



A Note on Clean Elements and Inverses Along an Element

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Abstract. Let R be an associative ring with unity 1 and let $a, d \in R$. An element $a \in R$ is called invertible along d if there exists unique $a^{\parallel d}$ such that $a^{\parallel d}ad = d = daa^{\parallel d}$ and $a^{\parallel d} \in dR \cap Rd$ (see [6, Definition 4]). In this note, we present new characterizations for the existence of $a^{\parallel d}$ by clean decompositions of ad and da . As applications, existence criteria for the Drazin inverse and the group inverse are given.

1. Introduction

Throughout this paper, all rings R considered are assumed to be associative rings with unity 1. We call $a \in R$ regular if there exists some $x \in R$ such that $a = axa$. Such an x is called an inner inverse of a , and is denoted by a^- . An element $a \in R$ is said to be group invertible if there exists an element $b \in R$ such that $ab = ba$, $aba = a$ and $bab = b$. Such a b is called a group inverse of a . It is unique if it exists, and is denoted by $a^\#$. It is known that $a^\#$ exists if and only if there exist $x, y \in R$ such that $a = a^2x = ya^2$. In this case, $a^\# = yax = y^2a = ax^2$.

Let R be a ring with an involution $x \mapsto x^*$ satisfying $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(a + b)^* = a^* + b^*$ for all $a, b \in R$. An element $a \in R$ (with involution) is Moore-Penrose invertible [13] if there exists some $x \in R$ such that $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$. Such an x is called a Moore-Penrose inverse of a , it is unique if it exists, and is denoted by a^\dagger . The standard notion of Drazin inverses can be found in mathematical literature [4]. The symbols R^{-1} , $R^\#$, R^D and R^\dagger denote the sets of all invertible, group invertible, Drazin invertible and Moore-Penrose invertible elements in R , respectively.

Green's preorders (see [5]) in a ring R are defined by: (i) $a \leq_L b$ denotes $a \in Rb$; (ii) $a \leq_R b$ denotes $a \in bR$; (iii) $a \leq_{\mathcal{H}} b$ denotes $a \in bR \cap Rb$. Given $a, d \in R$, following [6], a is invertible along d if there exists an element $b \in R$ such that $bad = d = dab$ and $b \leq_{\mathcal{H}} d$. Such a b is unique if it exists, it is called the inverse of a along d , and is denoted by $a^{\parallel d}$. Furthermore, Mary [6] showed that (i) $a \in R^\#$ if and only if it is invertible along a ; (ii) $a \in R^D$ if and only if it is invertible along a^m , for some positive integer m ; (iii) $a \in R^\dagger$ if and only if it is invertible along a^* . In these cases, $a^\# = a^{\parallel a}$, $a^D = a^{\parallel a^m}$ and $a^\dagger = a^{\parallel a^*}$. More results on the inverse along an

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element can be referred to [2, 7, 17, 18]. It follows from [6] that if $a^{\parallel d}$ exists then both $1 - aa^{\parallel d}$ and $1 - a^{\parallel d}a$ are idempotents. Also, by [7], we know that $a^{\parallel d}$ exists implies $da + 1 - dd^- \in R^{-1}$ and $ad + 1 - d^-d \in R^{-1}$, where the regularity of d is ensured by the existence of $a^{\parallel d}$ (see [6, Theorem 7]).

Recall that an element of a ring R is called clean if it can be written as the sum of an idempotent e and a unit u . A clean ring is one whose each element is clean, which dates back to the paper of Nicholson [8]. Given a clean decomposition $a = e + u$, it is called special clean [1] if $aR \cap eR = 0$, and is strongly clean [9] if it is a clean decomposition and $eu = ue$. Several scholars [3, 10, 14] paid attention to the cleanness of elements in rings. Since clean elements can be written as the sum of an idempotent and a unit, they has close relations with the inverse along an element (idempotents and units can be constructed by this types of generalized inverses). However, few articles are presented about the connections between the cleanness of elements and their generalized inverses.

It should be noted that the classical invertibility constructed by generalized inverses usually is the form $ek + 1 - e$, where e is an idempotent. For instance, for the case of the inverse of a along d , taking $e = dd^-$ and $k = da$. For the group inverse of a regular element d , setting $e = dd^-$ and $k = d$. However, a key issue for investigating the clean decomposition of the inverse along an element is that we need the unit of the form $ek - 1 + e$. So, at the beginning of Section 2, we illustrate that $ek - 1 + e$ is a unit if and only if $ek + 1 - e$ is a unit.

In this note, we firstly give an existence criterion for the inverse along an element, which slightly differs from the result of Mary and Patrício [11]. Then, we characterize the inverse of the product of triple elements along an element by using one-sided inverses along an element. Moreover, the formulae relating them are given. Also, new characterizations for the inverse of a along d are obtained by clean decompositions of ad and da . As special cases, existence criteria for the group inverse and the Drazin inverse are given.

2. The Cleanness of Elements and the Inverse Along an Element

We first begin with the following lemmas, which play an important role in the sequel.

Lemma 2.1. [7, Theorem 2.1] *Let $a, d \in R$. Then the following conditions are equivalent:*

- (i) $a^{\parallel d}$ exists.
 - (ii) $d \leq_{\mathcal{R}} da$ and $(da) \in R^{\#}$.
 - (iii) $d \leq_{\mathcal{L}} ad$ and $(ad) \in R^{\#}$.
- In this case, $a^{\parallel d} = d(ad)^{\#} = (da)^{\#}d$.*

Lemma 2.2. (Jacobson’s Lemma) *Let $a, b \in R$. Then we have*

- (i) *If $(1 - ab) \in R^{-1}$, then $(1 - ba) \in R^{-1}$ and $(1 - ba)^{-1} = 1 + b(1 - ab)^{-1}a$.*
- (ii) *If $(ab - 1) \in R^{-1}$, then $(ba - 1) \in R^{-1}$ and $(ba - 1)^{-1} = b(ab - 1)^{-1}a - 1$.*

Mary and Patrício [7] presented the existence criterion for the inverse along an element by units in a ring, i.e., the equivalences (i) \Leftrightarrow (iv) \Leftrightarrow (v) in Proposition 2.3 below. We next show that $da + (1 - dd^-) \in R^{-1}$ is equivalent to $da - (1 - dd^-) \in R^{-1}$.

Let $e, k \in R$ with e idempotent. Assume that $ek + 1 - e$ is a unit in R . Then, by Lemma 2.2, $u = eke + 1 - e$ is also a unit in R . Consequently, $eke = eue$ is a unit in eRe , which gives that $e(-k)e$ is a unit, and then $e(-k)e + 1 - e$ is a unit in R . Therefore, $-eke + 1 - e$ is a unit, which implies that $eke - 1 + e$ is a unit and hence $ek - 1 + e$ is a unit by Lemma 2.2. Dually, if $ek - 1 + e$ is a unit then so is $ek + 1 - e$. More details on corner rings can be referred to [12].

We remind the reader that there must be a connection with k and e in the first summand x , that is, e must be somehow in x . If it contains no e , then $x + (1 - e) \in R^{-1}$ does not imply $x - (1 - e) \in R^{-1}$ in general.

Such as, let $R = \mathbb{R}_{2 \times 2}$ be the ring of all 2 by 2 real matrices. Suppose $x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $e = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$x + (1 - e) = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \in R^{-1}. \text{ But } x - (1 - e) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \notin R^{-1}.$$

Let $d \in R$ be regular. If $e = dd^-$ and $k = da$, then $ek + 1 - e = da + 1 - dd^- \in R^{-1}$ if and only if $ek - 1 + e = da - 1 + dd^- \in R^{-1}$. Dually, setting $f = d^-d$ and $h = ad$, then $hf + 1 - f = ad + 1 - d^-d \in R^{-1}$ if and only if $hf - 1 + f = ad - 1 + d^-d \in R^{-1}$. We hence add another two characterizations for the inverse along an element.

Proposition 2.3. *Let $a, d \in R$ with d regular. Then the following conditions are equivalent:*

- (i) $a^{\parallel d}$ exists.
 - (ii) $u = da - 1 + dd^- \in R^{-1}$.
 - (iii) $v = ad - 1 + d^-d \in R^{-1}$.
 - (iv) $u' = da + 1 - dd^- \in R^{-1}$.
 - (v) $v' = ad + 1 - d^-d \in R^{-1}$.
- In this case, $a^{\parallel d} = u^{-1}d = dv^{-1} = (u')^{-1}d = d(v')^{-1}$.

Proof. By a direct check, we know that $u^{-1}a = av^{-1}$ is the inverse of a along d . \square

Proposition 2.3 above can also be proved by the following equivalence: a is invertible along d if and only if $-a$ is invertible along d . Applying [7, Theorem 3.2], then $da + 1 - dd^-$ is invertible if and only if $da - 1 + dd^-$ is invertible.

It follows from [17, p.168] that $a \in R^\#$ if and only if 1 is invertible along a , for a regular element $a \in R$. Thus, we obtain the existence criterion for the group inverse of a regular element.

Corollary 2.4. *Let $a \in R$ be regular. Then the following conditions are equivalent:*

- (i) $a \in R^\#$.
 - (ii) $u = a - 1 + aa^- \in R^{-1}$.
 - (iii) $v = a - 1 + a^-a \in R^{-1}$.
 - (iv) $u' = a + 1 - aa^- \in R^{-1}$.
 - (v) $v' = a + 1 - a^-a \in R^{-1}$.
- In this case, $a^\# = u^{-1}a = av^{-1} = (u')^{-1}a = a(v')^{-1}$.

Let us recall that [17] $a \in R^\dagger$ if and only if a^* is invertible along a . Hence, we get the following result concerning the existence criterion of the Moore-Penrose inverse.

Corollary 2.5. *Let R be a ring with involution and let $a \in R$ be regular. Then the following conditions are equivalent:*

- (i) $a \in R^\dagger$.
 - (ii) $u = aa^* - 1 + aa^- \in R^{-1}$.
 - (iii) $v = a^*a - 1 + a^-a \in R^{-1}$.
 - (iv) $u' = aa^* + 1 - aa^- \in R^{-1}$.
 - (v) $v' = a^*a + 1 - a^-a \in R^{-1}$.
- In this case, $a^\dagger = (u^{-1}a)^* = (av^{-1})^* = ((u')^{-1}a)^* = (a(v')^{-1})^*$.

Let R be a ring with involution and let $a, x \in R$. If x satisfies $axa = a$ and $(ax)^* = ax$, then x is a $\{1,3\}$ -inverse of a , and is denoted by $a^{(1,3)}$. If x satisfies $axa = a$ and $(xa)^* = xa$, then x is a $\{1,4\}$ -inverse of a , and is denoted by $a^{(1,4)}$. It is well known that a is Moore-Penrose invertible if and only if it is both $\{1,3\}$ -invertible and $\{1,4\}$ -invertible. Moreover, $a^\dagger = a^{(1,4)}aa^{(1,3)}$.

Suppose we are given any p, a, q in involutory rings R . The present author Patrício [11] illustrated that paq is Moore-Penrose invertible if and only if pa is $\{1,3\}$ -invertible and aq is $\{1,4\}$ -invertible, provided that $p'pa = a = aqq'q'$ for some $p', q' \in R$. In this case, $(paq)^\dagger = (aq)^{(1,4)}a(pa)^{(1,3)}$.

Inspired by the above mentioned author's work, we present the relations among the inverse of paq along d and a left inverse of aq along certain element and a right inverse of pa along certain element.

Let us now recall some definitions and properties of one-sided inverses along an element.

Lemma 2.6. [16] Let $a, d \in R$.

(i) An element $b \in R$ is called a left inverse of a along d if $bad = d$ and $b \leq_{\mathcal{L}} d$. Moreover, a is left invertible along d if and only if $d \leq_{\mathcal{L}} dad$.

(ii) An element $b \in R$ is called a right inverse of a along d if $dab = d$ and $b \leq_{\mathcal{R}} d$. Moreover, a is right invertible along d if and only if $d \leq_{\mathcal{R}} dad$.

Our notation follows [16, 17]. For instance, the symbol $a_l^{\parallel d}$ (resp., $a_r^{\parallel d}$) denotes a left (resp., right) inverse of a along d .

It follows from [16, Corollary 2.5] that a is invertible along d if and only if it is both left and right invertible along d . In particular, by Lemma 2.6, a is invertible along d if and only if $d \leq_{\mathcal{H}} dad$. However, the present authors [16] did not present the formula between the inverse along an element and one-sided inverses along an element.

Theorem 2.7. Let $p, a, q, d \in R$. If there exist $p', q' \in R$ such that $dpp' = d = q'qd$, then the following conditions are equivalent:

(i) paq is invertible along d .

(ii) aq is left invertible along dp and pa is right invertible along qd .

In this case, $(paq)^{\parallel d} = (aq)_l^{\parallel dp} a (pa)_r^{\parallel qd}$.

Proof. (i) \Rightarrow (ii) Since paq is invertible along d , by Lemma 2.6, we have $d \leq_{\mathcal{L}} dpaqd$ and $d \leq_{\mathcal{R}} dpaqd$, and consequently $dp \leq_{\mathcal{L}} dpaqdp$ and $qd \leq_{\mathcal{R}} qdpaqd$. Again, from Lemma 2.6, it follows that aq is left invertible along dp and pa is right invertible along qd .

(ii) \Rightarrow (i) As aq is left invertible along dp , then there is $x \in R$ such that $(aq)_l^{\parallel dp} = xdp$. Hence, $(aq)_l^{\parallel dp} a (pa)_r^{\parallel qd} = xdpa(pa)_r^{\parallel qd} = xq'qdpa(pa)_r^{\parallel qd} = xq'qd = xd$, which implies $(aq)_l^{\parallel dp} a (pa)_r^{\parallel qd} paqd = xdpaqd = (aq)_l^{\parallel dp} aqd = (aq)_l^{\parallel dp} aqdpp' = dpp' = d$.

Since pa is right invertible along qd , we have $(pa)_r^{\parallel qd} = qdy$ for some $y \in R$. A straightforward calculation gives $(aq)_l^{\parallel dp} a (pa)_r^{\parallel qd} = dy$, we hence get $dpaq(aq)_l^{\parallel dp} a (pa)_r^{\parallel qd} = dpaqdy = dpa(pa)_r^{\parallel qd} = d$.

Finally, $(aq)_l^{\parallel dp} a (pa)_r^{\parallel qd} = xd = dy \leq_{\mathcal{H}} d$.

Therefore, paq is invertible along d and $(paq)^{\parallel d} = (aq)_l^{\parallel dp} a (pa)_r^{\parallel qd}$. \square

Taking $p = q = 1$ in Theorem 2.7, it follows that

Corollary 2.8. [18, Proposition 2.3] Let $a, d \in R$. Then the following conditions are equivalent:

(i) a is invertible along d .

(ii) a is both left and right invertible along d .

In this case, $a^{\parallel d} = a_l^{\parallel d} = a_r^{\parallel d} = a_l^{\parallel d} a a_r^{\parallel d}$.

Next, we give another existence criterion of the inverse along an element by the cleanness of elements.

Theorem 2.9. Let $a, d \in R$. Then the following conditions are equivalent:

(i) $a^{\parallel d}$ exists.

(ii) $d \leq_{\mathcal{R}} da$, there exist $e^2 = e \in R$ and $u \in R^{-1}$ such that $da = e + u$ is both a strongly clean decomposition and a special clean decomposition.

(iii) $d \leq_{\mathcal{L}} ad$, there exist $f^2 = f \in R$ and $v \in R^{-1}$ such that $ad = f + v$ is both a strongly clean decomposition and a special clean decomposition.

In this case, $a^{\parallel d} = u^{-2}dad = dadv^{-2}$.

Proof. (i) \Rightarrow (ii) Suppose that $a^{\parallel d}$ exists. Then $d \leq_{\mathcal{R}} da$ and $da \in R^{\#}$ by Lemma 2.1. Set $e = 1 - da(da)^{\#}$ and $u = da - 1 + da(da)^{\#}$. Then $e^2 = e$, and $u \in R^{-1}$ from Corollary 2.4. We compute $eu = ue = -e$ and hence $da = e + u$ is a strongly clean decomposition. Let $b \in daR \cap eR$. Then there are $x, y \in R$ such that $b = dax = ey = edax = 0$. So, $da = e + u$ is a special clean decomposition.

(ii) \Rightarrow (i) Since $da = e + u$ is both a strongly clean decomposition and a special clean decomposition, it follows $dae = eda \in daR \cap eR = 0$. Multiplying $da = e + u$ by da on both left and right sides gives $(da)^2 = uda = dau$, and hence $da = u^{-1}(da)^2 = (da)^2u^{-1}$. So, $da \in R^\#$ and $(da)^\# = u^{-2}da$, which together with Lemma 2.1 ensure that $a^{\parallel d}$ exists since $d \leq_{\mathcal{R}} da$. Moreover, $a^{\parallel d} = (da)^\#d = u^{-2}dad$.

(i) \Leftrightarrow (iii) is similar to the proof of (i) \Leftrightarrow (ii). \square

Remark 2.10. Theorem 2.9 above can also be proved by [3, Corollary 2.4 and Proposition 2.5]. Indeed, $a \in R$ is group invertible (strongly regular) if and only if it has both a strongly clean decomposition and a special clean decomposition. As a consequence, da (ad) is group invertible if and only if da (ad) has both a strongly clean decomposition and a special clean decomposition. Applying Lemma 2.1, the result follows.

It is known that $a \in R^D$ if and only if a is invertible along a^m for some positive integer m . Hence, in the characterization of Drazin inverses, the condition $d \leq_{\mathcal{R}} da$ of Theorem 2.9 can be reduced to $a^{n-1} \leq_{\mathcal{R}} a^n$ for some positive integer n . Herein, we claim that the condition $a^{n-1} \leq_{\mathcal{R}} a^n$ can be dropped. Indeed, if $a^n = e + u$ is both a strongly clean decomposition and a special clean decomposition for some positive integer n , then $ea^n = a^n e \in eR \cap a^n R = 0$. Hence, $a^n = u^{-1}a^{2n} = a^{2n}u^{-1}$, which implies $a^n \in a^{n+1}R \cap Ra^{n+1}$.

The following result gives the characterization of the Drazin inverse by a strongly clean decomposition and a special clean decomposition of a^n , where n is a positive integer.

Corollary 2.11. *Let $a \in R$. Then the following conditions are equivalent:*

(i) $a \in R^D$.

(ii) *There exist $e^2 = e \in R$ and $u \in R^{-1}$ such that $a^n = e + u$ is both a strongly clean decomposition and a special clean decomposition, for some positive integer n .*

In this case, $a^D = u^{-2}a^{2n-1} = a^{2n-1}u^{-2}$.

By substituting “a projection ($p^2 = p = p^*$)” for “an idempotent” in the appropriate concepts, it follows notions of the strongly $*$ -clean decomposition and special $*$ -clean decomposition in $*$ -ring (see [15]).

It follows from (see e.g. [19, Lemma 2.2]) that $a^{(1,3)}$ exists if and only if $a \leq_{\mathcal{L}} a^*a$, and $a^{(1,4)}$ exists if and only if $a \leq_{\mathcal{R}} aa^*$. We next give characterizations for the Moore-Penrose inverse by $*$ -clean properties.

Corollary 2.12. *Let R be a ring with involution and let $a \in R$. Then the following conditions are equivalent:*

(i) $a \in R^\dagger$.

(ii) *$a^{(1,3)}$ exists, there exist $p^2 = p = p^* \in R$ and $u \in R^{-1}$ such that $a^*a = p + u$ is both a strongly $*$ -clean decomposition and a special $*$ -clean decomposition.*

(iii) *$a^{(1,4)}$ exists, there exist $q^2 = q = q^* \in R$ and $v \in R^{-1}$ such that $aa^* = q + v$ is both a strongly $*$ -clean decomposition and a special $*$ -clean decomposition.*

*In this case, $a^\dagger = u^{-2}a^*aa^* = a^*aa^*v^{-2}$.*

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