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On Pseudo-Valuations on BCK-Algebras

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Abstract. In this paper, we study some properties of pseudo-valuations and their induced quasi metrics. The continuity of operation of a BCK-algebra was studied with topology induced by a pseudo-valuation. Moreover, we show that product of finite number of this pseudo metric spaces is a pseudo metric space. Also, we prove that if a BCK-algebra X has a pseudo-valuation, then every quotient space of X has a pseudo metric. The completion of this spaces has been investigated in the present study.

1. Introduction

A BCK-algebra is one of important of logical algebras introduced by Y. Imai and K. Iseki in 1966 [8]. This notation is originated from two different ways: one of them is based on set theory, the other is from classical and non-classical propositional calculi. The BCK-operator $*$ is an analogue of the set theoretical difference. As is well known, there is a close relation between the notions of the set difference in set theory and the implication functor in logical systems. Busneag in [2] defined a pesudo-valuation on a Hilbert algebra, and proved that every pseudo-valuation induces a pseudo-metric on a Hilbert algebra. Doh and Kang [3] by using the model of Hilbert algebra introduced the notion of pseudo-valuation on a BCK/BCI-algebra and provided several theorems of pseudo-valuations. In this paper, in section 3, we study some properties of pseudo-valuations on BCK-algebrs and completion $(\widetilde{X}, \widetilde{d_\varphi})$ of pseudo metric space (X, d_φ). In section 4, we introduced some pseudo-valuations on quotient BCK-algebra X/I_φ and study the induced pseudo metric by this pseudo-valuations. Moreover, we show that for each pseudo-valuation on a BCK-algebra X there is an ideal J different with I_φ such that X/J is pseudo metrizable.

2. Preliminaries

2.1. BCK-algebras

An algebra $(X, *, 0)$ of type $(2, 0)$ is called a *BCK-algebra* if it satisfies the following axioms: for any $x, y, z \in X$,

(1) $((x * y) * (x * z)) * (z * y) = 0$,

(2) $(x * (x * y)) * y = 0$,

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(3) $x * x = 0$,

(4) $x * y = y * x = 0 \Rightarrow x = y$,

(5) $0 * x = 0$.[See,[4]]

In BCK-algebra X if we define $\leqslant$ by $x \leqslant y$ if and only if $x * y = 0$, then $\leqslant$ is a partial order and the following conclusions hold:

(6) $(x * y) * (x * z) \leqslant (z * y)$ and $(y * x) * (z * x) \leqslant (y * z)$,

(7) $x * (x * (x * y)) = x * y$,

(8) $(x * y) * z = (x * z) * y$,

(9) $x * 0 = x$,

(10) $x * y \leqslant x$,

(11) $x \leqslant y$ implies $x * z \leqslant y * z$ and $z * y \leq z * x$,

(12) $(x * y) * z \leq x * z \leq x * (z * u)$.

Let $(X, *, 0)$ be a BCK-algebra and $x \wedge y = y * (y * x)$. Then X is called *commutative BCK-algebra* if $x \wedge y = y \wedge x$. If X is commutative BCK-algebra, then $\inf\{x, y\} = x \wedge y$.

If there is an element 1 of a BCK-algebra $(X, *, 0)$ such that $x \leq 1$ for all $x \in X$, then $(X, *, 0)$ is said to be *bounded* BCK-algebra. [See, [4]]

Definition 2.1. *[4] Let X be a BCK-algebra. An* ideal *is a nonempty set $I \subseteq X$ such that*

(a) $0 \in I$,

(b) $x * y \in I,\ y \in I \Rightarrow x \in I$.

Proposition 2.2. *[4] Let I be an ideal in a BCK-algebra $(X, *, 0)$. Then:*

(i) If $x \leqslant y$ and $y \in I$, then $x \in I$.

(ii) the relation

$$x \equiv^I y \Leftrightarrow x * y,\ \ y * x \in I$$

is a congruence relation on X, i.e. it is an equivalence relation on X such that for each $a, b, c, d \in X$, if $a \equiv^I b$ and $c \equiv^I d$, then $a * c \equiv^I b * d$,

(iii) if $\frac{x}{I} = \{y \in X : x \equiv^I y\}$ and $\frac{X}{I} = \{\frac{x}{I} : x \in X\}$, then $\frac{X}{I}$ is a BCK-algebra under the binary operation

$$\frac{x}{I} * \frac{y}{I} = \frac{x * y}{I}.$$

In this case $\frac{X}{I}$ is said to be a quotient BCK-algebra,

(iv) the mapping $\pi_I : X \hookrightarrow \frac{X}{I}$ by $\pi_I(x) = x/I$ is an epimorphism and for each $S \subseteq X$,

$$(\pi_I^{-1} \circ \pi_I)(S) = \bigcup_{x \in S} \frac{x}{I}.$$

π_I is also called a canonical epimorphism.

2.2. **Pseudo-valuations**

Definition 2.3. *[3] A real-valued function φ on a BCK-algebra X is called a* weak pseudo-valuation *on X if for all $x, y \in X$,*

$$\varphi(x * y) \leq \varphi(x) + \varphi(y). \quad (15)$$

Definition 2.4. *[3] A real-valued function φ on a BCK-algebra X is called a* pseudo-valuation *on X if*

(i) $\varphi(0) = 0$,

(ii) $\varphi(x) - \varphi(y) \leq \varphi(x * y)$, *for all $x, y \in X$.*

A pseudo-valuation φ on a BCK-algebra X is said to be *valuation* if

$$\varphi(x) = 0 \Rightarrow x = 0.$$

Let φ be a pseudo-valuation on a BCK-algebra X. Then for all $x, y, z \in X$,

(16) $\varphi(x) \geq 0$,

(17) $x \leq y \Rightarrow \varphi(x) \leq \varphi(y)$,

(18) $\varphi(x * z) \leq \varphi(x * y) + \varphi(y * z)$.

In a BCK-algebra, every pseudo-valuation is a weak pseudo-valuation.[See, [3]]

Proposition 2.5. *Let φ be a pseudo-valuation on X. Then $I_\varphi = \{x \in X : \varphi(x) = 0\}$ is an ideal of X.*

Theorem 2.6. *[3] Let φ be a pseudo-valuation on a BCK-algebra X. Define $d_\varphi : X \times X \to X$ by*

$$d_\varphi(x, y) = \varphi(x * y) + \varphi(y * x)$$

for all $(x, y) \in X \times X$. Then d_φ is a pseudo-metric, i.e. for evey $x, y, z \in X$ we have:

(i) $d_\varphi(x, x) = 0$,

(ii) $d_\varphi(x, y) = d_\varphi(y, x)$,

(iii) $d_\varphi(x, y) \leq d_\varphi(x, z) + d_\varphi(z, y)$.

If (X, d) is a pseudo-metric space, then:
(i) for each $x \in X$ and $\varepsilon > 0$, the set $B_\varepsilon(x) = \{y \in X : d(y, x) < \varepsilon\}$ is called a *ball of radius ε with center at x*,
(ii) the set $U \subseteq X$ is open in (X, d) if for each $x \in U$, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$,
(iii) the topology τ_d induced by d is the collection of all open sets in (X, d).

Theorem 2.7. *[3] Let φ be a pseudo-valuation on a BCK-algebra X. Then a map $\varphi : X \to \mathbb{R}$ is a valuation if and only if (X, d_φ) is a metric space.*

Proposition 2.8. *[3] Let φ be a pseudo-valuation on a BCK-algebra X. Then:*

(19) $d_\varphi(x, y) \geq d_\varphi(z * x, z * y)$,

(20) $d_\varphi(x, y) \geq d_\varphi(x * z, y * z)$,

(21) $d_\varphi(x * y, z * w) \leq d_\varphi(x * y, z * y) + d_\varphi(z * y, z * w)$,

for all $x, y, z, w \in X$.

3. Pseudo-valuations on BCK-algebras

Proposition 3.1. *Let φ and ψ be pseudo-valuations on BCK-algebra X. Then*

(i) $d_\varphi((x \wedge z), (y \wedge z)) \leq d_\varphi(x, y)$,

(ii) $|\varphi(x) - \varphi(y)| \leq d_\varphi(x, y)$,

(iii) $\varphi : X \to \mathbb{R}$ *is continuous,*

(iv) I_φ *is a closed subset of* X,

(v) *for each* $x \in X$, $\varphi + \psi : X \to \mathbb{R}$ *defined by* $(\varphi + \psi)(x) = \varphi(x) + \psi(x)$ *is a pseudo-valuation on X. Moreover,* $(t\varphi)(x) = t(\varphi(x))$ *is a pseudo-valuation on X for any* $t \in \mathbb{R}^+$ *and* $x \in X$.

Proof. (i) We have $d_\varphi((x \wedge z), (y \wedge z)) = \varphi((x \wedge z) * (y \wedge z)) + \varphi((y \wedge z) * (x \wedge z))$. By (6),

$$(x \wedge z) * (y \wedge z) = (z * (z * x)) * (z * (z * y)) \leq (z * y) * (z * x) \leq (x * z).$$

Similarly, we have $(y \wedge z) * (x \wedge z) \leq (y * x)$. By (17), $\varphi((z * y) * (z * x)) \leq \varphi(x * z)$ and $\varphi((y \wedge z) * (x \wedge z)) \leq \varphi(y * x)$. Hence

$$d_\varphi((x \wedge z), (y \wedge z)) = \varphi((x \wedge z) * (y \wedge z)) + \varphi((y \wedge z) * (x \wedge z)) \leq \varphi(x * y) + \varphi(y * x) = d_\varphi(x, y).$$

(ii) Let $x, y \in X$. Then:

$$\varphi(y) - \varphi(x) \leq \varphi(y * x)) \leq \varphi(y * x) + \varphi(x * y) = d_\varphi(x, y).$$

and

$$\varphi(x) - \varphi(y) \leq \varphi(x * y) \leq \varphi(x * y) + \varphi(y * x) = d_\varphi(x, y).$$

Hence $-d_\varphi(x, y) \leq \varphi(x) - \varphi(y) \leq d_\varphi(x, y)$. Thus $|\, \varphi(x) - \varphi(y)\, | \leq d_\varphi(x, y)$.
(iii) Let $\{x_n\}$ be a sequence in X such that $x_n \longrightarrow x \in X$. Then $d_\varphi(x_n, x) \longrightarrow 0$ in $\mathbb{R}$. The desired result follows by part (ii).
(iv) Since $I_\varphi = \{x \in X : \varphi(x) = 0\} = \varphi^{-1}(\{0\})$, by part ($iii$) the proof is clear.
(v) By definition, $(\varphi + \psi)(0) = \varphi(0) + \psi(0) = 0 + 0 = 0$. Suppose that $x, y \in X$. Then:

$$\begin{aligned}(\varphi + \psi)(x * y) &= \varphi(x * y) + \psi(x * y),\\ &\geq (\varphi(x) - \varphi(y)) + (\psi(x) - \psi(y)),\\ &= (\varphi(x) + \psi(x)) - (\varphi(y) + \psi(y)),\\ &= (\varphi + \psi)(x) - (\varphi + \psi)(y).\end{aligned}$$

Thus $\varphi + \psi$ is a pseudo-valuation on X. The proof for other case is similar. □

Proposition 3.2. *If* τ_φ *is a induced topology by* d_φ, *then* $(X, *, \tau_\varphi)$ *is a topological BCK-algebra.*

Proof. By Theorem 2.6, (X, d_φ) is a pseudo-metric space. Let $x * y \in B_\epsilon(x * y)$. We claim that $B_{\frac{\varepsilon}{2}}(x) * B_{\frac{\varepsilon}{2}}(y) \subseteq B_\varepsilon(x * y)$. Let $z \in B_{\frac{\varepsilon}{2}}(x) * B_{\frac{\varepsilon}{2}}(y)$. Then there exist $p \in B_{\frac{\varepsilon}{2}}(x)$ and $q \in B_{\frac{\varepsilon}{2}}(y)$ such that $z = p * q$. Hence $d_\varphi(x, p) \leq \frac{\varepsilon}{2}$ and $d_\varphi(y, q) \leq \frac{\varepsilon}{2}$. By (19) and (20) we have $d_\varphi(x * y, p * y) \leq d_\varphi(x, p)$ and $d_\varphi(p * y, p * q) \leq d_\varphi(y, q)$. By (21),

$$d_\varphi(x * y, p * q) \leq d_\varphi(x * y, p * y) + d_\varphi(p * y, p * q) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $z = p * q \in B_\varepsilon(x * y)$. Therefore $(X, *, \tau)$ is a topological BCK-algebra. □

Proposition 3.3. *A pseudo-valuation φ on the topological BCK-algebra (X, τ) is continuous iff, for each $\varepsilon > 0$ there exists a neighbourhood U of 0 such that $\varphi(z) < \varepsilon$, for each $z \in U$.*

Proof. Suppose $x \in X$ and ε is a positive number. Let U be a neighborhood of 0 such that $\varphi(z) < \varepsilon$, for each $z \in U$. Since $x * x = 0$, there are open neighborhoods V and W of x such that $V * W \subseteq U$. Put $P = V \cap W$. For each $y \in P$, $x * y, y * x \in P * P \subseteq U$ and so $\varphi(x * y), \varphi(y * x) < \varepsilon$. Thus $\varphi(y) - \varphi(x) < \varepsilon$ and $\varphi(x) - \varphi(y) < \varepsilon$. Hence $|\varphi(y) - \varphi(x)| < \varepsilon$. Thus φ is continuous. Conversely, If φ is continuous on X, then it is continuous in 0. Let ε be a positive number. Since $\varphi(0) = 0$, $(-\varepsilon, \varepsilon)$ is an open neighborhood of $\varphi(0)$ in $\mathbb{R}$. There is an open neighborhood U of 0 in X such that $\varphi(U) \subseteq (-\varepsilon, \varepsilon)$. Therefore $\varphi(z) < \varepsilon$, for each $z \in U$. □

A function between two metric spaces will be called isometry if it preserves distances. Let φ_X and φ_Y be pseudo-valuations on BCK-algebras X and Y respectively. Then $f : X \to Y$ will be called *pseudo-valuation preserving* if $\varphi_Y \circ f = \varphi_X$.

Proposition 3.4. *Let X and Y be BCK-algebras and $\varphi_X : X \to \mathbb{R}$ and $\varphi_Y : Y \to \mathbb{R}$ be pseudo-valuations. If $f : X \to Y$ ia a homomorphism, then the following are equivalent:*

(*i*) *f is pseudo-valuation preserving,*

(*ii*) *f is an isometry.*

Proof. Assume that f is pseudo-valuation preserving. Then for each $x \in X$, $\varphi_Y(f(x)) = \varphi_X(x)$. for any $x, y \in X$ we have,

$$\begin{aligned} d_{\varphi_Y}(f(x), f(y)) &= \varphi_Y(f(x) * f(y)) + \varphi_Y(f(y) * f(x)), \\ &= \varphi_Y(f(x * y)) + \varphi_Y(f(y * x)), \\ &= \varphi_Y \circ f(x * y) + \varphi_Y \circ f(y * x), \\ &= \varphi_X(x * y) + \varphi_X(y * x), \\ &= d_{\varphi_X}(x, y). \end{aligned}$$

Hence f is isometry. Conversely, if f is an isometry, then for any $x \in X$,

$$\varphi_X(x) = d_{\varphi_X}(x, 0) = d_{\varphi_Y}(f(x), f(0)) = \varphi_Y(f(x)) + \varphi_Y(f(0)) = \varphi_Y(f(x)).$$

Therefore f is pseudo-valuation preserving. □

Proposition 3.5. *Let f be an isomorphism from BCK-algebra $(X, *, 0_X)$ to BCK-algebra $(Y, \star, 0_Y)$. If φ is a pseudo-valuation on X, then $\psi : Y \to \mathbb{R}$ defined by $\psi(y) = \varphi \circ f^{-1}(y)$ for any $y \in Y$ is a pseudo-valuation on Y.*

Proof. Since $f(0_X) = 0_Y$, $\psi(0_Y) = \varphi \circ f^{-1}(0_Y) = \varphi(0_X) = 0$. Let $y, y' \in Y$. Since f is bijection, there are $x, x' \in X$ such that $f(x) = y$ and $f(x') = y'$. Hence

$$\begin{aligned} \psi(y \star y') &= \varphi(f^{-1}(y \star y')), \\ &= \varphi(f^{-1}(y) * f^{-1}(y')), \\ &= \varphi(x * x'), \\ &\geq \varphi(x) - \varphi(x'), \\ &= \varphi(f^{-1}(y)) - \varphi(f^{-1}(y'), \\ &= \psi(y) - \psi(y'). \end{aligned}$$

Therefore ψ is a pseudo-valuation on Y. □

Proposition 3.6. *Let f be an isomorphism from BCK-algebra $(X, *, 0_X)$ to BCK-algebra $(Y, \star, 0_Y)$. If ψ is a pseudo-valuation on Y, then $\varphi : X \to \mathbb{R}$ defined by $\varphi(x) = \psi \circ f(x)$ for any $x \in X$ is a pseudo-valuation on X.*

Proof. Since $f(0_X) = 0_Y$, $\varphi(0_X) = \psi \circ f(0_X) = \psi(0_Y) = 0$. For any $x, y \in X$ we have

$$\varphi(x * y) = \psi(f(x * y)) = \psi(f(x) \star f(y)) \geq \psi(f(x)) - \psi(f(y)) = \varphi(x) - \varphi(y).$$

Thus φ is a pseudo-valuation on X. □

Proposition 3.7. *Let φ be a pseudo-valuation on X and $A \subseteq X$. Let $x \in X$. If there is a $y \in A$ such that $x \equiv^{I_\varphi} y$, then $x \in \overline{A}$. The converse is also true, when I_φ is a neighborhood of* 0.

Proof. Let there is a $y \in A$ such that $x \equiv^{I_\varphi} y$. Then $\varphi(x * y) = \varphi(y * x) = 0$. Thus $\varphi(x * y) + \varphi(y * x) < \varepsilon$ for any $\varepsilon > 0$. Hence for each $\varepsilon > 0$, $B_\varepsilon(x) \cap A \neq \emptyset$ and so $x \in \overline{A}$. Conversely, let I_φ be a neighborhood of 0 and $x \in \overline{A}$. There is a sequence $\{x_n\}$ in A such that $x_n \to x$. Since $*$ is continuous, $x * x_n \to 0$ and $x_n * x \to 0$. Hence there is a positive integer n_0 such that $x_{n_0} * x \in I_\varphi$ and $x * x_{n_0} \in I_\varphi$. Thus $x_{n_0} \equiv^{I_\varphi} x$. □

Proposition 3.8. *Let φ be a pseudo-valuation on a BCK-algebra X. If $m_\varphi(x) = lim_{r\to 0^+} inf\{\varphi(z) : z \in B_r(x)\}$ and $M_\varphi(x) = lim_{r\to 0^+} sup\{\varphi(z) : z \in B_r(x)\}$, then:*

(*i*) *for each $x \in X$, $m_\varphi(x)$ and $M_\varphi(x)$ are pseudo-valuations on X,*

(*ii*) *$m_\varphi(x) \leq M_\varphi(x)$ for any $x \in X$,*

(*iii*) *$M_\varphi(x) - m_\varphi(x) \leq M_\varphi(0)$ for any $x \in X$,*

(*iv*) *if $M_\varphi(0) < \infty$, then for all $x \in X$, $M_\varphi(x), m_\varphi(x) < \infty$.*

Proof. (*i*) Let r be a positive number and $z \in B_r(0)$. Then $d_\varphi(z, 0) < r$ and so $\varphi(0) = 0 \leq \varphi(z) < r$. Thus $m_\varphi(0) = 0$. Let $x, y \in X$ and r be a positive number. We show that $m_\varphi(x) \leq m_\varphi(x * y) + m_\varphi(y)$. If $u \in B_r(x)$, then

$$inf\{\varphi(z) : z \in B_r(x)\} \leq \varphi(u) \leq \varphi(u * v) + \varphi(v)$$

for any $v \in B_r(y)$. Hence

$$inf\{\varphi(z) : z \in B_r(x)\} \leq inf\{\varphi(u * v) + \varphi(v) : v \in B_r(y)\}.$$

Since for each $x \in X$, $\varphi(x) \geq 0$, we get

$$inf\{\varphi(u * v) + \varphi(v) : v \in B_r(y)\} = inf\{\varphi(w) : w \in u * B_r(y)\} + inf\{\varphi(v) : v \in B_r(y)\}.$$

From $u * v \in u * B_r(y) \subseteq B_r(x) * B_r(y) \subseteq B_{2r}(x * y)$, we conclude that

$$inf\{\varphi(z) : z \in B_r(x)\} \leq inf\{\varphi(w) : w \in B_{2r}(x * y)\} + inf\{\varphi(v) : v \in B_r(y)\}.$$

Now, the result follows on taking limits as $r \to 0^+$. For other case, since $\{\varphi(z) : z \in B_r(0)\} = \{\varphi(z) : 0 \leq \varphi(z) < r\}$, we get $0 \leq sup\{\varphi(z) : 0 \leq \varphi(z) < r\} \leq r$. Taking limits as $r \to 0^+$, we have $M_\varphi(0) = 0$. Now by similar argument the desired result will obtain.
(*ii*) The proof is clear.
(*iii*) For $m_\varphi(x) < a$ and $b < M_\varphi(x)$ there exist $u, v \in B_r(x)$ with $m_\varphi(x) \leq \varphi(u) < a$ and $b < \varphi(v) \leq M_\varphi(x)$. Hence

$$b - a < \varphi(v) - \varphi(u) \leq \varphi(v * u) = d_\varphi(u * v, 0) < 2r$$

beacuse $v * u \in B_r(x) * B_r(x) \subseteq B_{2r}(x * x) = B_{2r}(0)$. Thus $\varphi(v * u) \leq sup\{\varphi(z) : z \in B_{2r}(0)\}$. Hence, with r fixed, taking a, b to respective limits,

$$M_\varphi(x) - m_\varphi(x) \leq sup\{\varphi(z) : z \in B_{2r}(0)\}.$$

Taking limits as $r \to 0^+$, we obtain the inequality.
(*iv*) Clearly, $0 \leq M_\varphi(x) - m_\varphi(x) \leq M_\varphi(0)$, hence both $M_\varphi(x)$ and $m_\varphi(x)$ are finite for every x. □

Let X be a BCK-algebra. Then X is called positive implicative BCK-algebra if $(x * y) * z = (x * z) * (y * z)$. The sets of the form

$$[0, c] = \{x \in X : 0 \leq x \leq c\} = \{x \in X : x \leq c\}$$

is called initial segment.

Proposition 3.9. *If φ is a pseudo-valuation on positive implicative BCK-algebra X and $a \in X$, then $\varphi_a(x) = \varphi(x * a)$ is a pseudo-valuation on X, for any $x \in X$. Moreover, if φ is a valuation, then φ_a is a valuation if and only if I_{φ_a} is an initial segment.*

Proof. It is easy to prove that $\varphi_a(0) = 0$. Let $x, y, a \in X$. Then

$$\varphi_a(x) - \varphi_a(y) = \varphi(x * a) - \varphi(y * a) \leq \varphi((x * a) * (y * a)) = \varphi((x * y) * a) = \varphi_a(x * y).$$

Hence φ_a is a pseudo-valuation on X. Now, we have

$$\varphi_a(x) = 0 \Leftrightarrow \varphi(x * a) = 0 \Leftrightarrow x * a = 0 \Leftrightarrow x \leq a \Leftrightarrow x \in [0, a].$$

□

Proposition 3.10. *Let X and Y be two BCK-algebras and φ be a pseudo-valuation on X. If $f : X \to Y$ is a surjective homomorphism, then $\phi(y) = inf\{\varphi(x) : f(x) = y\}$ is a pseudo-valuation on Y.*

Proof. It is easy to prove that $\phi(0) = 0$. Let $x, y \in Y$. Then there are $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. Since f is a homomorphism, $f(a * b) = x * y$. Thus

$$\begin{aligned} \phi(x * y) + \phi(y) &= inf\{\varphi(a * b) : f(a * b) = x * y\} + inf\{\varphi(b) : f(b) = y\}, \\ &\geq inf\{\varphi(a * b) + \varphi(b) : f(a * b) = x * y, f(b) = y\}, \\ &\geq inf\{\varphi(a) : f(a) = x\} = \phi(x). \end{aligned}$$

Therefore ϕ is a pseudo-valuation on Y. □

Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two BCK-algebras and $X = X_1 \times X_2$. Let $\pi_i : X \to X_i (i = 1, 2)$ be a projection from X to X_i. Then for any $x = (x_1, x_2), y = (y_1, y_2) \in X$ we have

$$\pi_i(x * y) = \pi_i(x_1 *_1 y_1, x_2 *_2 y_2) = x_i *_i y_i = \pi_i(x) *_i \pi_i(y).$$

Proposition 3.11. *Let $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ be two BCK-algebras and $X = X_1 \times X_2$. Then X has a pseudo-valuation φ if and only if X_i have a pseudo-valuation for each $i = 1, 2$. Moreover, φ is continuous.*

Proof. Let X has a pseudo-valuation. Since $\pi_i : X \to X_i$ is an epimorphism, X_i has a pseudo-valuation for $i = 1, 2$ by Proposition 3.10. Conversely, let φ_1 and φ_2 be pseudo-valuations on X_1 and X_2, respectively. Let $x = (x_1, x_2)$ define $\varphi : X \to \mathbb{R}$ by $\varphi(x) = \varphi_1(x_1) + \varphi_2(x_2)$, then $\varphi(0) = \varphi((0_1, 0_2)) = \varphi_1(0_1) + \varphi_2(0_2) = 0$. Let $x = (x_1, x_2), y = (y_1, y_2) \in X$. Then we have

$$\begin{aligned} \varphi(x * y) &= \varphi(x_1 *_1 y_1, x_2 *_2 y_2), \\ &= \varphi_1(x_1 *_1 y_1) + \varphi_2(x_2 *_2 y_2), \\ &\geq \varphi_1(x_1) - \varphi_1(y_1) + \varphi_2(x_2) - \varphi_2(y_2), \\ &= \varphi_1(x_1) + \varphi_1(x_2) - (\varphi_2(y_1) + \varphi_2(y_2)) \\ &= \varphi(x) - \varphi(y). \end{aligned}$$

Hence φ is a pseudo-valuation on X. Now, let $\{x_n\}$ and $\{y_n\}$ be converges sequences to x and y in X_1 and X_2, respectively. Since φ_1, φ_2 and $*$ are continuous, $\varphi_1(x_n *_1 x) \to 0$ and $\varphi_2(y_n *_2 y) \to 0$. Hence

$$\varphi((x_n, y_n) * (x, y)) = \varphi(x_n *_1 x, y_n *_2 y) = \varphi_1(x_n *_1 x) + \varphi_2(y_n *_2 y) \to 0.$$

Thus φ is continuous. □

Proposition 3.12. *Let φ_1 and φ_2 be two pseudo-valuations on BCK-algebras X_1 and X_2, respectively. For each $(x, y), (a, b) \in X_1 \times X_2$ define*

$$d((x, y), (a, b)) = d_{\varphi_1}(x, a) + d_{\varphi_2}(y, b).$$

Then d is a pseudo metric on $X_1 \times X_2$.

Proof. For any $(x, y), (a, b) \in X_1 \times X_2$, we have

$$d((x, y), (x, y)) = d_{\varphi_1}(x, x) + d_{\varphi_2}(y, y) = 0 + 0 = 0.$$

and

$$d((x, y), (a, b)) = d_{\varphi_1}(x, a) + d_{\varphi_2}(y, b) = d_{\varphi_1}(a, x) + d_{\varphi_2}(b, y) = d((a, b), (x, y)).$$

Let $(x, y), (a, b), (u, v) \in X_1 * X_2$. Then

$$\begin{aligned} d((x, y), (u, v)) &= d_{\varphi_1}(x, u) + d_{\varphi_2}(y, v), \\ &\leq [d_{\varphi_1}(x, a) + d_{\varphi_1}(a, u)] + [d_{\varphi_2}(y, b) + d_{\varphi_2}(b, v)], \\ &= [d_{\varphi_1}(x, a) + d_{\varphi_2}(y, b)] + [d_{\varphi_1}(a, u) + d_{\varphi_2}(b, v)], \\ &= d((x, y), (a, b)) + d((a, b), (u, v)). \end{aligned}$$

Therefore $(X_1 \times X_2, d)$ is a pseudo metric space. □

Corollary 3.13. *If φ_1 and φ_2 are two valuations on BCK-algebras X_1 and X_2, respectively, then $(X_1 \times X_2, d)$ is a metric space.*

Proposition 3.14. *Let φ_1 and φ_2 be two pseudo-valuations on BCK-algebras $(X_1, *_1, 0_1)$ and $(X_2, *_2, 0_2)$ respectively. If $X = X_1 \times X_2$, then $* : X \times X \to X$ is continuous.*

Proof. Let $(x, y), (a, b) \in X$. We show that

$$B_{\frac{\varepsilon}{2}}((a, b)) * B_{\frac{\varepsilon}{2}}((x, y)) \subseteq B_\varepsilon((a, b) * (x, y)) = B_\varepsilon((a *_1 x, b *_2 y)).$$

Let $(s, t) \in B_{\frac{\varepsilon}{2}}((a, b)) * B_{\frac{\varepsilon}{2}}((x, y))$. Then $(s, t) = (\alpha *_1 \gamma, \beta *_2 \lambda) = (\alpha, \beta) * (\gamma, \lambda)$ such that $(\alpha, \beta) \in B_{\frac{\varepsilon}{2}}((a, b))$ and $(\gamma, \lambda) \in B_{\frac{\varepsilon}{2}}((x, y))$. Hence $d((\alpha, \beta), (a, b)) < \frac{\varepsilon}{2}$ and $d((\gamma, \lambda), (x, y)) < \frac{\varepsilon}{2}$. By (19) and (20) we have,

$$\begin{aligned} d((s, t), (a, b) * (x, y)) &= d((\alpha, \beta) * (\gamma, \lambda), (a, b) * (x, y)), \\ &= d((\alpha *_1 \gamma, \beta *_2 \lambda), (a *_1 x, b *_2 y)), \\ &= d_{\varphi_1}((\alpha *_1 \gamma), (a *_1 x)) + d_{\varphi_2}((\beta *_2 \lambda), (b *_2 y)), \\ &\leq [d_{\varphi_1}(\alpha *_1 \gamma, a *_1 \gamma) + d_{\varphi_1}(a *_1 \gamma, a *_1 x)] \\ &+ [d_{\varphi_2}(\beta *_2 \lambda, \beta *_2 y) + d_{\varphi_2}(\beta *_2 y, b *_2 y)], \\ &\leq [d_{\varphi_1}(\alpha, a) + d_{\varphi_1}(\gamma, x)] + [d_{\varphi_2}(\lambda, y) + d_{\varphi_2}(\beta, b)], \\ &= [d_{\varphi_1}(\alpha, a) + d_{\varphi_2}(\beta, b)] + [d_{\varphi_1}(\gamma, x) + d_{\varphi_2}(\lambda, y)], \\ &= d((\alpha, \beta), (a, b)) + d((\gamma, \lambda), (x, y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $*$ is continuous. □

A sequence $\{x_n\} \subseteq X$ is a d_φ*-cauchy* if it is a cauchy sequence of the pseudo-metric (X, d_φ). The space (X, d_φ) is d_φ-complete if any d_φ-cauchy converges to an element of X. Let $\{x_n\}$ and $\{y_n\}$ be d_φ-cauchy sequences. Then the sequence $\{d_\varphi(x_n, y_n)\}$ is convergent, because it is a cauchy sequence in $\mathbb{R}$.

Proposition 3.15. *Let φ be a pseudo-valuation on a BCK-algebra X. Define the relation $\sim$ by:*

$$\{x_n\} \sim \{y_n\} \Leftrightarrow d_\varphi(x_n, y_n) \longrightarrow 0$$

for all d_φ-cauchy sequences $\{x_n\}$ and $\{y_n\}$ in X. Then $\sim$ is a congruence relation on the set of all d_φ-cauchy sequences in X.

Proof. It is easy to prove that $\sim$ is an equivalence relation on X. Let $\{x_n\} \sim \{y_n\}$ and $\{a_n\} \sim \{b_n\}$. Then $d_\varphi(x_n, y_n) \longrightarrow 0$ and $d_\varphi(a_n, b_n) \longrightarrow 0$. By (19) and (20) we have $d_\varphi(x_n * a_n, y_n * a_n) \longrightarrow 0$ and $d_\varphi(y_n * a_n, y_n * b_n) \longrightarrow 0$. By (21) we have $d_\varphi(x_n * y_n, a_n * b_n) \longrightarrow 0$ and so $\{x_n\} * \{y_n\} \sim \{a_n\} * \{b_n\}$. Therefore $\sim$ is a congruence relation on X. □

Definition 3.16. *Let φ be a pseudo-valuation on a BCK-algebra X. The set of all equivalence classes $\widetilde{\{x_n\}} = \{\{y_n\} : \{y_n\} \sim \{x_n\}\}$ is denoted by $\widetilde{X}$. On this set, we define $\widetilde{\{x_n\}} * \widetilde{\{y_n\}} = \widetilde{\{x_n * y_n\}}$.*

Proposition 3.17. *Let φ be a pseudo-valuation on a BCK-algebra X. Then $(\widetilde{X}, *, \widetilde{\{0\}})$ is a BCK-algebra and the pseudo-metric d_φ induces a metric $\widetilde{d_\varphi}$ on $\widetilde{X}$ as follows:*

$$\widetilde{d_\varphi}(\widetilde{\{x_n\}}, \widetilde{\{y_n\}}) = lim_n d_\varphi(x_n, y_n)$$

for all $\widetilde{\{x_n\}}, \widetilde{\{y_n\}} \in \widetilde{X}$.

Proof. It is easy to prove that $(\widetilde{X}, *, \widetilde{\{0\}})$ is a BCK-algebra and $\widetilde{d_\varphi}$ is a pseudo-metric on $\widetilde{X}$. Let $\widetilde{\{x_n\}}, \widetilde{\{y_n\}} \in \widetilde{X}$ and $\widetilde{d_\varphi}(\widetilde{\{x_n\}}, \widetilde{\{y_n\}}) = 0$. Then $d_\varphi(x_n, y_n) \longrightarrow 0$ and so $\{x_n\} \sim \{y_n\}$. Hence $\widetilde{\{x_n\}} = \widetilde{\{y_n\}}$. Therefore $(\widetilde{X}, \widetilde{d_\varphi})$ is a metric space. □

Proposition 3.18. *Let φ be a pseudo-valuation on a BCK-algebra X. Then*

(*i*) *If $\{x_n\}$ is a d_φ-cauchy sequence in X, then $\{\varphi(x_n)\}$ is a cauchy sequence in $\mathbb{R}$.*

(*ii*) *the mapping $\pi_\varphi : X \to \widetilde{X}$ by $\pi_\varphi(x) = \widetilde{\{x\}}$ where $\widetilde{\{x\}}$ is the equivalence class of the constant sequence with any element equal to x, is an homomorphism.*

Proof. (*i*) By Proposition 3.1 (*ii*), the proof is clear.
(*ii*) The proof is clear. □

Proposition 3.19. *Let φ be a pseudo-valuation on a BCK-algebra X. Then the mapping $\widetilde{\varphi} : \widetilde{X} \to \mathbb{R}$ by $\widetilde{\varphi}(\widetilde{\{x_n\}}) = lim_n \varphi(x_n)$ for each d_φ-cauchy sequence in X, is a pseudo-valuation on $\widetilde{X}$.*

Proof. It is easy to prove that $\widetilde{\varphi}(\widetilde{\{0\}}) = 0$. Let $\{x_n\}$ and $\{y_n\}$ be d_φ-cauchy sequences in X. Then

$$\widetilde{\varphi}(\widetilde{\{x_n\}}) = Lim_n \varphi(x_n) \leq lim_n \varphi(x_n * y_n) + lim_n \varphi(y_n) = \widetilde{\varphi}(\widetilde{\{x_n\}} * \widetilde{\{y_n\}}) + \widetilde{\varphi}(\widetilde{\{y_n\}}).$$

Hence $\widetilde{\varphi}$ is a pseudo-valuation on $\widetilde{X}$. □

Corollary 3.20. *The metric space $(\widetilde{X}, \widetilde{d_\varphi})$ is $\widetilde{d_\varphi}$-complete.*

Proposition 3.21. *If $\widetilde{X}, \widetilde{\varphi}, \pi_\varphi$ and $\widetilde{d}$ are defined as above, then following properties hold:*

(*i*) *$\widetilde{\varphi} \circ \pi_\varphi = \varphi$ and hence π_φ is pseudo-valuation preserving.*

(*ii*) *φ is a valuation iff, $\pi_\varphi(x) = \widetilde{\{0\}}$ implies that $x = 0$.*

(*iii*) *$\widetilde{d_\varphi} = d_{\widetilde{\varphi}}$.*

(*iv*) *π_φ is continuous.*

Proof. (i) For any $x \in X$, $\widetilde{\varphi} \circ \pi_\varphi(x) = \widetilde{\varphi}(\pi_\varphi(x)) = \lim_n \varphi(x) = \varphi(x)$.
(ii) Let φ be a valuation and $\pi_\varphi(x) = \widetilde{\{0\}}$. Then $\widetilde{\{x\}} = \widetilde{\{0\}}$ and so $\{x\} \sim \{0\}$. Hence $\varphi(x) = d_\varphi(x, 0) = 0$. Since φ is a valuation, $x = 0$. Conversely, if $\varphi(x) = 0$ for any $x \in X$, then $d_\varphi(x, 0) = \varphi(x) = 0$ and so $\pi_\varphi(x) = \widetilde{\{x\}} = \widetilde{\{0\}}$. Hence $x = 0$. Thus φ is a valuation.
(iii) For any $\widetilde{\{x_n\}}, \widetilde{\{y_n\}} \in \widetilde{X}$ we have

$$\begin{aligned} d_{\widetilde{\varphi}}(\widetilde{\{x_n\}}, \widetilde{\{y_n\}}) &= \widetilde{\varphi}(\widetilde{\{x_n * y_n\}}) + \widetilde{\varphi}(\widetilde{\{y_n * x_n\}}), \\ &= lim_n \varphi(x_n * y_n) + lim_n \varphi(y_n * x_n), \\ &= lim_n d_\varphi(x_n, y_n) \\ &= \widetilde{d_\varphi}(\widetilde{\{x_n\}}, \widetilde{\{y_n\}}). \end{aligned}$$

(iv) If $x_n \longrightarrow x$ in (X, d_φ), then $lim_n d_\varphi(x_n, x) = 0$ in $\mathbb{R}$. Since

$$\begin{aligned} d_{\widetilde{\varphi}}(\pi_\varphi(x_n), \pi_\varphi(x)) &= \widetilde{\varphi}(\pi_\varphi(x_n * x)) + \widetilde{\varphi}(\pi_\varphi(x * x_n)), \\ &= \varphi(x_n * x) + \varphi(x * x_n), \\ &= d_\varphi(x_n, x). \end{aligned}$$

Hence $\pi_\varphi(x_n) \longrightarrow \pi_\varphi(x)$ in $(\widetilde{X}, \widetilde{d_\varphi})$. □

Proposition 3.22. *Let ψ be a pseudo-valuation on a BCK-algebra Y such that (Y, d_ψ) is a d_ψ-complete space. If φ is a pseudo-valuation on a BCK-algebra X and $f : X \to Y$ is a pseudo-valuation preserving homomorphism, then there exists a unique pseudo-valuation preserving homomorphism $\widetilde{f} : \widetilde{X} \to Y$ such that $\widetilde{f} \circ \pi_\varphi = f$.*

Proof. Suppose that $f : X \to Y$ is a pseudo-valuation preserving homomorphism. By Proposition 3.4, f is an isometry. If $\{x_n\}$ is a d_φ-cauchy sequence in X, then $\{f(x_n)\}$ is a d_ψ-cauchy sequence in Y. Since Y is d_ψ-complete, $f(x_n) \to y$ for some $y \in Y$. Define $\widetilde{f}(\widetilde{\{x_n\}}) = y$. We show that $\widetilde{f}$ is the unique isometry such that $\widetilde{f} \circ \pi_\varphi = f$. Let $\widetilde{\{x_n\}}, \widetilde{\{y_n\}} \in \widetilde{X}$, $f(x_n) \to x$ and $f(y_n) \to y$. Then

$$\begin{aligned} d_{\widetilde{\varphi}}(\widetilde{\{x_n\}}, \widetilde{\{y_n\}}) &= \widetilde{d_\varphi}(\widetilde{\{x_n\}}, \widetilde{\{y_n\}}), \\ &= lim_n \varphi(x_n * y_n) + lim_n \varphi(y_n * x_n), \\ &= lim_n \psi \circ f(x_n * y_n) + lim_n \psi \circ f(y_n * x_n), \\ &= lim_n \psi(f(x_n * y_n)) + lim_n \psi(f(y_n * x_n)), \\ &= lim_n \psi(f(x_n) * f(y_n)) + lim_n \psi(f(y_n) * f(x_n)), \\ &= lim_n \psi(x * y) + lim_n \psi(y * x), \\ &= \psi(x * y) + \psi(y * x), \\ &= \psi(\widetilde{f}(\widetilde{\{x_n\}}) * \widetilde{f}(\widetilde{\{y_n\}})) + \psi(\widetilde{f}(\widetilde{\{y_n\}}) * \widetilde{f}(\widetilde{\{x_n\}})), \\ &= d_\psi(\widetilde{f}(\widetilde{\{x_n\}}), \widetilde{f}(\widetilde{\{y_n\}})). \end{aligned}$$

The uniqueness is obvious. Since the BCK-algebra operation Y is continuous respect to d_ψ, we get that $\widetilde{f}$ is a homomorphism. Finally, for each $x \in X$, $\widetilde{f} \circ \pi_\varphi(x) = \widetilde{f}(\widetilde{\{x\}}) = f(x)$. Thus $\widetilde{f} \circ \pi_\varphi = f$. □

4. Pseudo-valuations on Quotient BCK-algebras

Proposition 4.1. *Let I be an ideal in a BCK-algebra X. Then:*

(i) *If φ is a pseudo-valuation on a BCK-algebra X, then $\overline{\varphi}(x/I) = inf\{\varphi(z) : z \in x/I\}$ is a pseudo-valuation on X/I.*

(ii) *If $\overline{\varphi}$ is a pseudo-valuation on X/I, then $\varphi(x) = \overline{\varphi}(x/I)$ is a pseudo-valuatuon on X. Moreover, $\overline{\varphi}$ is a valuation on X if and only if $I = I_\varphi$.*

Proof. (i) This is Proposition 3.10 with $y = x/I$ and $f = \pi_I$.

(ii) Let $\overline{\varphi}$ be a pseudo-valuation on X/I. It is easy to prove that the mapping $\overline{\varphi}(x/I) = \varphi(x)$ is a pseudo-valuation on X. Let $\overline{\varphi}$ be a valuation on X/I. If $x \in I$, then $x/I = 0/I$ and so $\varphi(x) = \overline{\varphi}(x/I) = \overline{\varphi}(0/I) = 0$. Hence $I \subseteq I_\varphi$. If $x \in I_\varphi$, then $\varphi(x) = 0$ and so $\overline{\varphi}(x/I) = 0$. Thus $x/I = 0/I$ and hence $x \in I$. Therefore $I_\varphi \subseteq I$. Conversly, let $I_\varphi = I$ and $\overline{\varphi}(x/I) = 0$. Then $\varphi(x) = 0$ and so $x \in I$. Hence $x/I = 0/I$. Thus $\overline{\varphi}$ is a valuation on X/I. □

Corollary 4.2. *Let φ be a valuation on a BCK-algebra X. If for each $x \in X$, the set x/I has a minimum, then $\overline{\varphi}(x/I) = inf\{\varphi(z) : z \in x/I\}$ is a valuation on X/I.*

Proof. By Proposition 4.1 (i), $\overline{\varphi}$ is a pseudo-valuation. Let for some $x \in X$, $\overline{\varphi}(x/I) = 0$. By assumption, there is an $a \in X$ such that $a = minx/I$. Since for each $z \in x/I$, $a \leq z$, we get that $\varphi(a) \leq \varphi(z) = \overline{\varphi}(z/I) = \overline{\varphi}(x/I)$ and so $\varphi(a) = 0$. Since φ is a valuation, $a = 0$. Hence $x/I = 0/I$. □

Proposition 4.3. *Let φ be a pseudo-valuation on a BCK-algebra X. Then $I \subseteq I_\varphi$ if and only if there exists a pseudo-valuation $\phi : X/I \to \mathbb{R}$ such that $\phi \circ \pi_I = \varphi$.*

Proof. Let $\phi : X/I \to \mathbb{R}$ be a pseudo-valuation on X/I such that $\phi \circ \pi_I = \varphi$. If $x \in I$, then $x/I = 0/I$. Hence

$$\varphi(x) = \phi \circ \pi_I(x) = \phi(\pi_I(x)) = \phi(x/I) = \phi(0/I) = \phi \circ \pi_I(0) = \varphi(0) = 0.$$

Thus $x \in I_\varphi$ and hence $I \subseteq I_\varphi$. Conversely, let $I = I_\varphi$. Define $\phi(x) = \varphi(x)$ for any $x \in X$. If $x, y \in X$ and $x/I = y/I$, then $x * y, y * x \in I$. Since $\phi(x) = \varphi(x)$, $\varphi(x * y) = \varphi(y * x) = 0$. Therefore $0 = \varphi(x * y) \geq \varphi(x) - \varphi(y)$ and $0 = \varphi(y * x) \geq \varphi(y) - \varphi(x)$. Thus $\varphi(x) = \varphi(y)$ and hence ϕ is well defined. We have $\phi(0/I) = \varphi(0) = 0$ and

$$\phi(x/I * y/I) = \phi(x * y/I) = \varphi(x * y) \geq \varphi(x) - \varphi(y) = \phi(x/I) - \phi(y/I).$$

Thus ϕ is a pseudo-valuation on X/I. It is easy to prove that $\phi \circ \pi_I = \varphi$. □

Proposition 4.4. *Let φ be pseudo-valuation on a BCK-algebra X and $I_\varphi = \{x \in X : \varphi(x) = 0\}$. If d_φ is the induced pseudo-metric by φ, Then $D(x/I_\varphi, y/I_\varphi) = d_\varphi(x, y)$ is a metric on X/I_φ.*

Proof. First we show that D is well defined. Let x, y, a and b be in X and $x/I_\varphi = a/I_\varphi$ and $y/I_\varphi = b/I_\varphi$. Then $x * a, a * x, y * b, b * y \in I_\varphi$ and so $\varphi(x * a) = \varphi(a * x) = \varphi(y * b) = \varphi(b * y) = 0$. By (6), $(x * y) * (x * a) \leq (a * y)$ and $(a * y) * (b * y) \leq (a * b)$. Hence

$$\begin{aligned}
\varphi(x * y) - \varphi(x * a) \leq \varphi((x * y) * (x * a)) &\leq \varphi(a * y) \\
&= \varphi(a * y) - \varphi(b * y) \\
&\leq \varphi((a * y) * (b * y)) \leq \varphi(a * b).
\end{aligned}$$

Hence $\varphi(x * y) \leq \varphi(a * b)$. By similar argument we have $\varphi(a * b) \leq \varphi(x * y)$ and so $\varphi(x * y) = \varphi(a * b)$. In a similar fashion we have $\varphi(y * x) = \varphi(b * a)$. Therefore $D(x/I_\varphi, y/I_\varphi) = D(a/I_\varphi, b/I_\varphi)$ and so D is well defined. It is easy to prove that D is a pseudo-metric. To prove that D is a metric, let $D(x/I_\varphi, y/I_\varphi) = 0$. Then $\varphi(x * y) = \varphi(y * x) = 0$ and so $x * y, y * x \in I_\varphi$. Thus $x/I_\varphi = y/I_\varphi$. Hence D is a metric on X/I_φ. □

Proposition 4.5. *Let φ be pseudo-valuation on a BCK-algebra X and $I_\varphi = \{x \in X : \varphi(x) = 0\}$. If τ_D is the induced topology by D on X/I_φ and τ is the quotient topology on X/I_φ, then:*

(i) *the epimorphism $\pi_{I_\varphi} : (X, \tau_\varphi) \to (X/I_\varphi, \tau_D)$ is an open map,*

(ii) $\tau_D = \tau$,

(iii) *if φ is a valuation, then π_{I_φ} is a homeomorphism.*

Proof. (*i*) It is enough to show that $\pi_{I_\varphi}(B_\varepsilon(x)) \in \tau_D$ for each $x \in X$ and $\varepsilon > 0$. We have

$$\begin{aligned}\pi_{I_\varphi}(B_\varepsilon(x)) &= \{\pi_{I_\varphi}(y) : y \in B_\varepsilon(x)\} = \{y/I_\varphi : d_\varphi(y,x) < \varepsilon\},\\ &= \{y/I_\varphi : D(y/I_\varphi, x/I_\varphi) < \varepsilon\},\\ &= B^D_\varepsilon(x/I_\varphi) \in \tau_D.\end{aligned}$$

(*ii*) It is clear that the map $\pi_{I_\varphi} : (X, \tau_\varphi) \to (X/I_\varphi, \tau_D)$ is continuous, becaus $D(x/I_\varphi, y/I_\varphi) = d_\varphi(x, y)$. Thus $\tau_D \subseteq \tau$. If $U \in \tau$, then $\pi^{-1}_\varphi(U) \in \tau_\varphi$. Hence $\pi^{-1}_{I_\varphi}(U) = \cup_{x \in \pi^{-1}_{I_\varphi}(U)} B_\varepsilon(x)$. Since π_{I_φ} is an epimorphism, $U = \pi_{I_\varphi}(\pi^{-1}_{I_\varphi}(U)) = \pi_{I_\varphi}(\cup_{x\in\pi^{-1}_{I_\varphi}(U)} B_\varepsilon(x)) = \cup_{x\in\pi^{-1}_{I_\varphi}(U)} B^D_\varepsilon(x/I_\varphi) \in \tau_D$. Thus $U \in \tau_D$. Therefore $\tau_D = \tau$.
(*iii*) It is enough to show that π_{I_φ} is injective. Let $x, y \in X$ and $\pi_{I_\varphi}(x) = \pi_{I_\varphi}(y)$. Then $x/I_\varphi = y/I_\varphi$ and so $x * y, y * x \in I_\varphi$. Thus $\varphi(x * y) = \varphi(y * x) = 0$. Since φ is a valuation, $x * y = y * x = 0$. By (4), $x = y$. Hence π_{I_φ} is a homeomorphism. □

Proposition 4.6. *Let φ be a pseudo-valuation on a BCK-algebra X. If $x/I_\varphi = y/I_\varphi$, then $\varphi(x) = \varphi(y)$ for any $x, y \in X$.*

Proof. Let $x/I_\varphi = y/I_\varphi$. Then $x \equiv^{I_\varphi} y$ and so $\varphi(x * y) = \varphi(y * x) = 0$. By Proposition 3.1, we have

$$|\varphi(x) - \varphi(y)| \leq d_\varphi(x, y) = \varphi(x * y) + \varphi(y * x) = 0.$$

Thus $\varphi(x) = \varphi(y)$. □

Theorem 4.7. *Let φ be a pseudo-valuation on a BCK-algebra X and for each $x \in X$ the set x/I_φ has a minimum. Then there is a pseudo-valuation ϕ on X/I_φ such that $(X/I_\varphi, d_\phi)$ is a metric space. Moreover, if τ_ϕ is the induced topology by d_ϕ, then τ_ϕ is weaker than the quotient topology on X/I_φ.*

Proof. Let $x \in X$. By assumption, there is a $x_0 \in x/I_\varphi$ such that $x_0 = min\, x/I_\varphi$. Define $\phi(x/I_\varphi) = \varphi(x_0)$. We show that ϕ is a pseudo-valuation on X/I_φ. Since $0 \in I_\varphi = 0/I_\varphi$, $\phi(0/I_\varphi) = \varphi(0) = 0$. Let $x, y \in X$, $x_0 = min\, x/I_\varphi$, $y_0 = min\, y/I_\varphi$ and $z_0 = min\, (x * y)/I_\varphi$. Since $x_0 * y_0 \in (x * y)/I_\varphi$, $x_0 * y_0 \equiv^{I_\varphi} z_0$ and so $(x_0 * y_0)/I_\varphi = z_0/I_\varphi$. By Proposition 4.6, $\varphi(x_0 * y_0) = \varphi(z_0)$. Thus

$$\phi(x/I_\varphi) = \varphi(x_0) \leq \varphi(x_0 * y_0) + \varphi(y_0) = \varphi(z_0) + \varphi(y_0) = \phi((x * y)/I_\varphi) + \phi(y/I_\varphi).$$

Hence ϕ is a pseudo-valuation on X/I_φ. By Theorem 2.6, $d_\phi = \phi((x*y)/I_\varphi)+\phi((y*x)/I_\varphi)$ is a pseudo-valuation on X/I_φ. Now, we show that d_ϕ is a metric. Let $x \in X$ and $x_0 = \min\ x/I_\varphi$. If $\phi(x/I_\varphi) = 0$, then $\varphi(x_0) = 0$ and so $x_0 \in I_\varphi$. Hence $x/I_\varphi = x_0/I_\varphi = 0/I_\varphi$. Thus d_ϕ is a metric on X/I_φ. Finally, we show that τ_ϕ is weaker than the quotient topology on X/I_φ. For this, let $a_0 = \min\, (x * y)/I_\varphi$ and $b_0 = min\, (y * x)/I_\varphi$. Then $a_0 \leq x * y$ and $b_0 \leq y * x$ we have

$$d_\phi(x/I_\varphi, y/I_\varphi) = \phi((x * y)/I_\varphi) + \phi((y * x)/I_\varphi) = \varphi(a_0) + \varphi(b_0) \leq \varphi(x * y) + \varphi(y * x) = d_\varphi(x, y).$$

Now it is easy to prove that the mapping $\pi_{I_\varphi} : X \to X/I_\varphi$ by $\pi_{I_\varphi}(x) = x/I_\varphi$ is continuous. Therefore τ_ϕ is weaker than the quotient topology on X/I_φ. □

Theorem 4.8. *Let φ be a valuation on a BCK-algebra X. If (X, d_φ) is a d_φ-complete, then for each closed ideal I, X/I is a metric space.*

Proof. Let I be a closed ideal in (X, d_φ). By Proposition 4.1, the mapping $\overline{\varphi}(x/I) = inf\{\varphi(z) : z \in x/I\}$ is a pseudo-valuation on X/I. We prove that $\overline{\varphi}$ is a valuation. For this let $\overline{\varphi}(x/I) = 0$ for some $x \in X$. Since $\overline{\varphi}(x/I) = inf\{\varphi(z) : z \in x/I\}$, there is a sequence $\{z_n\} \subseteq x/I$ such that the sequence $\{\varphi(z_n)\}$ converges to 0. We show that $\{z_n\}$ is a d_φ-cauchy sequence. Let $\varepsilon > 0$. There is a $n_0 \in \mathbb{N}$ such that for each $n \geq n_0$, $\varphi(z_n) < \frac{\varepsilon}{2}$. Now by (17), for each $n, m \geq n_0$, we have

$$d_\varphi(z_n, z_m) = \varphi(z_n * z_m) + \varphi(z_m * z_n) \leq \varphi(z_n) + \varphi(z_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence the sequence $\{z_n\}$ is d_φ-cauchy sequence and so converges to a $z \in X$. Since φ is continuous, the sequence $\{\varphi(z_n)\}$ converges to $\varphi(z)$. Hence $\varphi(z) = 0$ and since φ is a valuation on X, we get $z = 0$. On oter hand, since the sequence $\{z_n\}$ is converges to z, then $z \in \overline{x/I}$. Since I is closed in (X, d_φ) and $(X, *, \tau_\varphi)$ is a topological BCK-algebra, by [[10], Proposition 3.8] x/I is closed in (X, d_φ) and so $0 = z \in x/I$. Thus $\overline{\varphi}$ is a valuation on X/I. Now by Proposition 2.7, X/I is a metric space. □

Theorem 4.9. *Let φ be a pseudo-valuation on a BCK-algebra X. Then there exists a closed ideal J on X such that the quotient BCK-algebra X/J is pseudo-metrizable.*

Proof. We define a binary relation $\sim$ for elements $a, b \in X$ by the rule $a \sim b$ if $\varphi((x * a) * y) = \varphi(x * b) * y)$ for all $x, y \in X$. It is immediate from definition that this relation is an equivalence relation. Let J be the class containing $0 \in X$. Let us show that J is a closed ideal of X and for each $x \in X$, $x/J \subseteq x/\sim$. Clearly,

$$J = \{a \in X : \varphi((x * a) * y) = \varphi(x * 0) * y) = \varphi(x * y) \text{ for all } x, y \in X\}$$

For $x, y \in X$ define a function $f_{x,y} : X \to \mathbb{R}$ by $f_{x,y}(z) = \varphi((x * z) * y)$ for each $z \in X$. Since the function $f_{x,y}$ is continuous, the set $J = \cap_{x,y \in X} f^{-1}_{x,y}(f_{x,y}(0))$ is closed in X. To show that J is an ideal of X, let $a * b, b \in J$. Then $\varphi((x * (a * b) * y) = \varphi(x * y)$ and $\varphi((x * b) * y) = \varphi(x * y)$. Replacing x by $x * b$ in the frist equality, by (6) we obtain

$$\varphi((x * b) * y) = \varphi(((x * b) * (a * b)) * y) \leq \varphi((x * a) * y).$$

Thus $\varphi(x * y) \leq \varphi((x * a) * y)$. On the other hand, (8) and (10) imply $(x * a) * y = (x * y) * a \leq x * y$. By (17), $\varphi((x * a) * y) \leq \varphi(x * y)$. Therefore $\varphi((x * a) * y) = \varphi(x * y)$ and so $a \in J$. Thus J is an ideal of X.

Let $d \in c/J$. Then $c * d, d * c \in J$. Since $\varphi((x * (c * d)) * y) = \varphi(x * y)$ and $\varphi((x * (d * c)) * y) = \varphi(x * y)$, replacing x by $x * d$ in frist equality, we obtain

$$\varphi((x * d) * y) = \varphi(((x * d) * (c * d)) * y) \leq \varphi((x * c) * y).$$

Similarly, replacing x by $x * c$ in second equality, we obtain $\varphi((x * c) * y) = \varphi(((x * c) * (d * c)) * y) \leq \varphi((x * d) * y)$. Thus $\varphi((x * d) * y) = \varphi((x * c) * y)$ which implies that $c \sim d$. Hence $d \in c/\sim$. Therefore $c/J \subseteq c/\sim$. Since for any $x, y \in X$, the function $\varphi((x * a) * y)$ with argument a is constant on the set a/J, so for any $a, b \in X$, we can define

$$\rho(a/J, b/J) = sup_{x,y \in X} |\varphi((x * a) * y) - \varphi((x * b) * y)|.$$

We claim that ρ is a pseudo-metric on X/J. Clearly, $\rho(a/J, b/J) \geq 0$ for each $a, b \in X$. It is clear that $\rho(a/J, b/J) = \rho(b/J, a/J)$. To verify triangle inequality, let $a, b, c \in X$. Then

$$\begin{aligned}
\rho(a/J, c/J) &= sup_{x,y \in X} |\varphi((x * a) * y) - \varphi((x * c) * y)| \\
&\leq sup_{x,y \in X} (|\varphi((x * a) * y) - \varphi((x * b) * y)| + |\varphi((x * b) * y) - \varphi((x * c) * y)|) \\
&\leq sup_{x,y \in X} |\varphi((x * a) * y) - \varphi((x * b) * y)| + sup_{x,y \in X} |\varphi((x * b) * y) - \varphi((x * c) * y)| \\
&= \rho(a/J, b/J) + \rho(b/J, c/J).
\end{aligned}$$

□

5. Conclusion

In this paper, we studied some properties of pseudo-valuations and their induced metrics on a BCK-algebra and we showed that there are many pseudo-valuations on a BCK-algebra. The set of all pseudo-valuations on a BCK-algebra is a BCK-algebra, too. Next the researchers can study properties of this BCK-algebra. Moreover, since the power set of a non-empty set is a BCK-algebra using of pseudo-valuations can be useful in the study of theory of sets.

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