively, of both sides of (31), we get

$$\limsup_{n\to\infty}\left(t_n^{(1)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s))\right)\leq 0. \tag{32}$$

This time, applying the Lemma 2.12 (ii) to $(t_n^{(1)}(\omega^{(r)}(s)))$, we get

$$t_n^{(1)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s)) = \frac{\ell_{[n^\lambda]}}{\ell_n-\ell_{[n^\lambda]}}\left(t_n^{(2)}(\omega^{(r)}(s))-t_{[n^\lambda]}^{(2)}(\omega^{(r)}(s))\right)+\frac{1}{\ell_n-\ell_{[n^\lambda]}}\sum_{k=[n^\lambda]+1}^{n}\frac{t_n^{(1)}(\omega^{(r)}(s))-t_k^{(1)}(\omega^{(r)}(s))}{k+1}. \quad (33)$$

If $0<\lambda<1$,

$$\frac{\lambda}{2(1-\lambda)} \leq \frac{\ell_{[n^\lambda]}}{\ell_n-\ell_{[n^\lambda]}} \leq \frac{3\lambda}{2(1-\lambda)}. \quad (34)$$

Then, from (33) and (34)

$$t_n^{(1)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s)) \geq \frac{\lambda}{2(1-\lambda)}\left(t_n^{(2)}(\omega^{(r)}(s))-t_{[n^\lambda]}^{(2)}(\omega^{(r)}(s))\right)+\min_{[n^\lambda]\leq k<n}\left(t_n^{(1)}(\omega^{(r)}(s))-t_k^{(1)}(\omega^{(r)}(s))\right). \quad (35)$$

Taking the infimum limit as $n\to\infty$ and letting $\lambda\to 1^-$, respectively, of both sides of (35), we have

$$\liminf_{n\to\infty}\left(t_n^{(1)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s))\right)\geq 0. \quad (36)$$

Combining (32) and (36),

$$\lim_{n\to\infty}\left(t_n^{(1)}(\omega^{(r)}(s))-t_n^{(2)}(\omega^{(r)}(s))\right)=0.$$

Last limit and (28) necessiate that

$$\lim_{n\to\infty} t_n^{(1)}(\omega^{(r)}(s))=0. \quad (37)$$

From (24), $st-\lim t_n^{(1)}(\omega^{(r-1)}(s))=0$ and so

$$\lim_{n\to\infty} t_n^{(1)}(\omega^{(r-1)}(s)=0 \quad (38)$$

by Theorem 2.6. Also, from (24), $st-\lim t_n^{(1)}(\omega^{(r-2)}(s))=0$. Then, once again from Theorem 2.6, we have

$$\lim_{n\to\infty} t_n^{(1)}(\omega^{(r-2)}(s))=0. \quad (39)$$

Taking (37), (38) and (39) into consideration and proceeding likewise, we accomplish

$$\lim_{n\to\infty} t_n^{(1)}(\omega^{(0)}(s))=\lim_{n\to\infty} v_n^{(0)}(\Delta s)=0.$$

The proof therefore follows from Theorem 2.6. □

Corollary 3.5. *Let (s_n) be a bounded sequence which is statistically convergent to ξ. If for any nonnegative integer r, $(\omega^{(r)}(s))$ is slowly decreasing in the $(\ell,1)$ sense, then (s_n) converges to ξ.*

Corollary 3.6. *Let (s_n) be a bounded sequence which is statistically convergent to ξ. If $(v_n^{(0)}(\Delta s))$ is slowly decreasing in the $(\ell,1)$ sense, then (s_n) converges to ξ.*

Corollary 3.7. *Let (s_n) be a bounded sequence which is statistically convergent to ξ. If*

$$\alpha_n\Delta v_n^{(0)}(\Delta s)\geq -H,$$

then (s_n) converges to ξ.

4. Tauberian Theorems for Statistical (ℓ, m) Summability

In this section we give extensions of some Tauberian theorems to statistical (ℓ, m) summability. The next result generalize the Theorem 2.3 due to Móricz.

Theorem 4.1. *Let (s_n) be statistically (ℓ, m) summable to ξ. If*

$$(s_n) \text{ is slowly decreasing in the } (\ell, 1) \text{ sense}, \tag{40}$$

then (s_n) converges to ξ.

Proof. Since (s_n) is slowly decreasing, by Lemma 2.10

$$(t_n^{(k)}(s)) \text{ is slowly decreasing for each integer } k \geq 1. \tag{41}$$

From the assumption, we have $st - \lim t_n^{(m)}(s) = \xi$. Choosing $k = m$ in (41), Theorem 2.3 implies

$$\lim_{n\to\infty} t_n^{(m)}(s) = \xi.$$

This means that

$$\lim_{n\to\infty} t_n^{(m-1)}(s) = \xi(\ell, 1).$$

Now, taking $k = m - 1$ in (41), by Theorem 2.1

$$\lim_{n\to\infty} t_n^{(m-1)}(s) = \xi,$$

which is equivalent to

$$\lim_{n\to\infty} t_n^{(m-2)}(s) = \xi(\ell, 1).$$

Repeating the same reasoning $m - 2$ more times we obtain

$$\lim_{n\to\infty} s_n = \xi(\ell, 1).$$

By the hypothesis and Theorem 2.1 we conclude

$$\lim_{n\to\infty} s_n = \xi.$$

□

The following two theorems are the extensions of Theorem 3.1 and Theorem 3.4 to statistical (ℓ, m) summability.

Theorem 4.2. *Let (s_n) be a bounded sequence which is statistically (ℓ, m) summable to ξ. If for any integer $r \geq 0$*

$$\omega_n^{(r)}(s) \geq -H, \tag{42}$$

then (s_n) converges to ξ.

Proof. Let take k-fold logarithmic mean of $(\omega_n^{(r)}(s))$, then the identity

$$t_n^{(k)}(\omega^{(r)}(s)) = \omega_n^{(r)}(t^{(k)}(s))$$

holds. Hence, from the hypothesis for each integer $k \geq 0$

$$\omega_n^{(r)}(t^{(k)}(s)) \geq -H. \tag{43}$$

From the assumption we have

$$st - \lim_{n\to\infty} t_n^{(m)}(s) = \xi.$$

Then, using (43) for $k = m$ together with Theorem 3.1 yields

$$\lim_{n\to\infty} t_n^{(m)}(s) = \xi \text{ or equivalently } \lim_{n\to\infty} t_n^{(m-1)}(s) = \xi(\ell, 1). \tag{44}$$

Taking (44) and Lemma 1.1 into account, it follows

$$st - \lim_{n\to\infty} t_n^{(m-1)}(s) = \xi.$$

Now, considering (43) for $k = m - 1$ together with Theorem 3.1, we obtain

$$\lim_{n\to\infty} t_n^{(m-1)}(s) = \xi \text{ or equivalently } \lim_{n\to\infty} t_n^{(m-2)}(s) = \xi(\ell, 1),$$

which also implies by Lemma 1.1 that

$$st - \lim_{n\to\infty} t_n^{(m-2)}(s) = \xi.$$

Thus, continuing the proof in the same manner we deduce

$$st - \lim_{n\to\infty} s_n = \xi.$$

Therefore, since $\omega_n^{(r)}(s) \geq -H$, the proof follows from Theorem 3.1. □

Theorem 4.3. *Let (s_n) be a bounded sequence which is statistically (ℓ, m) summable to ξ. If for any integer $r \geq 0$*

$$(t_n^{(1)}(\omega^{(r)}(s))) \text{ is slowly decreasing in the } (\ell, 1) \text{ sense}, \tag{45}$$

then (s_n) converges to ξ.

Proof. Suppose (45) holds, then the sequence

$$(t_n^{(k)}(t^{(1)}(\omega^{(r)}(s)))) = (t_n^{(1)}(t^{(k)}(\omega^{(r)}(s))))$$

is slowly decreasing in the $(\ell, 1)$ sense for every integer $k \geq 0$. Now, considering the identity

$$t_n^{(k)}(\omega^{(r)}(s)) = \omega_n^{(r)}(t^{(k)}(s)),$$

we further obtain that

$$(t_n^{(1)}(\omega_n^{(r)}(t^{(k)}(s)))) \text{ is slowly decreasing in the } (\ell, 1) \text{ sense}, \tag{46}$$

for all integer $k \geq 0$. After taking $k = m$ in (46) it follows from Theorem 3.4 that

$$\lim_{n\to\infty} t_n^{(m-1)}(s) = \xi(\ell, 1),$$

which implies by Lemma 1.1,

$$st - \lim_{n\to\infty} t_n^{(m-1)}(s) = \xi. \tag{47}$$

Then, by using (46) for $k = m - 1$ together with Theorem 3.4 we get

$$\lim_{n\to\infty} t_n^{(m-2)}(s) = \xi(\ell, 1).$$

This also implies by Lemma 1.1 that

$$st - \lim_{n\to\infty} t_n^{(m-2)}(s) = \xi. \tag{48}$$

Considering (47) and (48) and applying the same reasoning $m - 2$ more times we find

$$st - \lim_{n\to\infty} s_n = \xi.$$

Therefore, by the hypothesis and Theorem 3.4 the proof is completed. □

References

[1] H. Fast, Sur la convergence statistique, Colloquium Math. 2 (1951), 241–244.
[2] J. A. Fridy, On statistical convergence, Analysis 5 (1985), 301–313.
[3] J. A. Fridy and H. I. Miller, A matrix characterization of statistical convergence, Analysis 11 (1991), 59–66.
[4] K. Ishiguro, Tauberian theorems concerning the summability methods of logarithmic type, Proc. Japan Acad. 39 (1963), 156–159.
[5] J. Korevaar, Tauberian Theory: A Century of Developments, Springer-Verlag, Berlin, 2004.
[6] B. Kwee, A Tauberian theorem for the logarithmic method of summation, Proc. Cambridge Philos. Soc. 63 (1967), 401–405.
[7] B. Kwee, Some Tauberian theorems for the logarithmic method of summability, Canad. J. Math. 20 (1968), 1324–1331.
[8] F. Móricz, Theorems relating to statistical harmonic summability and ordinary convergence of slowly decreasing or oscillating sequences, Analysis (Munich). 24 (2004), 127–145.
[9] I. J. Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly 66 (1959), 361–375.
[10] S. A. Sezer and İ. Çanak, Power series methods of summability for series of fuzzy numbers and related Tauberian Theorems, Soft Comput. 21 (2017), 1057–1064.
[11] S. A. Sezer and İ. Çanak, Tauberian theorems for the summability methods of logarithmic type, Bull. Malays. Math. Sci. Soc. 41 (4) (2018), 1977-1994.
[12] Ü. Totur and M. A. Okur, On logarithmic averages of sequences and its applications, Kuwait J. Sci. 43 (2016), 56–67.
[13] Ü. Totur and M. A. Okur, On Tauberian conditions for the logarithmic methods of integrability, Bull. Malays. Math. Sci. Soc. 41 (2) (2018), 879-892.
[14] E. Yavuz and H. Coşkun, On the logarithmic summability method for sequences of fuzzy numbers, Soft Comput. 21 (19) (2017), 5779-5785.
[15] A. Zygmund, Trigonometric series, Cambridge University Press, 1959.