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On Endomorphisms of Crossed Products

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Abstract. In this paper, we will study the endomorphisms of a certain crossed product.

1. Introduction

In the theory of Hopf algebras, crossed products introduced by Doi and Takeuchi ([8]) are very important algebraic objects, which are associated to cleft extensions of Hopf algebras: indeed, a cleft extension is the same as a crossed product with an invertible cocycle([3],[8]). The main properties of the crossed product in the category of Hopf algebras were investigated by Agore ([2]), which pointed out that the crossed product was a new Hopf algebra containing a normal Hopf subalgebra.

Radford's biproducts are important Hopf algebras, which account for many examples of semisimple Hopf algebras. In ([12]), Radford characterized the endomorphisms (resp. automorphisms) of biproducts. Inspired by the Radford's ideas in ([12]), the aim of this paper is to discuss the endomorphisms of the crossed products.

The paper is organized as follows.

In Sec.2, we recall the notion of crossed products and other useful notations which we often use. The focus of this paper is to characterize the endomorphisms of crossed products in Sec.3. As the application of the main result of Sec.3, we shall give a concrete example, then using the primary method, we characterize its endomorphisms and furthermore automorphisms.

2. Preliminaries

Throughout k is a field and all vector spaces are over k , though we use the redundant expression "over k"quite often. For vector spaces U and V, we drop the subscript k from $\mathrm{Hom}_k(U,V)$, $\mathrm{End}_k(U)$ and $U \otimes_k V$ and use id_U to denote the identity map of U.

Let $(C, \Delta_C, \varepsilon_C)$ be a coalgebra. We use a shorthand version of the Heyneman-Sweedler notation for expressing the coproduct in writing $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ for $c \in C$. For a coalgebra C and an algebra D over k, we let "$\star$" denote the convolution product of $\mathrm{Hom}(C,D)$. Refer to ([7]-[13]) for more knowledge about Hopf algebras.

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Let H be a Hopf algebra and A an algebra. H measures A, if there is a linear map $\phi : H\otimes A \to A$, written by $\phi(h\otimes a) = h\cdot a$, such that

$$h\cdot 1_A = \varepsilon_H(h)1_A \quad \text{and} \quad h\cdot(ab) = (h_{(1)}\cdot a)(h_{(2)}\cdot b),$$

for all $h\in H, a, b\in A$.

Assume that H measures A and that σ is an invertible map in $\mathrm{Hom}(H\otimes H, A)$. The crossed product $A\sharp_\sigma H$ of A with H is the set $A\otimes H$ as a vector space, with multiplication

$$(a\sharp h)(b\sharp g) = a(h_{(1)}\cdot b)\sigma(h_{(2)}, g_{(1)})\sharp h_{(3)}g_{(2)},$$

for all $h, g\in H$ and $a, b\in A$. Here we have written $a\sharp h$ for the tensor $a\otimes h$. It can be proved that $A\sharp_\sigma H$ is an associative algebra with identity element $1_A\sharp 1_H$ if and only if the following compatibility conditions hold: $1_H\cdot a = a$, $\sigma(1_H, h) = \sigma(h, 1_H) = \varepsilon_H(h)1_A$ and

$$h\cdot(g\cdot a) = \sigma(h_{(1)}, g_{(1)})(h_{(2)}g_{(2)}\cdot a)\sigma^{-1}(h_{(3)}, g_{(3)}), \tag{1}$$

$$[h_{(1)}\cdot\sigma(g_{(1)}, l_{(1)})]\sigma(h_{(2)}, g_{(2)}l_{(2)}) = \sigma(h_{(1)}, g_{(1)})\sigma(h_{(2)}g_{(2)}, l), \tag{2}$$

for all $h, g, l\in H$ and $a\in A$. Suppose that A is also a Hopf algebra and ϕ, σ are coalgebra maps, then $A\sharp_\sigma H$ is a Hopf algebra if and only if the following two compatibility conditions hold:

$$h_{(1)}\otimes h_{(2)}\cdot a = h_{(2)}\otimes h_{(1)}\cdot a, \tag{3}$$

$$h_{(1)}g_{(1)}\otimes\sigma(h_{(2)}, g_{(2)}) = h_{(2)}g_{(2)}\otimes\sigma(h_{(1)}, g_{(1)}), \tag{4}$$

for all $h, g\in H$ and $a\in A$. The antipode of $A\sharp_\sigma H$ is given by

$$S(a\sharp h) = (S_A[\sigma(S_H(h_{(2)}), h_{(3)})]\sharp S_H(h_{(1)}))(S_A(a)\sharp 1_H),$$

for all $h\in H, a\in A$.

3. Factorization of Certain Crossed Product Endomorphisms

Let $A\sharp_\sigma H$ be the crossed product. We define $\pi : A\sharp_\sigma H\to H$ by $\pi(a\sharp h) = \varepsilon_A(a)h$ for $a\in A$ and $h\in H$ and $j : H\to A\sharp_\sigma H$ by $j(h) = 1_A\sharp h$ for $h\in H$ are Hopf algebra maps which satisfy $\pi\circ j = id_H$. We use $\mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$ to denote the set of Hopf algebra endomophisms F of $A\sharp_\sigma H$ satisfying $\pi\circ F = \pi$.

We define $\Pi : A\sharp_\sigma H\to A$ and $J : A\to A\sharp_\sigma H$ by $\Pi(a\sharp h) = a\sigma(h_{(1)}, S_H(h_{(2)}))$, for all $a\in A, h\in H$ and $J(a) = a\sharp 1_H$, for all $b\in A$. There is a fundamental relationship between these four maps given by:

$$J\circ\Pi = id_{A\sharp_\sigma H}\star(j\circ S_H\circ\pi). \tag{5}$$

The factorization of F is given in terms of $F_l : A\to A$ and $F_r : H\to A$ defined by:

$$F_l = \Pi\circ F\circ J \quad \text{and} \quad F_r = \Pi\circ F\circ j. \tag{6}$$

Lemma 3.1. *Let $F\in\mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$. Then:*

(a) for all $a\in A$,

$$F_l(a)\sharp 1_H = F(a\sharp 1_H). \tag{7}$$

(b) for all $h\in H$,

$$F_r(h)\sharp 1_H = F(1_A\sharp h_{(1)})(1_A\sharp S_H(h_{(2)})). \tag{8}$$

(c) if H is cocommutative, for all $a \in A$ and $h \in H$, we have

$$F(a\sharp h) = F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))\sharp h_{(4)}. \tag{9}$$

Proof. We need to calculate $J \circ \Pi \circ F$. For $a \in A$ and $h \in H$, we compute

$$\begin{aligned}
&(J \circ \Pi)(F(a\sharp h))\\
(5) &= F((a\sharp h)_{(1)})(j \circ S_H \circ \pi)(F((a\sharp h)_{(2)}))\\
&= F((a\sharp h)_{(1)})(j \circ S_H \circ \pi)((a\sharp h)_{(2)})\\
&= F(a_{(1)}\sharp h_{(1)})(j \circ S_H \circ \pi)(a_{(2)}\sharp h_{(2)})\\
&= F(a\sharp h_{(1)})(1_A\sharp S_H(h_{(2)})).
\end{aligned}$$

Thus,

$$(J \circ \Pi)(F(a\sharp h)) = F(a\sharp h_{(1)})(1_A\sharp S_H(h_{(2)})),$$

for all $a \in A$ and $h \in H$. Equations (7) and (8) follow from the above equation. As for (9), we calculate

$$\begin{aligned}
F(a\sharp h) &= F(a\sharp 1_H)F(1_A\sharp h)\\
&= F(a\sharp 1_H)F(1_A\sharp h_{(1)})(1_A\sharp S_H(h_{(5)}))(\sigma^{-1}(h_{(3)}, S(h_{(2)}))\sharp h_{(4)})\\
&= F(a\sharp 1_H)F(1_A\sharp h_{(1)})(1_A\sharp S_H(h_{(2)}))(\sigma^{-1}(h_{(3)}, S(h_{(4)}))\sharp h_{(5)})\\
&= (F_l(a)\sharp 1_H)(F_r(h_{(1)})\sharp 1_H)(\sigma^{-1}(h_{(2)}, S(h_{(3)}))\sharp h_{(4)})\\
&= F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))\sharp h_{(4)},
\end{aligned}$$

as desired. □

By (7) and (8) of Lemma 3.1:

$$(id_{A\sharp_\sigma H})_l = id_A \quad \text{and} \quad (id_{A\sharp_\sigma H})_r = \sigma(h_{(1)}, S_H(h_{(2)})). \tag{10}$$

Since $F_l(1_A) = 1_A$ by (7) of Lemma 3.1. By (9) of Lemma 3.1, we have

$$F(1_A\sharp h) = F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))\sharp h_{(4)}, \tag{11}$$

for all $h \in H$. We are now able to compute the factors of a composite.

Corollary 3.2. *Let $F, G \in \mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$. Assume that H is cocommutative. Then*

(1) $(F \circ G)_l = F_l \circ G_l$,
(2) $(F \circ G)_r = (F_l \circ (G_r \star (\sigma^{-1} \circ (id_H \otimes S_H) \circ \Delta_H))) \star F_r$

Proof. For $b \in A$, by (7) of Lemma 3.1, we have

$$(F \circ G)_l(b)\sharp 1_H = (F \circ G)(b\sharp 1) = F(G_l(b)\sharp 1_H) = (F_l \circ G_l)(b)\sharp 1_H.$$

Thus, it follows that part (1) holds. Let $h \in H$. Using (11) and the fact that F is multiplicative, and part (1) of (7), we obtain that:

$$\begin{aligned}
&(F\circ G)_r(h_{(1)})\sigma^{-1}(h_{(2)}, S_H(h_{(3)}))\sharp h_{(4)}\\
&= F \circ G(1_A\sharp h)\\
&= F(G_r(h_{(1)})\sigma^{-1}(h_{(2)}, S_H(h_{(3)}))\sharp h_{(4)})\\
&= (F_l(G_r(h_{(1)})\sigma^{-1}(h_{(2)}, S_H(h_{(3)})))\sharp 1_H)(F_r(h_{(4)})\sigma^{-1}(h_{(5)}, S_H(h_{(6)}))\sharp h_{(7)})\\
&= F_l(G_r(h_{(1)})\sigma^{-1}(h_{(2)}, S_H(h_{(3)})))F_r(h_{(4)})\sigma^{-1}(h_{(5)}, S_H(h_{(6)}))\sharp h_{(7)}.
\end{aligned}$$

Applying $id_A \otimes \varepsilon_H$ to both sides of the above equation, we can get part (2). □

Lemma 3.3. *Let $F \in \mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$. Assume that H is cocommutative. Then:*

(1) $F_l : A \to A$ is an algebra endomorphism.
(2) $\varepsilon_A \circ F_l = \varepsilon_A$.
(3) for all $b \in A$,

$$\Delta(F_l(b)) = F_l(b_{(1)}) \otimes F_l(b_{(2)}). \tag{12}$$

(4) for all $b \in A$ and $h \in H$,

$$F_l(h_{(1)} \cdot b)F_r(h_{(2)}) = F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S_H(h_{(3)}))(h_{(4)} \cdot F_l(b))\sigma(h_{(5)}, S_H(h_{(6)})). \tag{13}$$

Proof. For $a, b \in A$, we have

$$\begin{aligned} F_l(ab) &= (\Pi \circ F \circ J)(ab) \\ &= \Pi(F(ab\sharp 1_H)) = \Pi(F(a\sharp 1_H)F(b\sharp 1_H)) \\ (7)\ &= \Pi((F_l(a)\sharp 1_H)(F_l(b)\sharp 1_H)) \\ &= F_l(a)F_l(b). \end{aligned}$$

We compute the coproduct of $F_l(b)\sharp 1_H = F(b\sharp 1_H)$ in two ways. First of all,

$$\Delta(F_l(b)\sharp 1_H) = (F_l(b)_{(1)}\sharp 1_H) \otimes (F_l(b)_{(2)}\sharp 1_H)$$

and secondly, since F is a coalgebra map, we have

$$\begin{aligned} &\Delta(F(b\sharp 1_H)) \\ &= F((b\sharp 1_H)_{(1)}) \otimes F((b\sharp 1_H)_{(2)}) \\ &= F(b_{(1)}\sharp 1_H) \otimes F(b_{(2)}\sharp 1_H) \\ &= F_l(b_{(1)})\sharp 1_H \otimes F_l(b_{(2)})\sharp 1_H. \end{aligned}$$

It follows that

$$(F_l(b)_{(1)}\sharp 1_H) \otimes (F_l(b)_{(2)}\sharp 1_H) = F_l(b_{(1)})\sharp 1_H \otimes F_l(b_{(2)})\sharp 1_H. \tag{14}$$

Applying $id_A \otimes \varepsilon_H \otimes id_A \otimes \varepsilon_H$ to both sides of (14) yields (12). It follows easily that $\varepsilon_A \circ F_r = \varepsilon_A$ from (11). For $a, b \in A$ and $h, g \in H$, we have

$$\begin{aligned} &F((a\sharp h)(b\sharp g)) \\ &= F(a(h_{(1)} \cdot b)\sigma(h_{(2)}, g_{(1)})\sharp h_{(3)}g_{(2)}) \\ &= F_l(a(h_{(1)} \cdot b)\sigma(h_{(2)}, g_{(1)}))F_r(h_{(3)}g_{(2)})\sigma^{-1}(h_{(4)}g_{(3)}, S(h_{(5)}g_{(4)}))\sharp h_{(6)}g_{(5)}. \end{aligned}$$

On the other hand, since F preserves the multiplication, we compute:

$$\begin{aligned} &F((a\sharp h)(b\sharp g)) \\ &= F(a\sharp h)F(b\sharp g) \\ &= (F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))\sharp h_{(4)})(F_l(b)F_r(g_{(1)})\sigma^{-1}(g_{(2)}, S(g_{(3)}))\sharp g_{(4)}) \\ &= F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)})) \\ &\quad \times (h_{(4)} \cdot F_l(b)F_r(g_{(1)})\sigma^{-1}(g_{(2)}, S(g_{(3)})))\sigma(h_{(5)}, g_{(4)})\sharp h_{(6)}g_{(5)} \\ &= F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))(h_{(4)} \cdot F_l(b)F_r(g_{(1)})) \\ &\quad \times (h_{(5)} \cdot \sigma^{-1}(g_{(2)}, S(g_{(3)})))\sigma(h_{(6)}, g_{(4)})\sharp h_{(7)}g_{(5)}. \end{aligned}$$

Applying $id_A \otimes \varepsilon_H$ to both expressions for $F((a\sharp h)(b\sharp g))$, we obtain

$$\begin{aligned} &F_l(a(h_{(1)} \cdot b)\sigma(h_{(2)}, g_{(1)}))F_r(h_{(3)}g_{(2)})\sigma^{-1}(h_{(4)}g_{(3)}, S(h_{(5)}g_{(4)})) \\ &\quad = F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))(h_{(4)} \cdot F_l(b)F_r(g_{(1)})) \\ &\quad\quad \times (h_{(5)} \cdot \sigma^{-1}(g_{(2)}, S(g_{(3)})))\sigma(h_{(6)}, g_{(4)}). \end{aligned} \tag{15}$$

Taking $a = 1_A$ and $g = 1_H$ in (15) yields (13). □

Lemma 3.4. *Let $F \in \mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$. Assume that H is cocommutative. Then,*

(1) $F_r(1_H) = 1_A$.
(2) for all $h, g \in H$,

$$\begin{aligned} F_r(hg) =& F_l(\sigma^{-1}(h_{(1)}, g_{(1)}))F_r(h_{(2)})\sigma^{-1}(h_{(3)}, S_H(h_{(4)})) \\ &\times (h_{(5)} \cdot F_r(g_{(2)})\sigma^{-1}(g_{(3)}, S_H(g_{(4)})))\sigma(h_{(6)}, g_{(5)})\sigma(h_{(7)}g_{(6)}, S_H(h_{(8)}g_{(7)})), \end{aligned} \tag{16}$$

(3) $F_r : H \to A$ is a coalgebra map,

Proof. Taking $a = b = 1_A$ in (15) yields (16). For $h \in H$, we compute $\Delta(F(1_A\sharp h))$ in two ways as follows.

$$\begin{aligned} \Delta(F(1_A\sharp h)) =& F(1_A\sharp h_{(1)}) \otimes F(1_A\sharp h_{(2)}) \\ =& F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))\sharp h_{(4)} \otimes F_r(h_{(5)})\sigma^{-1}(h_{(6)}, S(h_{(7)}))\sharp h_{(8)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \Delta(F(1_A\sharp h)) =& \Delta(F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))\sharp h_{(4)}) \\ =& (F_r(h_{(1)})_{(1)}\sigma^{-1}(h_{(2)}, S(h_{(3)}))_{(1)}\sharp h_{(4)}) \\ &\otimes (F_r(h_{(1)})_{(2)}\sigma^{-1}(h_{(2)}, S(h_{(3)}))_{(2)}\sharp h_{(5)}). \end{aligned}$$

Applying $id_A \otimes \varepsilon_H \otimes id_A \otimes \varepsilon_H$ to the expressions for $\Delta(F(1_B\sharp h))$ gives part (3). □

The following theorem characterizes the element of $\mathrm{End}_{\mathrm{Hom}}(A\sharp_\sigma H)$.

Theorem 3.5. *Let $A\sharp_\sigma H$ be a crossed product and H a cocommutative Hopf algebra, let $\pi : A\sharp_\sigma H \to H$ be the projection from $A\sharp_\sigma H$ onto H, and let $\mathcal{F}_{A,H}$ be the set of pairs $(\mathcal{L}, \mathcal{R})$, where $\mathcal{L} : A \to A$, $\mathcal{R} : H \to A$ are maps which satisfy the conclusions of Lemma 3.3 and Lemma 3.4 for F_l and F_r, respectively. Then the function $\Phi : \mathcal{F}_{A,H} \to \mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$, described by $(\mathcal{L}, \mathcal{R}) \mapsto F$, where*

$$F(a\sharp h) = F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)}, S(h_{(3)}))\sharp h_{(4)},$$

for all $a \in A$ and $h \in H$, is a bijection. Furthermore, $F_l = \mathcal{L}$ and $F_r = \mathcal{R}$.

Proof. We define $\Psi : \mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi) \to \mathcal{F}_{A,H}$ by $\Psi(F) = (\Pi \circ F \circ J, \Pi \circ F \circ j)$. It is easily proved that Φ and Ψ are mutually inverse.

It is easy to see that $\pi \circ F = \pi$. Note that $F(1_A\sharp 1_H) = 1_A\sharp 1_H$ and

$$\begin{aligned} \varepsilon(F(a\sharp h)) =& \varepsilon(\mathcal{L}(\alpha^{-1}(a))\mathcal{R}(h_{(1)})\sharp\beta(h_{(2)})) \\ =& \varepsilon_A(\mathcal{L}(\alpha^{-1}(a))\mathcal{R}(h_{(1)}))\varepsilon_H(\beta(h_{(2)})) \\ =& \varepsilon_A(\mathcal{L}(\alpha^{-1}(a)))\varepsilon_A(\mathcal{R}(h_{(1)}))\varepsilon_H(\beta(h_{(2)}) \\ =& \varepsilon_A(a)\varepsilon_H(h), \end{aligned}$$

for $a \in A$ and $h \in H$ which means $\varepsilon \circ F = \varepsilon$.

For $a, b \in A$ and $h, g \in H$, we have

$$
\begin{aligned}
&F((a\sharp h)(b\sharp g))\\
=&F(a(h_{(1)}\cdot b)\sigma(h_{(2)},g_{(1)})\sharp h_{(3)}g_{(2)})\\
=&F_l(a(h_{(1)}\cdot b)\sigma(h_{(2)},g_{(1)}))F_r(h_{(3)}g_{(2)})\sigma^{-1}(h_{(4)}g_{(3)},S_H(h_{(5)}g_{(4)}))\sharp h_{(6)}g_{(5)}\\
=&F_l(a)F_l(h_{(1)}\cdot b)F_l(\sigma(h_{(2)},g_{(1)}))F_r(h_{(3)}g_{(2)})\sigma^{-1}(h_{(4)}g_{(3)},S_H(h_{(5)}g_{(4)}))\sharp h_{(6)}g_{(5)}\\
=&F_l(a)F_l(h_{(1)}\cdot b)F_r(h_{(2)})\sigma^{-1}(h_{(3)},S_H(h_{(4)}))\\
&\times(h_{(5)}\cdot F_r(g_{(1)})\sigma^{-1}(g_{(2)},S_H(h_{(6)})))\sigma(h_{(7)},g_{(3)})\sharp h_{(8)}g_{(4)}\\
=&F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)},S_H(h_{(3)}))\\
&\times(h_{(4)}\cdot F_l(b))(h_{(5)}\cdot F_r(g_{(1)}))\sigma^{-1}(g_{(2)},S_H(h_{(6)}))\sigma(h_{(7)},g_{(3)})\sharp h_{(8)}g_{(4)}\\
=&F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)},S_H(h_{(3)}))\\
&\times(h_{(4)}\cdot F_l(b)F_r(g_{(1)}))\sigma^{-1}(g_{(2)},S_H(h_{(5)}))\sigma(h_{(6)},g_{(3)})\sharp h_{(7)}g_{(4)}\\
=&F(a\sharp h)F(b\sharp g)
\end{aligned}
$$

Therefore, F is an algebra morphism. Since

$$
\begin{aligned}
&\Delta(F(a\sharp h))\\
=&\Delta(F_l(a)F_r(h_{(1)})\sigma^{-1}(h_{(2)},S(h_{(3)}))\sharp h_{(4)})\\
=&F_l(a_{(1)})F_r(h_{(1)})\sigma^{-1}(h_{(2)},S(h_{(3)}))\sharp h_{(4)}\otimes F_l(a_{(2)})F_r(h_{(5)})\sigma^{-1}(h_{(6)},S(h_{(7)}))\sharp h_{(8)}\\
=&F(b_{(1)}\sharp h_{(1)})F(b_{(2)}\sharp h_{(2)}),
\end{aligned}
$$

we have shown that $\Delta\circ F=(F\otimes F)\circ\Delta$. The other conditions which make $F\in\mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H,\pi)$ can be checked easily. Thus the proof is completed. □

4. The Special Crossed Product

In this section, we shall construct a special crossed product, and describe its endomorphisms.

Example 4.1. *Let A be the Sweedler's Hopf algebra over the complex number field $\mathbb{C}$ which is described as follows:*

$$A=\mathbb{C}<1_A,g,x,gx|g^2=1,x^2=0,xg=-gx>$$

with coalgebra structure $\Delta_A(g)=g\otimes g$, $\Delta_A(x)=x\otimes 1+g\otimes x$, $\varepsilon_A(g)=1$, $\varepsilon_A(x)=0$, $S_A(g)=g=g^{-1}$ and $S_A(x)=-gx$. Let $H=\mathbb{C}<1_A,h>$ be the group Hopf algebra with $h^2=1_H$, $\Delta_H(h)=h\otimes h$, $S_H(h)=h=h^{-1}$, $\varepsilon_H(h)=1$. Define the action of H on A as follows:

$$
\begin{aligned}
&1_H\cdot 1_A=1_A,1_H\cdot x=x,1_H\cdot g=g,1_H\cdot xg=xg,\\
&h\cdot 1_A=1_A,h\cdot g=g,h\cdot x=\frac{\sqrt{2}}{2}e^{\frac{3\pi}{4}i}x+\frac{\sqrt{2}}{2}e^{\frac{\pi}{4}i}xg,\\
&h\cdot xg=\frac{\sqrt{2}}{2}e^{\frac{\pi}{4}i}x+\frac{\sqrt{2}}{2}e^{\frac{3\pi}{4}i}xg.
\end{aligned}
$$

Define $\mathbb{C}$-bilinear maps $\sigma: H\otimes H\to A$ as follows:

$$\sigma(1_H,1_H)=\sigma(1_H,h)=\sigma(h,1_H)=1_A,\sigma(h,h)=g.$$

Then we have a crossed product $A\sharp_\sigma H$ which is Hopf algebra with the tensor coalgebra and the antipode S of $A\sharp_\sigma H$ given by

$$S(1_A\sharp 1_H) = 1_A\sharp 1_H, S(1_A\sharp h) = g\sharp h, S(x\sharp 1_H) = -gx\sharp 1_H,$$
$$S(x\sharp h) =(\frac{\sqrt{2}}{2}e^{\frac{\pi}{4}i}gx - \frac{\sqrt{2}}{2}e^{\frac{3\pi}{4}i}x)\sharp h, S(g\sharp 1_H) = g\sharp 1_H,$$
$$S(g\sharp h) = 1_A\sharp h, S(gx\sharp 1_H) = x\sharp 1_H, S(gx\sharp h) = (\frac{\sqrt{2}}{2}e^{\frac{3\pi}{4}i}gx - \frac{\sqrt{2}}{2}e^{\frac{\pi}{4}i}x)\sharp h.$$

Now, we shall characterize the element of $\mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$. Take a base of $\mathrm{End}(A)$ as follows:

$$\begin{aligned}
\mathcal{L}_1 :&1_A \mapsto 1_A, g \mapsto 0, x \mapsto 0, gx \mapsto 0,\\
\mathcal{L}_2 :&1_A \mapsto 0, g \mapsto 1_A, x \mapsto 0, gx \mapsto 0,\\
\mathcal{L}_3 :&1_A \mapsto 0, g \mapsto 0, x \mapsto 1_A, gx \mapsto 0,\\
\mathcal{L}_4 :&1_A \mapsto 0, g \mapsto 0, x \mapsto 0, gx \mapsto 1_A,\\
\mathcal{L}_5 :&1_A \mapsto g, g \mapsto 0, x \mapsto 0, gx \mapsto 0,\\
\mathcal{L}_6 :&1_A \mapsto 0, g \mapsto g, x \mapsto 0, gx \mapsto 0,\\
\mathcal{L}_7 :&1_A \mapsto 0, g \mapsto 0, x \mapsto g, gx \mapsto 0,\\
\mathcal{L}_8 :&1_A \mapsto 0, g \mapsto 0, x \mapsto 0, gx \mapsto g,\\
\mathcal{L}_9 :&1_A \mapsto x, g \mapsto 0, x \mapsto 0, gx \mapsto 0,\\
\mathcal{L}_{10} :&1_A \mapsto 0, g \mapsto x, x \mapsto 0, gx \mapsto 0,\\
\mathcal{L}_{11} :&1_A \mapsto 0, g \mapsto 0, x \mapsto x, gx \mapsto 0,\\
\mathcal{L}_{12} :&1_A \mapsto 0, g \mapsto 0, x \mapsto 0, gx \mapsto x,\\
\mathcal{L}_{13} :&1_A \mapsto gx, g \mapsto 0, x \mapsto 0, gx \mapsto 0,\\
\mathcal{L}_{14} :&1_A \mapsto 0, g \mapsto gx, x \mapsto 0, gx \mapsto 0,\\
\mathcal{L}_{15} :&1_A \mapsto 0, g \mapsto 0, x \mapsto gx, gx \mapsto 0,\\
\mathcal{L}_{16} :&1_A \mapsto 0, g \mapsto 0, x \mapsto 0, gx \mapsto gx.
\end{aligned}$$

Next, we shall consider $F_l \in \mathrm{End}(A)$ which satisfies the conditions of Lemma 3.3. Let

$$F_l = \sum_{i=1}^{16} k_i\mathcal{L}_i.$$

So we have

$$\begin{aligned}
F_l(1_A) &= k_1 1_A + k_5 g + k_9 x + k_{13} gx,\\
F_l(g) &= k_2 1_A + k_6 g + k_{10} x + k_{14} gx,\\
F_l(x) &= k_3 1_A + k_7 g + k_{11} x + k_{15} gx,\\
F_l(gx) &= k_4 1_A + k_8 g + k_{12} x + k_{16} gx.
\end{aligned}$$

First, by (2) of Lemma 3.3 and applying to g, we have $k_2 + k_6 = 1$. By (1) of Lemma 3.3, $F_l(1_A) = 1_A$. Thus it follows that $k_1 = 1, k_5 = k_9 = k_{13} = 0$. Since F_l preserves the multiplication, we have $1_A = F_l(gg) = F_l(g)F_l(g)$, which yields the following equations:

$$\begin{cases} k_2^2 + k_6^2 = 1,\\ k_2k_6 = 0,\\ k_2k_{10} = 0,\\ k_2k_{14} = 0. \end{cases} \qquad (R1)$$

That $F_l(x)F_l(x) = 0$ yields

$$\begin{cases} k_3^2 + k_7^2 = 0, \\ k_3k_7 = 0, \\ k_3k_{11} = 0, \\ k_3k_{15} = 0. \end{cases} \quad (R2)$$

That $F_l(gx)F_l(gx) = 0$ yields

$$\begin{cases} k_4^2 + k_8^2 = 0, \\ k_4k_8 = 0, \\ k_4k_{12} = 0, \\ k_4k_{16} = 0. \end{cases} \quad (R3)$$

That $F_l(g)F_l(x) = F_l(gx)$ yields

$$\begin{cases} k_2k_3 + k_6k_7 = k_4, \\ k_2k_7 + k_6k_3 = k_8, \\ k_2k_{11} + k_6k_{15} + k_3k_{10} - k_7k_{14} = k_{12}, \\ k_2k_{15} + k_6k_{11} + k_3k_{14} - k_{10}k_7 = k_{16}. \end{cases} \quad (R4)$$

That $F_l(x)F_l(g) = -F_l(gx)$ yields

$$\begin{cases} k_2k_3 + k_6k_7 = -k_4, \\ k_2k_7 + k_6k_3 = -k_8, \\ k_2k_{11} - k_6k_{15} + k_3k_{10} + k_7k_{14} = -k_{12}, \\ k_2k_{15} - k_6k_{11} + k_3k_{14} + k_{10}k_7 = -k_{16}. \end{cases} \quad (R5)$$

By (R5) and (R4), we can get $k_4 = k_8 = 0$, thus (R3) naturally holds. That $F_l(x)F_l(gx) = F_l(gx)F_l(x) = 0$ yield

$$\begin{cases} k_3k_{12} + k_7k_{16} = 0, \\ k_3k_{16} + k_7k_{12} = 0, \\ k_3k_{12} - k_7k_{16} = 0, \\ k_7k_{12} - k_3k_{16} = 0. \end{cases} \quad (R6)$$

That $F_l(g)F_l(gx) = F_l(x)$ and $F_l(gx)F_l(g) = -F_l(x)$ yield

$$\begin{cases} k_2k_{12} + k_6k_{16} = k_{11}, \\ k_2k_{16} + k_6k_{12} = k_{15}, \\ k_3 = k_7 = 0, \\ k_2k_{12} - k_6k_{16} = -k_{11}, \\ k_2k_{16} - k_6k_{12} = -k_{15}. \end{cases} \quad (R7)$$

Applying part (3) of Lemma 3.3 to g, we have the following relations:

$$\begin{cases} k_{10} = k_{14} = 0, \\ k_2^2 = k_2, \\ k_6^2 = k_6, \\ k_2k_6 = 0. \end{cases} \quad (R8)$$

Applying part (3) of Lemma 3.3 to gx yields

$$\begin{cases} k_{12} = 0, \\ k_{16} = k_{16}k_6, \\ k_2k_{16} = 0. \end{cases} \quad (R9)$$

Applying part (3) of Lemma 3.3 to x yields

$$\begin{cases} k_{15} = 0, \\ k_{11} = k_{11}k_6, \qquad (R10) \\ k_2k_{11} = 0. \end{cases}$$

By (R1)-(R10), we can get $k_1 = 1, k_3 = k_4 = k_5 = k_7 = k_8 = k_9 = k_{10} = k_{12} = k_{13} = k_{14} = k_{15} = 0$, and

$$\begin{cases} k_2^2 + k_6^2 = 1, \\ k_2k_6 = k_2k_{11} = k_2k_{16} = 0, \\ k_{11} = k_6k_{11} = k_{16}k_6 = k_{16}. \end{cases}$$

Thus

$$F_l : 1_A \mapsto 1_A, g \mapsto k_2 1_A + k_6 g, x \mapsto k_{11}x, gx \mapsto k_{16}gx.$$

Case 1: If $k_2 = 0$, then $k_6 = 1$ and $k_{11} = k_{16}$ are arbitrary complex number. Thus

$$F_l : 1_A \mapsto 1_A, g \mapsto g, x \mapsto tx, gx \mapsto tgx,$$

where $t \in \mathbb{C}$.
Case 2: If $k_6 = 0$, then we have $k_2 = 1$ and $k_6 = k_{11} = k_{16} = 0$. Thus

$$F_l : 1_A \mapsto 1_A, g \mapsto 1_A, x \mapsto 0, gx \mapsto 0.$$

Next, we shall describe all $F_r \in$Hom(H, A) which satisfy the conditions of Lemma 3.4. Take a base of End(H, A) as follows:

$\mathcal{R}_1 : 1_H \mapsto 1_A, h \mapsto 0,$
$\mathcal{R}_2 : 1_H \mapsto 0, h \mapsto 1_A,$
$\mathcal{R}_3 : 1_H \mapsto g, h \mapsto 0,$
$\mathcal{R}_4 : 1_H \mapsto 0, h \mapsto g,$
$\mathcal{R}_5 : 1_H \mapsto x, h \mapsto 0,$
$\mathcal{R}_6 : 1_H \mapsto 0, h \mapsto x,$
$\mathcal{R}_7 : 1_H \mapsto gx, h \mapsto 0,$
$\mathcal{R}_8 : 1_H \mapsto 0, h \mapsto gx.$

Let

$$F_r = \sum_{i=1}^{8} k_i \mathcal{R}_i.$$

By part (1) of Lemma 3.4, it follows that $k_1 = 1, k_3 = k_5 = k_7 = 0$. Thus we have

$$F_r(h) = k_2 1_A + k_4 g + k_6 x + k_8 gx.$$

Using $\varepsilon_A \circ F_r = \varepsilon_H$, we have $k_2 + k_4 = 1$. Applying part (3) of Lemma 3.4 to h, we can gain the following relations:

$$\begin{cases} k_6 = k_8 = 0, \\ k_2 = k_2^2, \\ k_4 = k_4^2. \end{cases}$$

Furthermore, such F_r which satisfy the above relations will be

$$(F_r)_1 = \mathcal{R}_1 + \mathcal{R}_2, (F_r)_2 = \mathcal{R}_1 + \mathcal{R}_4.$$

Concretely,

$$(F_r)_1 : 1_H \mapsto 1_A, h \mapsto 1_A,$$
$$(F_r)_2 : 1_H \mapsto 1_A, h \mapsto g.$$

Now, we shall consider the pair (F_l, F_r) which satifies the part (4) of Lemma 3.3. After careful discussion, we can get the following pairs:

$$(1^\circ)\begin{cases} F_l : 1_A \mapsto 1_A, g \mapsto 1_A, x \mapsto 0, gx \mapsto 0, \\ F_r : 1_H \mapsto 1_A, h \mapsto 1_A. \end{cases}$$
$$(2^\circ)\begin{cases} F_l : 1_A \mapsto 1_A, g \mapsto 1_A, x \mapsto 0, gx \mapsto 0, \\ F_r : 1_H \mapsto 1_A, h \mapsto g. \end{cases}$$
$$(3^\circ)\begin{cases} F_l : 1_A \mapsto 1_A, g \mapsto g, x \mapsto 0, gx \mapsto 0, \\ F_r : 1_H \mapsto 1_A, h \mapsto 1_A. \end{cases}$$
$$(4^\circ)\begin{cases} F_l : 1_A \mapsto 1_A, g \mapsto g, x \mapsto tx, gx \mapsto tgx, \\ F_r : 1_H \mapsto 1_A, h \mapsto g. \end{cases}$$

Observe that (3°) and (4°) satisfy the condition (3.12). By Theorem 3.5, we can get the elements of $\mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$ as follows:

$$\begin{aligned} F : 1_A\sharp 1_H &\mapsto 1_A\sharp 1_H, \\ 1_A\sharp h &\mapsto g\sharp h, \\ g\sharp 1_H &\mapsto g\sharp 1_H, \\ g\sharp h &\mapsto 1_A\sharp h, \\ x\sharp 1_H &\mapsto 0, \\ x\sharp h &\mapsto 0, \\ gx\sharp 1_H &\mapsto 0, \\ gx\sharp h &\mapsto 0. \end{aligned}$$

and

$$\begin{aligned} F : 1_A\sharp 1_H &\mapsto 1_A\sharp 1_H, \\ 1_A\sharp h &\mapsto 1_A\sharp h, \\ g\sharp 1_H &\mapsto g\sharp 1_H, \\ g\sharp h &\mapsto g\sharp h, \\ x\sharp 1_H &\mapsto tx\sharp 1_H, \\ x\sharp h &\mapsto tx\sharp h, \\ gx\sharp 1_H &\mapsto tgx\sharp 1_H, \\ gx\sharp h &\mapsto tgx\sharp h. \end{aligned}$$

Furthermore, the matrices of the elements of $\mathrm{End}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$ under the base $1_A\sharp 1_H, 1_A\sharp h, g\sharp 1_H, g\sharp h, x\sharp 1_H, x\sharp h, gx\sharp 1_H, gx\sharp h$ are

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&t&0&0&0\\ 0&0&0&0&0&t&0&0\\ 0&0&0&0&0&0&t&0\\ 0&0&0&0&0&0&0&t \end{pmatrix}.$$

Thus $\mathrm{Aut}_{\mathrm{Hopf}}(A\sharp_\sigma H, \pi)$ is isomorphic to

$$\{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \end{pmatrix} | 0 \neq t \in \mathbb{C}\}.$$

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