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Approximation by Stancu-Durrmeyer Type Operators Based on Pólya-Eggenberger Distribution

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Abstract. In the present paper we introduce the Durrmeyer type modification of Stancu operators based on Pólya-Eggenberger distribution. For these new operators some indispensable auxiliary results are established in the second section. Our further study focuses on a Voronovskaja type asymptotic formula and some estimates of the rate of approximation involving modulus of smoothness, respectively Ditzian-Totik modulus of smoothness. The rate of convergence for differential functions whose derivatives are of bounded variation is also obtained.

1. Introduction

In 1923, Eggenberger and Pólya [11] devised the original Pólya-Eggenberger urn model (usually simplified as Pólya urn) to study processes such as the spread of contagious diseases. In one of its simplest form, the Pólya-Eggenberger urn model contains w white balls and b black balls. A ball is drawn at random and then replaced together with s balls of the same color. This procedure is repeated n times and noting the distribution of the random variable X representing the number of times a white ball is drawn. The distribution of X is given by

$$Pr(X=k)=\binom{n}{k}\frac{w(w+s)\cdot\ldots\cdot(w+\overline{k-1}s)b(b+s)\cdot\ldots\cdot(b+\overline{n-k-1}s)}{(w+b)(w+b+s)\cdot\ldots\cdot(w+b+\overline{n-1}s)},\tag{1}$$

for $k=0,1,\cdots,n$ and $\overline{k-1}s=(k-1)s$. The distribution (1) is known as Pólya-Eggenberger distribution with parameters (n,w,b,s) and contains binomial, respectively hypergeometric distribution as particular cases.

Based on Pólya-Eggenberger distribution (1), Stancu [21] introduced a new class of positive linear operators associated to a real-valued function $f:[0,1]\to\mathbb{R}$, given by

$$P_n^{[\alpha]}(f;x)=\sum_{k=0}^{n}p_{n,k}^{[\alpha]}(x)f\left(\frac{k}{n}\right)=\sum_{k=0}^{n}\binom{n}{k}\frac{\prod_{\nu=0}^{k-1}(x+\nu\alpha)\prod_{\mu=0}^{n-k-1}(1-x+\mu\alpha)}{(1+\alpha)(1+2\alpha)\cdots(1+(n-1)\alpha)}f\left(\frac{k}{n}\right),\tag{2}$$

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where $p_{n,k}^{[\alpha]}$ are the fundamental Stancu polynomials and α is a non-negative parameter which may depend only on the natural number n. In the case when $\alpha = 0$ operators (2) reduce, obviously, to the original Bernstein operators [4] and for $\alpha = \frac{1}{n}$ we get a special case

$$P_n^{\left[\frac{1}{n}\right]}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\left[\frac{1}{n}\right]}(x) f\left(\frac{k}{n}\right) = \frac{2(n!)}{(2n)!} \sum_{k=0}^{n} \binom{n}{k} \prod_{\nu=0}^{k-1}(nx+\nu) \prod_{\mu=0}^{n-k-1}(n-nx+\mu) f\left(\frac{k}{n}\right), \tag{3}$$

introduced by Lupaş and Lupaş [15]. Concerning the operators (2) and (3), the reader is invited to see two recent papers [16], [17], where some results of the recalled operators are revised. Taking into account the period in which the Stancu operators (2) were introduced, we remark that there exists a huge interest to study them, respectively generalizations of them until nowadays. Some representative examples in this sense could be the papers of Razi [20], Finta [9], [10], Wang et al. [22], Abel et al. [1], Agrawal et al. [2], [3], Gupta et al. [13], [5], [14] and Deo et al [6].

Denote by $L_B[0,1]$ the space of bounded Lebesgue integrable functions on $[0,1]$ and by Π_n the space of polynomials of degree at most $n \in \mathbb{N}$. In 2007, Păltănea [19] has introduced the following class of operators $U_{n,\rho} : L_B[0,1] \to \Pi_n$, given by

$$U_{n,\rho}(f;x) = \sum_{k=0}^{n} p_{n,k}(x) F_{n,k}^{\rho}(f) = (1-x)^n f(0) + x^n f(1) + \sum_{k=1}^{n-1} p_{n,k}(x) \left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)} f(t)dt \right), \tag{4}$$

where $\rho > 0$, $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ are the well-known Bernstein's fundamental polynomials and $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}$, for $x, y > 0$ is Euler's Beta function. Further investigations concerning a recursion formula of the moments and estimates for simultaneous approximation of derivatives for the presented operators (4) were made by Gonska and Păltănea [12]. The authors showed that operators $U_{n,\rho} f$ constitute a link between the well-known Bernstein operators and their genuine Bernstein-Durrmeyer variants.

Inspired by the above two articles, we introduce the Stancu-Durrmeyer type operators $U_{n,\rho}^{[\alpha]} : L_B[0,1] \to \Pi_n$, defined by

$$\begin{aligned} U_{n,\rho}^{[\alpha]}(f;x) &= \sum_{k=0}^{n} p_{n,k}^{[\alpha]}(x) F_{n,k}^{\rho}(f) \\ &= \frac{(1-x)^{[n,-\alpha]} f(0)}{1^{[n,-\alpha]}} + \frac{x^{[n,-\alpha]} f(1)}{1^{[n,-\alpha]}} + \sum_{k=1}^{n-1} p_{n,k}^{[\alpha]}(x) \left(\int_0^1 \frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)} f(t)dt \right), \end{aligned} \tag{5}$$

where $\rho > 0$, $p_{n,k}^{[\alpha]}(x) = \binom{n}{k} \frac{1}{1^{[n,-\alpha]}} x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]}$ are the well-known Stancu's fundamental polynomials and $t^{[n,h]} = t(t-h) \cdot \ldots \cdot (t-(n-1)h)$ is the nth factorial power of t with increment h.

The aim of this paper is to introduce a new Durrmeyer type modification of Stancu operators based on Pólya-Eggenberger distribution. For these new operators some indispensable auxiliary results are established in the second section. Our further study focuses on the qualitative part of these new operators involving the uniform convergence and asymptotic behavior. In order to get the degree of approximation, some quantitative theorems will be established.

2. Auxiliary Results

Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The monomials $e_k(x) = x^k$, for $k \in \mathbb{N}_0$ called also test functions play an important role in uniform approximation by linear positive operators. In order to determine the images of the monomials by the Stancu-Durrmeyer type operators (5) we present a useful form of these operators.

Lemma 2.1. *For $\rho > 0$, $\alpha > 0$ and $x \in (0,1)$, we get*

$$U_{n,\rho}^{[\alpha]}(f;x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1} U_{n,\rho}(f;t)dt,$$

where $U_{n,\rho}f$ are defined at (4).

Proof. Using the relationship between Euler's functions

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(r)$ is Gamma function defined by $\Gamma(r) = \int_0^\infty u^{r-1}e^{-u}du,\ r>0$, with $\Gamma(r+n) = r(r+1)\cdot\ldots\cdot(r+n-1)\Gamma(r)$, for $n \in \mathbb{N}$, then it follows

$$B\left(\frac{x}{\alpha}+k, \frac{1-x}{\alpha}+n-k\right) = \frac{\Gamma\left(\frac{x}{\alpha}+k\right)\Gamma\left(\frac{1-x}{\alpha}+n-k\right)}{\Gamma\left(\frac{1}{\alpha}+n\right)} = p_{n,k}^{[\alpha]}(x)\binom{n}{k}^{-1} B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right).$$

Hence

$$p_{n,k}^{[\alpha]}(x) = \binom{n}{k}\left(B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)\right)^{-1} B\left(\frac{x}{\alpha}+k, \frac{1-x}{\alpha}+n-k\right)$$

and

$$\begin{aligned}
U_{n,\rho}^{[\alpha]}(f;x) &= \sum_{k=1}^{n-1}\binom{n}{k}\frac{B\left(\frac{x}{\alpha}+k, \frac{1-x}{\alpha}+n-k\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \frac{1}{B(k\rho,(n-k)\rho)}\int_0^1 s^{k\rho-1}(1-s)^{(n-k)\rho-1}f(s)ds \\
&\quad + \frac{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}+n\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}f(0) + \frac{B\left(\frac{x}{\alpha}+n, \frac{1-x}{\alpha}\right)}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}f(1) \\
&= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\left(\sum_{k=1}^{n-1}\binom{n}{k}\int_0^1 t^{\frac{x}{\alpha}+k-1}(1-t)^{\frac{1-x}{\alpha}+n-k-1}dt \cdot \frac{1}{B(k\rho,(n-k)\rho)}\int_0^1 s^{k\rho-1}(1-s)^{(n-k)\rho-1}f(s)ds\right. \\
&\quad \left. + f(0)\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}+n-1}dt + f(1)\int_0^1 t^{\frac{x}{\alpha}+n-1}(1-t)^{\frac{1-x}{\alpha}-1}dt\right) \\
&= \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}-1}U_{n,\rho}(f;t)dt.
\end{aligned}$$

□

Below, we present four results involving Stancu-Durrmeyer type operators (5) without proof, because for obtaining them we have to do only mechanical work. The images of the test functions $e_k(x) = x^k$, for $k \in \mathbb{N}_0$ by operators (5) are given in the following

Lemma 2.2. *For the Stancu-Durrmeyer type operators it holds that*

$U_{n,\rho}^{[\alpha]}(e_0;x) = 1;\qquad U_{n,\rho}^{[\alpha]}(e_1;x) = x;\qquad U_{n,\rho}^{[\alpha]}(e_2;x) = \frac{x^2\rho(n-1)}{(1+\alpha)(1+n\rho)} + \frac{x(1+\alpha+\rho+n\alpha\rho)}{(1+\alpha)(1+n\rho)};$

$U_{n,\rho}^{[\alpha]}(e_3;x) = \frac{(n-1)(n-2)x^3\rho^2}{(1+\alpha)(1+2\alpha)(1+n\rho)(2+n\rho)} + \frac{3(n-1)(1+\rho+\alpha(2+n\rho))x^2\rho}{(1+\alpha)(1+2\alpha)(1+n\rho)(2+n\rho)} + \frac{\big((1+\rho)(2+\rho)+3\alpha(1+\rho)(2+n\rho)+2\alpha^2(1+n\rho)(2+n\rho)\big)x}{(1+\alpha)(1+2\alpha)(1+n\rho)(2+n\rho)};$

$U_{n,\rho}^{[\alpha]}(e_4;x) = \frac{(n-1)(n-2)(n-3)x^4\rho^3}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n\rho)(2+n\rho)(3+n\rho)} + \frac{6(n-1)(n-2)(1+\rho+\alpha(3+n\rho))x^3\rho^2}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n\rho)(2+n\rho)(3+n\rho)}$

$+ \frac{(n-1)\big(11(1+\alpha(5+6\alpha))+18(1+3\alpha)(1+n\alpha)\rho+(1+n\alpha)(7+(11n-1)\alpha)\rho^2\big)x^2\rho}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n\rho)(2+n\rho)(3+n\rho)}$

$+ \frac{\big(6(1+\alpha)(1+2\alpha)(1+3\alpha)+11(1+n\alpha)(1+\alpha(5+6\alpha))\rho+6(1+3\alpha)(1+n\alpha)(1+2n\alpha)\rho^2+(1+n\alpha)(1+\alpha(6n(1+n\alpha)-1))\big)x\rho^3}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n\rho)(2+n\rho)(3+n\rho)}.$

In order to compute the central moments of the operators (5), for brevity we will write in the sequel $M^{[\alpha]}_{n,\rho,r}(x) := U^{[\alpha]}_{n,\rho}\left((e_1 - x)^r; x\right)$, where $n \geq 1$, $r \geq 0$ and $x \in [0, 1]$.

Lemma 2.3. *For the Stancu-Durrmeyer type operators it holds that*
$M^{[\alpha]}_{n,\rho,1}(x) = 0;\qquad M^{[\alpha]}_{n,\rho,2}(x) = \frac{(1+\alpha+\rho+n\alpha\rho)x(1-x)}{(1+\alpha)(1+n\rho)};$

$$\begin{aligned}M^{[\alpha]}_{n,\rho,4}(x) &= \frac{3x^4\big(2\alpha(1+\rho)((n-4)\rho-6)(3+n\rho)+\alpha^2((n-12)\rho-11)(2+n\rho)(3+n\rho)-6\alpha^3(1+n\rho)(2+n\rho)(3+n\rho)+(1+\rho)(-6+\rho(n-6+(n-2)\rho))\big)}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n\rho)(2+n\rho)(3+n\rho)}\\
&+ \frac{6x^3\big(6(1+\alpha)(1+2\alpha)(1+3\alpha)+(1+\alpha(5+6\alpha))(12+n(11\alpha-1))\rho+2(1+3\alpha)(1+n\alpha)(4+n(6\alpha-1))\rho^2+(1+n\alpha)(2+n(6\alpha-1)(1+n\alpha))\rho^3\big)}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n\rho)(2+n\rho)(3+n\rho)}\\
&- \frac{x^2\big(24(1+\alpha)(1+2\alpha)(1+3\alpha)+(1+\alpha(5+6\alpha))(47+n(44\alpha-3))\rho+6(1+3\alpha)(1+n\alpha)(5+n(8\alpha-1))\rho^2+(1+n\alpha)(7-\alpha+3n(8\alpha-1)(1+n\alpha))\rho^3\big)}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n\rho)(2+n\rho)(3+n\rho)}\\
&+ \frac{x\big(6(1+\alpha)(1+2\alpha)(1+3\alpha)+11(1+n\alpha)(1+\alpha(5+6\alpha))\rho+6(1+3\alpha)(1+n\alpha)(1+2n\alpha)\rho^2+(1+n\alpha)(1+\alpha(-1+6n(1+n\alpha)))\rho^3\big)}{(1+\alpha)(1+2\alpha)(1+3\alpha)(1+n\rho)(2+n\rho)(3+n\rho)}.\end{aligned}$$

Lemma 2.4. *For any $n \in \mathbb{N}$, we can write*

$$M^{[\alpha]}_{n,\rho,2}(x) = U^{[\alpha]}_{n,\rho}((e_1 - x)^2; x) \leq \frac{C^{[\alpha]}_{\rho}\ x(1-x)}{1+n\rho},$$

where $C^{[\alpha]}_{\rho}$ is a positive constant depending on ρ and α.

We recall that α is a non-negative parameter which may depend only on the natural number n.

Lemma 2.5. *If $\alpha \to 0$ as $n \to \infty$ and $\lim\limits_{n\to\infty} n\alpha = c \in \mathbb{R}$, then*

$$\lim_{n\to\infty} n \cdot M^{[\alpha]}_{n,\rho,1}(x) = 0,$$

$$\lim_{n\to\infty} n \cdot M^{[\alpha]}_{n,\rho,2}(x) = \frac{(1+\rho+c\rho)x(1-x)}{\rho},$$

$$\lim_{n\to\infty} n^2 \cdot M^{[\alpha]}_{n,\rho,4}(x) = \frac{x^4(3+3\rho(2+\rho+2c(1+\rho+c\rho)))}{\rho^2} - \frac{6x^3(1+\rho+c\rho)^2}{\rho^2} + \frac{3x^2(1+\rho)(1+\rho+2c\rho)}{\rho^2}.$$

3. Main Results

Our further studies focus on the qualitative part of Stancu-Durrmeyer type operators, involving the uniform convergence and asymptotic behaviors.

Theorem 3.1. *Let $f \in C[0, 1]$ and α be a non-negative parameter which may depend on $n \in \mathbb{N}$, with $\alpha \to 0$ as $n \to \infty$, then $\lim\limits_{n\to\infty} U^{[\alpha]}_{n,\rho}(f; x) = f(x)$ uniformly on $[0, 1]$.*

Proof. Since $U^{[\alpha]}_{n,\rho}(1; x) = 1$, $U^{[\alpha]}_{n,\rho}(e_1; x) = x$ and $U^{[\alpha]}_{n,\rho}(e_2; x) = x^2 + \frac{(1+\alpha+\rho+\alpha n\rho)x(1-x)}{(1+\alpha)(1+n\rho)}$, it follows

$$\lim_{n\to\infty} U^{[\alpha]}_{n,\rho}(e_i; x) = e_i(x), \text{ for } i = 0, 1, 2.$$

Applying the well-known Korovkin's theorem, we get

$$\lim_{n\to\infty} U^{[\alpha]}_{n,\rho}(f; x) = f(x) \text{ uniformly on } [0, 1].$$

□

The next result provides a Voronovskaja type theorem for the Stancu-Durrmeyer type operators.

Theorem 3.2. *Let $f : [0,1] \to \mathbb{R}$, $\alpha \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} n\alpha = c \in \mathbb{R}$. If $f \in C^2[0,2]$, then*

$$\lim_{n\to\infty} n\left(U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right) = \frac{(1+\rho+c\rho)x(1-x)}{2\rho} f''(x).$$

Proof. Using Taylor's expansion formula of function f, it follows

$$f(t) = f(x) + f'(x)(t-x) + \frac{1}{2} f''(x)(t-x)^2 + \varpi(t,x)(t-x)^2, \tag{6}$$

where $\varpi(t,x) := \varpi(t-x)$ is a bounded function and $\lim_{t\to x} \varpi(t,x) = 0$. Taking the linearity of Stancu-Durrmeyer type operators into account and then applying the operators $U_{n,\rho}^{[\alpha]}$ on both side of the above equation (6), we get

$$U_{n,\rho}^{[\alpha]}(f;x) - f(x) = U_{n,\rho}^{[\alpha]}((e_1-x);x)f'(x) + \frac{1}{2} U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right) f''(x) + U_{n,\rho}^{[\alpha]}\left(\varpi(t,x)\cdot(e_1-x)^2;x\right).$$

Therefore using Lemma 2.3, we get

$$\lim_{n\to\infty} n\left(U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right) = \frac{(1+\rho+c\rho)x(1-x)}{2\rho} f''(x) + \lim_{n\to\infty} n\left(U_{n,\rho}^{[\alpha]}\left(\varpi(t,x)\cdot(e_1-x)^2;x\right)\right). \tag{7}$$

We estimate the last term on the right-hand side of the above equality, applying the Cauchy-Schwarz inequality, such that

$$U_{n,\rho}^{[\alpha]}\left(\varpi(t,x)\cdot(e_1-x)^2;x\right) \le \sqrt{U_{n,\rho}^{[\alpha]}\left(\varpi^2(t,x);x\right)}\sqrt{U_{n,\rho}^{[\alpha]}\left((e_1-x)^4;x\right)}. \tag{8}$$

Because $\varpi^2(x,x) = 0$ and $\varpi^2(\cdot,x) \in C[0,1]$, using the convergence from Theorem 3.1, we get

$$\lim_{n\to\infty} U_{n,\rho}^{[\alpha]}\left(\varpi^2(t,x);x\right) = \varpi^2(x,x) = 0. \tag{9}$$

Therefore, taking Lemma 2.5 into account and from (8), respectively (9) yields

$$\lim_{n\to\infty} n\left(U_{n,\rho}^{[\alpha]}\left(\varpi(t,x)\cdot(e_1-x)^2;x\right)\right) = 0$$

and using (7) we obtain the asymptotic behavior of the Stancu-Durrmeyer type operators (5). □

The main tools to measure the degree of approximation of linear positive operators towards the identity operators are moduli of smoothness. For $f \in C[0,1]$ and $\delta \ge 0$ we know the definition of the moduli of smoothness of first, respectively second order, given by

$$\omega_1(f,\delta) := \sup\{|f(x+h) - f(x)| \,:\, x, x+h \in [0,1],\ 0 \le h \le \delta\}$$

and

$$\omega_2(f,\delta) := \sup\{|f(x+h) - 2f(x) + f(x-h)| \,:\, x, x\pm h \in [0,1],\ 0 \le h \le \delta\}.$$

Definition 3.3. *Let $f \in C_B[0,1]$ (the space of all real-valued functions continuous and bounded on $[0,1]$) endowed with the norm $\|f\| = \sup_{x\in[0,1]} |f(x)|$ and let us consider Peetre's K-functional*

$$K_2(f,\delta) = \inf\{\|f-g\| + \delta\|g''\| \,:\, g \in C^2[0,1]\},\ \text{for } \delta > 0. \tag{10}$$

There exists an absolute constant $M > 0$, such that

$$K_2(f,\delta) \le M\cdot\omega_2\left(f, \sqrt{\delta}\right), \tag{11}$$

conformable ([7], p. 177, Theorem 2.4).

Proposition 3.1. *Let f be a real-valued function continuous and bounded on* $[0,1]$*, with* $\|f\| = \sup_{x\in[0,1]}|f(x)|$*, then*

$$\left|U_{n,\rho}^{[\alpha]}(f;x)\right| \le \|f\|.$$

Proof. Taking the definition of Stancu-Durrmeyer type operators and Lemma 2.2 into account, it follows

$$\left|U_{n,\rho}^{[\alpha]}(f;x)\right| = \left|\sum_{k=0}^{n} p_{n,k}^{[\alpha]}(x)F_{n,k}^{\rho}(f)\right| \le \sum_{k=0}^{n} p_{n,k}^{[\alpha]}(x)F_{n,k}^{\rho}(|f|) \le \|f\|\cdot U_{n,\rho}^{[\alpha]}(e_0;x) = \|f\|.$$

□

In the following, we get direct estimates in terms of moduli of smoothness and Peetre's K-functional.

Theorem 3.4. *If $f \in C_B[0,1]$, then for any $x \in [0,1]$ and $\delta > 0$, it follows*

$$\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \le \frac{3}{2}\cdot\omega_1\left(f, \sqrt{\tfrac{1+\alpha+\rho+n\alpha\rho}{(1+\alpha)(1+n\rho)}}\right).$$

Proof. Using the well-known property of first modulus of smoothness (first modulus of continuity)

$$|f(t)-f(x)| \le \omega_1(f,|t-x|) \le \left(1+\delta^{-1}|t-x|\right)\omega_1(f,\delta)$$

and applying the linear positive Stancu-Durrmeyer type operators to the above inequality, it follows

$$\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \le \left(U_{n,\rho}^{[\alpha]}(e_0;x) + \frac{1}{\delta}U_{n,\rho}^{[\alpha]}(|e_1-x|;x)\right)\cdot\omega_1(f,\delta).$$

The Cauchy-Schwarz inequality for linear positive operators leads to

$$U_{n,\rho}^{[\alpha]}(|e_1-x|;x) \le \left(U_{n,\rho}^{[\alpha]}(e_0;x)\right)^{\frac{1}{2}}\cdot\left(U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}.$$

Knowing that Stancu-Durrmeyer type operators preserve constants and conformable with the results obtained in Lemma 2.3

$$M_{n,\rho,2}^{[\alpha]}(x) = U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right) = \frac{(1+\alpha+\rho+n\alpha\rho)x(1-x)}{(1+\alpha)(1+n\rho)},$$

we get

$$\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \le \left(1+\delta^{-1}\sqrt{\tfrac{(1+\alpha+\rho+n\alpha\rho)x(1-x)}{(1+\alpha)(1+n\rho)}}\right)\cdot\omega_1(f,\delta).$$

Taking the inequality $\sqrt{x(1-x)} \le \frac{1}{2}$ into account and choosing $\delta = \sqrt{\frac{1+\alpha+\rho+n\alpha\rho}{(1+\alpha)(1+n\rho)}}$ we get the desired result. □

Theorem 3.5. *If f is a differentiable function on $[0,1]$ and $f' \in C_B[0,1]$, then for any $x \in [0,1]$ and $\delta > 0$, it follows*

$$\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \le \frac{3\delta}{4}\cdot\omega_1(f',\delta), \quad \text{with } \delta = \sqrt{\tfrac{1+\alpha+\rho+n\alpha\rho}{(1+\alpha)(1+n\rho)}}.$$

Proof. Starting with the identity $f(t)-f(x) = f'(x)(t-x) + f(t) - f(x) - f'(x)(t-x)$, we get for ξ between t and x, $\left|f(t)-f(x)-f'(x)(t-x)\right| = \left|f'(\xi)-f'(x)\right|\cdot|t-x|$, using the Lagrange mean value theorem (there exists a ξ between t and x, such that $f(t)-f(x) = f'(\xi)(t-x)$). Because $|\xi - x| \le |t-x|$, it follows

$$|f'(\xi)-f'(x)| \le \omega_1(f',|t-x|) \le \left(1+\delta^{-1}(t-x)^2\right)\cdot\omega_1(f',\delta)$$

and

$$\left|f(t)-f(x)-f'(x)(t-x)\right| \le \left(|t-x|+\delta^{-1}(t-x)^2\right)\cdot\omega_1\left(f',\delta\right).$$

Applying the linear positive Stancu-Durrmeyer type operators to the inequality

$$|f(t)-f(x)| \le \left|f'(x)\right|\cdot|t-x|+\left(|t-x|+\delta^{-1}(t-x)^2\right)\cdot\omega_1\left(f',\delta\right),$$

obtained from the above relations, it follows

$$\left|U_{n,\rho}^{[\alpha]}(f;x)-f(x)\right| \le \left|f'(x)\right|\cdot\left|U_{n,\rho}^{[\alpha]}(e_1-x;x)\right|+\left(U_{n,\rho}^{[\alpha]}(|e_1-x|;x)+\frac{1}{\delta}U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)\cdot\omega_1\left(f',\delta\right).$$

The Cauchy-Schwarz inequality for linear positive operators

$$U_{n,\rho}^{[\alpha]}(|e_1-x|;x) \le \left(U_{n,\rho}^{[\alpha]}(e_0;x)\right)^{\frac{1}{2}}\cdot\left(U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}$$

and the results obtained in Lemma 2.2, respectively Lemma 2.3 leads to

$$\begin{aligned}\left|U_{n,\rho}^{[\alpha]}(f;x)-f(x)\right| &\le \left(\left(U_{n,\rho}^{[\alpha]}(e_0;x)\right)^{\frac{1}{2}}\cdot\left(U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}+\frac{1}{\delta}U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)\cdot\omega_1\left(f',\delta\right)\\ &\le \left(U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}\left(1+\frac{1}{\delta}\left(U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}\right)\cdot\omega_1\left(f',\delta\right).\end{aligned}$$

Because

$$\left(M_{n,\rho,2}^{[\alpha]}(x)\right)^{\frac{1}{2}}=\left(U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)^{\frac{1}{2}}=\sqrt{\tfrac{(1+\alpha+\rho+n\alpha\rho)x(1-x)}{(1+\alpha)(1+n\rho)}}\le\frac{1}{2}\sqrt{\tfrac{1+\alpha+\rho+n\alpha\rho}{(1+\alpha)(1+n\rho)}}$$

and choosing $\delta=\sqrt{\frac{1+\alpha+\rho+n\alpha\rho}{(1+\alpha)(1+n\rho)}}$, we get $\left|U_{n,\rho}^{[\alpha]}(f;x)-f(x)\right|\le\frac{3\delta}{4}\cdot\omega_1(f',\delta)$. □

Estimates using the combinations of the first and second order modulus of smoothness are more refined than estimates using only the first modulus of continuity.

Theorem 3.6. *If $f\in C[0,1]$, then for any $x\in[0,1]$ and $\delta>0$, it follows*

$$\left|U_{n,\rho}^{[\alpha]}(f;x)-f(x)\right|\le\frac{1}{2}\cdot\omega_1(f,\delta)+\frac{9}{8}\cdot\omega_2(f,\delta)\ \ \textit{with}\ \ \delta=\sqrt{\tfrac{1+\alpha+\rho+n\alpha\rho}{(1+\alpha)(1+n\rho)}}$$

Proof. Using Păltănea's result [18] established for a linear positive operator L

$$|L(f;x)-f(x)|\le|L(e_0;x)-1|\cdot|f(x)|+\frac{1}{\delta}|L(e_1-x;x)|\cdot\omega_1(f,\delta)+\left(L(e_0;x)+\frac{1}{2\delta^2}L\left((e_1-x)^2;x\right)\right)\cdot\omega_2(f,\delta)$$

we get for $U_{n,\rho}^{[\alpha]}:=L$ the estimate

$$\begin{aligned}\left|U_{n,\rho}^{[\alpha]}(f;x)-f(x)\right| \le&\left|U_{n,\rho}^{[\alpha]}(e_0;x)-1\right|\cdot|f(x)|+\frac{1}{\delta}\left|U_{n,\rho}^{[\alpha]}(e_1-x;x)\right|\cdot\omega_1(f,\delta)\\ &+\left(U_{n,\rho}^{[\alpha]}(e_0;x)+\frac{1}{2\delta^2}U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\right)\cdot\omega_2(f,\delta).\end{aligned}$$

Taking into account the results of Lemma 2.2, respectively Lemma 2.3 and choosing $\delta=\sqrt{\frac{1+\alpha+\rho+n\alpha\rho}{(1+\alpha)(1+n\rho)}}$ we get the desired result, using previously again the Cauchy-Schwarz inequality for linear positive operators. □

Theorem 3.7. *Let $f \in C[0,1]$, then for any $x \in [0,1]$ yields*

$$\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \leq M \cdot \omega_2\left(f, \tfrac{1}{2}\delta\right), \quad \textit{with} \quad \delta = \tfrac{1}{2}\left(\tfrac{C_\rho^{[\alpha]}}{1+n\rho}\right)^{\frac{1}{2}},$$

where M is an absolute constant.

Proof. For any $g \in C^2[0,1]$ and $t, x \in [0,1]$, by using the Taylor's expansion formula, we have

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Applying the Stancu-Durrmeyer type operators $U_{n,\rho}^{[\alpha]}$ on both sides of the above equation, we get

$$U_{n,\rho}^{[\alpha]}(g;x) - g(x) = g'(x) \cdot U_{n,\rho}^{[\alpha]}(e_1 - x; x) + U_{n,\rho}^{[\alpha]}\left(\int_x^t (t-u)g''(u)du; x\right) = U_{n,\rho}^{[\alpha]}\left(\int_x^t (t-u)g''(u)du; x\right),$$

taking the results of Lemma 2.3 into account. On the other hand

$$\left|\int_x^t (t-u)g''(u)du\right| \leq (t-x)^2 \cdot \|g''\|,$$

then having in mind the inequality established at Lemma 2.5

$$\left|U_{n,\rho}^{[\alpha]}(g;x) - g(x)\right| \leq \|g''\| \cdot U_{n,\rho}^{[\alpha]}\left((e_1-x)^2; x\right) \leq \tfrac{1+\alpha+\rho+n\alpha\rho}{4(1+\alpha)(1+n\rho)} \cdot \|g''\| \leq \tfrac{C_\rho^{[\alpha]}}{4(1+n\rho)} \cdot \|g''\| = \delta^2 \cdot \|g''\|.$$

For any $f \in C[0,1]$ and $g \in C^2[0,1]$, using the Proposition 3.1, it follows

$$\begin{aligned}\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| &\leq \left|U_{n,\rho}^{[\alpha]}(f-g;x)\right| + \left|U_{n,\rho}^{[\alpha]}(g;x) - g(x)\right| + |f(x) - g(x)| \\ &\leq 2 \cdot \|f-g\| + \delta^2 \cdot \|g''\| = 2\left(\|f-g\| + \tfrac{\delta^2}{2} \cdot \|g''\|\right).\end{aligned}$$

Now, taking the infimum on the right-hand side over all $g \in C^2[0,1]$ and using the relation (11), we get

$$\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \leq 2 \cdot K_2\left(f, \tfrac{1}{2}\delta^2\right) \leq M \cdot \omega_2\left(f, \tfrac{1}{2}\delta\right), \quad \text{with} \quad \delta = \tfrac{1}{2}\left(\tfrac{C_\rho^{[\alpha]}}{1+n\rho}\right)^{\frac{1}{2}}.$$

□

In order to prove a global approximation theorem for the Stancu-Durrmeyer type operators involving the Ditzian-Totik modulus of smoothness, we recall some results from [8]. For any $f \in C_B[0,1]$ and $\delta \geq 0$ we define the Ditzian-Totik moduli of smoothness of first, respectively second order by

$$\omega_1^\phi(f,\delta) = \sup_{|h|\leq\delta} \sup_{x\pm(h/2)\phi(x)\in[0,1]} \left|f\left(x + \tfrac{1}{2}h\phi(x)\right) - f\left(x - \tfrac{1}{2}h\phi(x)\right)\right|$$

and

$$\omega_2^\phi(f,\delta) = \sup_{|h|\leq\delta} \sup_{x\pm h\phi(x)\in[0,1]} \left|f\left(x + h\phi(x)\right) - 2f(x) + f\left(x - h\phi(x)\right)\right|, \tag{12}$$

with $\phi(x) = \sqrt{x(1-x)}$, $x \in [0,1]$. The appropriate K-functional of second order is given by

$$K_2^\phi\left(f, \delta^2\right) = \inf_{g' \in AC_{\mathrm{loc}}[0,1]} \left(\|f-g\| + \delta^2 \left\|\phi^2 g''\right\|\right), \tag{13}$$

where $g' \in AC_{\text{loc}}[0,1]$ means that g is differentiable and g' is absolutely continuous on every closed interval $[a,b] \subset [0,1]$. In [8], an inequality between K-functional (13) and second order modulus of smoothness (12), which is given for a positive constant N by

$$K_2^{\phi}\left(f,\delta^2\right) \leq N \cdot \omega_2^{\phi}(f,\delta) \tag{14}$$

is established. Now, we are able to prove the following

Theorem 3.8. *Let $f \in C[0,1]$, then for any $x \in [0,1]$ yields*

$$\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \leq N \cdot \omega_2^{\phi}(f,\delta),\ \textit{with}\ \delta = \left(\frac{C_\rho^{[\alpha]}}{2(1+n\rho)}\right)^{\frac{1}{2}},$$

where N is an absolute constant.

Proof. For any $g \in C^2[0,1]$ and $t,x \in [0,1]$, by using the Taylor's expansion formula we get as in Theorem 3.7 that

$$\left|U_{n,\rho}^{[\alpha]}(g;x) - g(x)\right| \leq U_{n,\rho}^{[\alpha]}\left(\int_x^t |t-u| \cdot |g''(u)| du; x\right). \tag{15}$$

Since $\phi^2(x)$ is a concave function on $[0,1]$, for $u = \lambda x + (1-\lambda)t$ with $t < u < x$ and $\lambda \in [0,1]$, it follows

$$\frac{|t-u|}{\phi^2(u)} = \frac{|t-\lambda x-(1-\lambda)t|}{\phi^2(\lambda x+(1-\lambda)t)} \leq \frac{\lambda|t-x|}{\lambda\phi^2(x)+(1-\lambda)\phi^2(t)} \leq \frac{|t-x|}{\phi^2(x)}.$$

Thus, using the above inequality in the relation (15) and Lemma 2.4, we get

$$\begin{aligned}\left|U_{n,\rho}^{[\alpha]}(g;x) - g(x)\right| &\leq U_{n,\rho}^{[\alpha]}\left(\int_x^t \frac{|t-u|}{\phi^2(u)} du; x\right) \cdot \left\|\phi^2 g''\right\| \leq \frac{1}{\phi^2(x)} \cdot \left\|\phi^2 g''\right\| \cdot U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right)\\ &\leq \frac{1}{\phi^2(x)} \cdot \left\|\phi^2 g''\right\| \cdot \frac{C_\rho^{[\alpha]} x(1-x)}{1+n\rho} = \left\|\phi^2 g''\right\| \cdot \frac{C_\rho^{[\alpha]}}{1+n\rho}.\end{aligned}$$

For any $f \in C[0,1]$ and $g \in AC_{\text{loc}}[0,1]$, using the above inequality and Proposition 3.1, it follows

$$\begin{aligned}\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| &\leq \left|U_{n,\rho}^{[\alpha]}(f-g;x)\right| + \left|U_{n,\rho}^{[\alpha]}(g;x) - g(x)\right| + |f(x)-g(x)|\\ &\leq 2 \cdot \|f-g\| + \frac{C_\rho^{[\alpha]}}{1+n\rho} \cdot \left\|\phi^2 g''\right\| = 2\left(\|f-g\| + \frac{C_\rho^{[\alpha]}}{2(1+n\rho)} \cdot \left\|\phi^2 g''\right\|\right).\end{aligned}$$

Taking the infimum on the right-hand side over all $g \in AC_{\text{loc}}[0,1]$ and using the relation (14), we get

$$\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \leq 2 \cdot K_2\left(f,\delta^2\right) \leq N \cdot \omega_2^{\phi}(f,\delta),\ \text{with}\ \delta = \left(\frac{C_\rho^{[\alpha]}}{2(1+n\rho)}\right)^{\frac{1}{2}}.$$

□

We establish the rate of convergence for differential functions whose derivatives are of bounded variation on $[0,1]$. Let $DBV[0,1]$ be the class of differentiable functions f defined on $[0,1]$, whose derivatives f' are of bounded variation on $[0,1]$. The functions $f \in DBV[0,1]$ could be represented

$$f(x) = \int_0^x g(t)dt + f(0),$$

where $g \in BV[0,1]$, which means that g is a function of bounded variation on $[0,1]$. Also, the operators $U_{n,\rho}^{[\alpha]}f$ admit the integral representation

$$U_{n,\rho}^{[\alpha]}(f;x) = \int_0^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)f(t)dt, \tag{16}$$

where the kernel $\mathcal{S}_{n,\rho}^{[\alpha]}$ is given by

$$\mathcal{S}_{n,\rho}^{[\alpha]}(x,t) = \sum_{k=1}^{n-1} p_{n,k}^{[\alpha]}(x)\frac{t^{k\rho-1}(1-t)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)} + \frac{(1-x)^{[n,-\alpha]}\delta(t)}{1^{[n,-\alpha]}} + \frac{x^{[n,-\alpha]}\delta(1-t)}{1^{[n,-\alpha]}},$$

$\delta(u)$ being the Dirac-delta function.

Lemma 3.9. *Let α be a non-negative parameter which may depend on $n \in \mathbb{N}$, with $\alpha \to 0$ as $n \to \infty$ and $\lim\limits_{n\to\infty} n\alpha = c \in \mathbb{R}$. For a fixed $x \in (0,1)$ and sufficiently large n, it follows*

i) $\lambda_{n,\rho}^{[\alpha]}(x,y) = \int_0^y \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \le \dfrac{C_\rho^{[\alpha]}}{(1+n\rho)}\dfrac{x(1-x)}{(x-y)^2},\ 0 \le y < x;$

ii) $1-\lambda_{n,\rho}^{[\alpha]}(x,z) = \int_z^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \le \dfrac{C_\rho^{[\alpha]}}{(1+n\rho)}\dfrac{x(1-x)}{(z-x)^2},\ x < z < 1.$

Proof.
i) Using Lemma 2.4, we get

$$\lambda_{n,\rho}^{[\alpha]}(x,y) = \int_0^y \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt \le \int_0^y \left(\frac{x-t}{x-y}\right)^2 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt = \frac{1}{(x-y)^2}\cdot U_{n,\rho}^{[\alpha]}\left((e_1-x)^2;x\right) \le \frac{C_\rho^{[\alpha]}}{(1+n\rho)}\frac{x(1-x)}{(x-y)^2}.$$

ii) The proof is immediately, hence the details are omitted. □

Theorem 3.10. *Let $f \in DBV[0,1]$, $\alpha \to 0$ as $n \to \infty$ and $\lim\limits_{n\to\infty} n\alpha = c \in \mathbb{R}$. Then for every $x \in (0,1)$ and sufficiently large n, we have*

$$\begin{aligned}\left|U_{n,\rho}^{[\alpha]}(f;x) - f(x)\right| \le{}& \sqrt{\frac{C_\rho^{[\alpha]}x(1-x)}{(1+n\rho)}}\frac{|f'(x+)-f'(x-)|}{2} + \frac{C_\rho^{[\alpha]}(1-x)}{(1+n\rho)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f_x') + \frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f_x') \\ &+ \frac{C_\rho^{[\alpha]}x}{(1+n\rho)}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+((1-x)/k)}(f_x') + \frac{(1-x)}{\sqrt{n}}\bigvee_{x}^{x+((1-x)/\sqrt{n})}(f_x'),\end{aligned}$$

where $\bigvee_a^b(f_x')$ denotes the total variation of f_x' on $[a,b]$ and f_x' is defined by

$$f_x'(t) = \begin{cases} f'(t)-f'(x-), & 0 \le t < x \\ 0, & t = x \\ f'(t)-f'(x+) & x < t < 1. \end{cases} \tag{17}$$

Proof. The Stancu-Durrmeyer type operators preserve constants and using (16), for every $x \in (0,1)$ we have

$$U_{n,\rho}^{[\alpha]}(f;x) - f(x) = \int_0^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)(f(t)-f(x))dt = \int_0^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t)\int_x^t f'(u)dudt. \tag{18}$$

For any $f \in DBV[0,1]$, from (17) we may write

$$f'(u) = f_x'(u) + \frac{f'(x+)+f'(x-)}{2} + \frac{f'(x+)-f'(x-)}{2}sgn(u-x) + \delta_x(u)\left(f'(u) - \frac{f'(x+)+f'(x-)}{2}\right), \tag{19}$$

where

$$\delta_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$$

Obviously,

$$\int_0^1 \left(\int_x^t \left(f'(u) - \frac{f'(x+) + f'(x-)}{2} \right) \delta_x(u) du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) dt = 0$$

and

$$\begin{aligned} \int_0^1 \left(\int_x^t \frac{f'(x+) + f'(x-)}{2} du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) dt &= \frac{f'(x+) + f'(x-)}{2} \int_0^1 (t-x) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) dt \\ &= \frac{f'(x+) + f'(x-)}{2} \cdot U_{n,\rho}^{[\alpha]}(e_1 - x; x) = 0. \end{aligned}$$

Applying Chauchy-Schwarz inequality for linear positive operators, it follows

$$\begin{aligned} \left| \int_0^1 \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) \left(\int_x^t \frac{f'(x+) - f'(x-)}{2} sgn(u-x) du \right) dt \right| &\leq \frac{\left| f'(x+) - f'(x-) \right|}{2} \int_0^1 |t-x| \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) dt \\ &\leq \frac{\left| f'(x+) - f'(x-) \right|}{2} \cdot U_{n,\rho}^{[\alpha]}(|t-x|; x) \\ &\leq \frac{\left| f'(x+) - f'(x-) \right|}{2} \left(U_{n,\rho}^{[\alpha]}((t-x)^2; x) \right)^{1/2}. \end{aligned}$$

Using Lemma 2.3, respectively Lemma 2.4 and the relations (18), (19) yields

$$\begin{aligned} \left| U_{n,\rho}^{[\alpha]}(f;x) - f(x) \right| &\leq \frac{|f'(x+) - f'(x-)|}{2} \sqrt{\frac{C_\rho^{[\alpha]} x(1-x)}{(1+n\rho)}} \\ &+ \left| \int_0^x \left(\int_x^t f_x'(u) du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) dt + \int_x^1 \left(\int_x^t f_x'(u) du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) dt \right|. \end{aligned} \tag{20}$$

Let $\mathcal{G}_{n,\rho}^{[\alpha]}(f_x', x) = \int_0^x \left(\int_x^t f_x'(u) du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) dt$ and $\mathcal{F}_{n,\rho}^{[\alpha]}(f_x', x) = \int_x^1 \left(\int_x^t f_x'(u) du \right) \mathcal{S}_{n,\rho}^{[\alpha]}(x,t) dt$. To complete the proof, it is sufficient to estimate $\mathcal{G}_{n,\rho}^{[\alpha]}$ and $\mathcal{F}_{n,\rho}^{[\alpha]}$. Since $\int_e^f d_t \lambda_{n,\rho}^{[\alpha]}(x,t) \leq 1$ for all $[a,b] \subseteq [0,1]$, applying the integration formula by parts and using Lemma 3.9 with $y = x - (x/\sqrt{n})$, we may write

$$\begin{aligned} \left| \mathcal{G}_{n,\rho}^{[\alpha]}(f_x', x) \right| &= \left| \int_0^x \left(\int_x^t f_x'(u) du \right) d_t \lambda_{n,\rho}^{[\alpha]}(x,t) \right| = \left| \int_0^x \lambda_{n,\rho}^{[\alpha]}(x,t) f_x'(t) dt \right| \\ &\leq \left(\int_0^y + \int_y^x \right) |f_x'(t)| \cdot |\lambda_{n,\rho}^{[\alpha]}(x,t)| dt \leq \frac{C_\rho^{[\alpha]} x(1-x)}{(1+n\rho)} \int_0^y \bigvee_t^x (f_x')(x-t)^{-2} dt + \int_y^x \bigvee_t^x (f_x') dt \\ &\leq \frac{C_\rho^{[\alpha]} x(1-x)}{(1+n\rho)} \int_0^y \bigvee_t^x (f_x')(x-t)^{-2} dt + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^x (f_x'). \end{aligned}$$

By the substitution of $u = x/(x-t)$, we get

$$\begin{aligned} \frac{C_\rho^{[\alpha]} x(1-x)}{(1+n\rho)} \int_0^{x-(x/\sqrt{n})} (x-t)^{-2} \bigvee_t^x (f_x') dt &= \frac{C_\rho^{[\alpha]}(1-x)}{(1+n\rho)} \int_1^{\sqrt{n}} \bigvee_{x-(x/u)}^x (f_x') du \\ &\leq \frac{C_\rho^{[\alpha]}(1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_{x-(x/k)}^x (f_x') du \leq \frac{C_\rho^{[\alpha]}(1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^x (f_x'). \end{aligned}$$

Thus

$$\left|\mathcal{G}_{n,\rho}^{[\alpha]}(f'_x,x)\right| \leq \frac{C_\rho^{[\alpha]}(1-x)}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_{x-(x/k)}^{x}(f'_x) + \frac{x}{\sqrt{n}} \bigvee_{x-(x/\sqrt{n})}^{x}(f'_x). \tag{21}$$

Using the integration formula by parts and applying Lemma 3.9 with $z = x + ((1-x)/\sqrt{n})$, we get

$$\begin{aligned}
\left|\mathcal{F}_{n,\rho}^{[\alpha]}(f'_x,x)\right| &= \left|\int_x^1 \left(\int_x^t f'_x(u)du\right)\mathcal{S}_{n,\rho}^{[\alpha]}(x,t)dt\right| \\
&= \left|\int_x^z \left(\int_x^t f'_x(u)du\right)d_t(1-\lambda_{n,\rho}^{[\alpha]}(x,t)) + \int_z^1 \left(\int_x^t f'_x(u)du\right)d_t(1-\lambda_{n,\rho}^{[\alpha]}(x,t))\right| \\
&= \left|\left[\int_x^t f'_x(u)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))du\right]_x^z - \int_x^z f'_x(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt + \int_z^1 \left(\int_x^t f'_x(u)du\right)d_t(1-\lambda_{n,\rho}^{[\alpha]}(x,t))\right| \\
&= \left|\int_x^z f'_x(u)du(1-\lambda_{n,\rho}^{[\alpha]}(x,z)) - \int_x^z f'_x(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt + \left[\int_x^t f'_x(u)du(1-\lambda_{n,\rho}^{[\alpha]}(x,t))\right]_z^1\right. \\
&\quad \left. - \int_z^1 f'_x(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt\right| \\
&= \left|\int_x^z f'_x(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt + \int_z^1 f'_x(t)(1-\lambda_{n,\rho}^{[\alpha]}(x,t))dt\right| \\
&\leq \frac{C_\rho^{[\alpha]}x(1-x)}{(1+n\rho)} \int_z^1 \bigvee_x^t (f'_x)(t-x)^{-2}dt + \int_x^z \bigvee_x^t (f'_x)dt \\
&= \frac{C_\rho^{[\alpha]}x(1-x)}{(1+n\rho)} \int_{x+((1-x)/\sqrt{n})}^1 \bigvee_x^t (f'_x)(t-x)^{-2}dt + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})}(f'_x).
\end{aligned}$$

By the substitution of $v = (1-x)/(t-x)$, we get

$$\begin{aligned}
\left|\mathcal{F}_{n,\rho}^{[\alpha]}(f'_x,x)\right| &\leq \frac{C_\rho^{[\alpha]}x(1-x)}{(1+n\rho)} \int_1^{\sqrt{n}} \bigvee_x^{x+((1-x)/v)} (f'_x)(1-x)^{-1}dv + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})}(f'_x) \\
&\leq \frac{C_\rho^{[\alpha]}x}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \int_k^{k+1} \bigvee_x^{x+((1-x)/v)} (f'_x)dv + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x)/\sqrt{n})}(f'_x) \\
&= \frac{C_\rho^{[\alpha]}x}{(1+n\rho)} \sum_{k=1}^{[\sqrt{n}]} \bigvee_x^{x+((1-x)/k)} (f'_x) + \frac{(1-x)}{\sqrt{n}} \bigvee_x^{x+((1-x))/\sqrt{n}}(f'_x).
\end{aligned} \tag{22}$$

Combining the estimates (20)–(22), we get the required result. □

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