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Improving some Operator Inequalities for Positive Linear Maps

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Abstract. Let $0 < mI \leq A \leq m'I \leq M'I \leq B \leq MI$ and $p \geq 1$. Then for every positive unital linear map Φ,

$$\Phi^{2p}(A\nabla_t B) \leq \Big(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\Big)^{2p}\Phi^{2p}(A\sharp_t B)$$

and

$$\Phi^{2p}(A\nabla_t B) \leq \Big(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\Big)^{2p}(\Phi(A)\sharp_t\Phi(B))^{2p},$$

where $t \in [0,1]$, $h = \frac{M}{m}$, $K(h,2) = \frac{(h+1)^2}{4h}$, $Q(t) = \frac{t^2}{2}(\frac{1-t}{t})^{2t}$ and $Q(0) = Q(1) = 0$. Moreover, we give an improvement for the operator version of Wielandt inequality.

1. Introduction

Throughout this paper, let m, m', M, M' be scalars and I be the identity operator. Other capital letters are used to denote the general elements of the C^* algebra $B(\mathcal{H})$ of all bounded linear operators acting on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. We write $A \geq 0$ to mean that the operator A is positive. If $A - B \geq 0$ ($A - B \leq 0$), then we say that $A \geq B$ ($A \leq B$). If $A, B \in B(\mathcal{H})$ are two positive operators, then the weighted arithmetic and geometric mean are respectively defined as:

$$A\nabla_\mu B = (1-\mu)A + \mu B, \quad A\sharp_\mu B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\mu A^{\frac{1}{2}},$$

where $\mu \in [0,1]$. When $\mu = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively, see [1] for more details. The Kantorovich constant is defined by $K(t,2) = \frac{(t+1)^2}{4t}$ for $t > 0$.

A linear map $\Phi : B(\mathcal{H}) \to B(\mathcal{H})$ is called positive (strictly positive) if $\Phi(A) \geq 0$ ($\Phi(A) > 0$) whenever $A \geq 0$ ($A > 0$), and Φ is said to be unital if $\Phi(I) = I$.

It is well known that for any two positive operators A, B,

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$$A \geq B \nRightarrow A^p \geq B^p$$

for $p > 1$.

Lin [10] showed that a reverse version of the operator AM-GM inequality can be squared: for $0 < mI \leq A, B \leq MI$,

$$\Phi^2(\tfrac{A+B}{2}) \leq K^2(h,2)\Phi^2(A\sharp B) \tag{1.1}$$

and

$$\Phi^2(\tfrac{A+B}{2}) \leq K^2(h,2)(\Phi(A)\sharp\Phi(B))^2, \tag{1.2}$$

where Φ is a unital positive linear map and $K(h,2) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$.

Zhang [14] generalized (1.1) and (1.2) when $p \geq 2$:

$$\Phi^{2p}(\tfrac{A+B}{2}) \leq \frac{(K(h,2)(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}\Phi^{2p}(A\sharp B) \tag{1.3}$$

and

$$\Phi^{2p}(\tfrac{A+B}{2}) \leq \frac{(K(h,2)(M^2+m^2))^{2p}}{16M^{2p}m^{2p}}(\Phi(A)\sharp\Phi(B))^{2p}. \tag{1.4}$$

Moradi et. al. [13] obtained a better bound than (1.1) and (1.2) as follows: for $0 < mI \leq A \leq m'I \leq M'I \leq B \leq MI$,

$$\Phi^2(\tfrac{A+B}{2}) \leq \frac{K^2(h,2)}{(1+\frac{(\log\frac{M'}{m'})^2}{8})^2}\Phi^2(A\sharp B) \tag{1.5}$$

and

$$\Phi^2(\tfrac{A+B}{2}) \leq \frac{K^2(h,2)}{(1+\frac{(\log\frac{M'}{m'})^2}{8})^2}(\Phi(A)\sharp\Phi(B))^2, \tag{1.6}$$

where Φ is a unital positive linear map and $K(h,2) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$.

Let $0 < mI \leq A \leq MI$ and Φ be a positive unital linear map. Lin [11] proved the following operator inequalities:

$$|\Phi(A^{-1})\Phi(A) + \Phi(A)\Phi(A^{-1})| \leq \frac{(M+m)^2}{2Mm}I \tag{1.7}$$

and

$$\Phi(A^{-1})\Phi(A) + \Phi(A)\Phi(A^{-1}) \leq \frac{(M+m)^2}{2Mm}I. \tag{1.8}$$

Fu [6] generalized (1.7) and (1.8) when $p \geq 1$:

$$|\Phi^p(A^{-1})\Phi^p(A) + \Phi^p(A)\Phi^p(A^{-1})| \leq \frac{(M+m)^{2p}}{2M^pm^p}I \tag{1.9}$$

and

$$\Phi^p(A^{-1})\Phi^p(A) + \Phi^p(A)\Phi^p(A^{-1}) \leq \frac{(M+m)^{2p}}{2M^pm^p}I \tag{1.10}$$

Bhatia and Davis [3] gave an operator version of Wielandt inequality and proved that if $0 < m \leq A \leq M$ and X, Y are two partial isometries on $\mathcal{H}$ whose final spaces are orthogonal to each other. Then for every 2-positive linear map Φ,

$$\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX) \leq (\frac{M-m}{M+m})^2\Phi(X^*AX).$$

Lin [11] conjectured that the following inequality could be true:

$$\|\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX)\Phi^{-1}(X^*AX)\| \leq (\frac{M-m}{M+m})^2. \tag{1.11}$$

Gumus [7] obtained a close upper bound to approximate the right side of (1.11) as follows:

$$\|\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX)\Phi^{-1}(X^*AX)\| \leq \frac{(M-m)^2}{2\sqrt{Mm}(M+m)}. \tag{1.12}$$

Moradi et. al. [13] refined (1.12) as follows: for $0 < mI \leq m'A^{-1} \leq A \leq MI$ and $m' > 1$,

$$\|\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX)\Phi^{-1}(X^*AX)\| \leq \frac{(M-m)^2}{2\sqrt{Mm}(M+m)(1+\frac{(\log m')^2}{8})}, \tag{1.13}$$

where X and Y are two isometries such that $X^*Y = 0$, Φ is an arbitrary 2-positive linear map.

Liao et. al. [12] also gave a close upper bound to approximate the right side of (1.11) below:

$$\|(\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(X^*AX)\| \leq \frac{(M-m)^p(M^\alpha+m^\alpha)^{\frac{p}{\alpha}}}{2^{2+\frac{p}{2}}(Mm)^{\frac{3}{4}}(M+m)^{\frac{p}{2}}} \tag{1.14}$$

for $1 \leq \alpha \leq 2$ and $p \geq 2\alpha$.

Recently, Kórus [9] gave a scalar inequality as follows:

$$(1 + Q(t)(\log a - \log b)^2)a^t b^{1-t} \leq ta + (1-t)b, \tag{1.15}$$

where $t \in [0, 1]$, $a, b > 0$, $Q(t) = \frac{t^2}{2}(\frac{1-t}{t})^{2t}$ and $Q(0) = Q(1) = 0$.

In this paper, we shall give some improvements of the inequalities mentioned above.

2. Main Results

Before we give the main results, let us present the following lemmas that will be useful later.

Lemma 2.1.(Choi inequality.) [4, p. 41] Let Φ be a unital positive linear map, then

(1) If $A > 0$ and $-1 \leq p \leq 0$, then $\Phi(A)^p \leq \Phi(A^p)$, in particular, $\Phi(A)^{-1} \leq \Phi(A^{-1})$;
(2) If $A \geq 0$ and $0 \leq p \leq 1$, then $\Phi(A)^p \geq \Phi(A^p)$;
(3) If $A \geq 0$ and $1 \leq p \leq 2$, then $\Phi(A)^p \leq \Phi(A^p)$.

Lemma 2.2. [2] Let Φ be a unital positive linear map and A, B be positive operators. Then for $\alpha \in [0, 1]$

$$\Phi(A\sharp_\alpha B) \leq \Phi(A)\sharp_\alpha\Phi(B).$$

Lemma 2.3. [5] Let $A, B \geq 0$. Then the following norm inequality holds:

$$\|AB\| \leq \frac{1}{4}\|A + B\|^2.$$

Lemma 2.4. [4, p. 28] Let $A, B \geq 0$. Then for $1 \leq r < +\infty$,

$$\|A^r + B^r\| \leq \|(A+B)^r\|.$$

Lemma 2.5. Let $A, B \in B(\mathcal{H})$ be two positive operators such $1 < m < M$ with the property $mA \leq B \leq MA$. Then

$$(1 + Q(t)(\log m)^2)A\sharp_t B \leq A\nabla_t B$$

for $t \in [0,1]$ and $Q(t)$ is from (1.15).

Proof. From the inequality (1.15), we know that for each $a, b > 0$ and $t \in [0,1]$,

$$(1 + Q(t)(\log a - \log b)^2)a^t b^{1-t} \leq ta + (1-t)b.$$

Note that if $0 < mb \leq a \leq Mb$ with $1 < m < M$, then by the monotonicity of logarithm function we obtain

$$(1 + Q(t)(\log m)^2)a^t b^{1-t} \leq ta + (1-t)b.$$

Taking $b = 1$ in the above inequality, we have

$$(1 + Q(t)(\log m)^2)a^t \leq ta + (1-t).$$

As $mI \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq MI$, on choosing a with the positive operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ in the above inequality, we obtain

$$(1 + Q(t)(\log m)^2)(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t \leq t(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) + (1-t)I.$$

Multiplying both side by $A^{\frac{1}{2}}$ yields the desired result. □

Lemma 2.6. Let $0 < mI \leq A \leq m'I \leq M'I \leq B \leq MI$ and $t \in [0,1]$. Then

$$A\nabla_t B + Mm(1 + Q(t)(\log \tfrac{M'}{m'})^2)(A\sharp_t B)^{-1} \leq (M+m)I,$$

where $Q(t)$ is from (1.15).

Proof. It is easy to see that

$$(1-t)(MI - A)(mI - A)A^{-1} \leq 0,$$

which is equivalent to

$$(1-t)A + (1-t)MmA^{-1} \leq (1-t)(M+m)I. \tag{2.1}$$

Similarly, we have

$$tB + tMmB^{-1} \leq t(M + m)I. \tag{2.2}$$

Summing up (2.1) and (2.2), we have

$$A\nabla_t B + MmA^{-1}\nabla_t B^{-1} \leq (M + m)I.$$

By $(A\sharp_t B)^{-1} = A^{-1}\sharp_t B^{-1}$ and Lemma 2.5, we have

$$\begin{aligned} A\nabla_t B + Mm(1 + Q(t)(\log \frac{M'}{m'})^2)(A\sharp_t B)^{-1} &= A\nabla_t B + Mm(1 + Q(t)(\log \frac{M'}{m'})^2)(A^{-1}\sharp_t B^{-1}) \\ &\leq A\nabla_t B + MmA^{-1}\nabla_t B^{-1} \\ &\leq (M + m)I, \end{aligned}$$

completing the proof. □

Theorem 2.7. Let $0 < mI \leq A \leq m'I \leq M'I \leq B \leq MI$ and $p \geq 1$. Then for every positive unital linear map Φ,

$$\Phi^{2p}(A\nabla_t B) \leq (\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log \frac{M'}{m'})^2)})^{2p}\Phi^{2p}(A\sharp_t B) \tag{2.3}$$

and

$$\Phi^{2p}(A\nabla_t B) \leq (\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log \frac{M'}{m'})^2)})^{2p}(\Phi(A)\sharp_t\Phi(B))^{2p}, \tag{2.4}$$

where $t \in [0, 1]$, $h = \frac{M}{m}$ and $Q(t)$ is from (1.15).

Proof. By computation, we can obtain

$$\begin{aligned} &\|\Phi^p(A\nabla_t B)M^p m^p(1 + Q(t)(\log \tfrac{M'}{m'})^2)^p\Phi^{-p}(A\sharp_t B)\| \\ &\leq \tfrac{1}{4}\|\Phi^p(A\nabla_t B) + M^p m^p(1 + Q(t)(\log \tfrac{M'}{m'})^2)^p\Phi^{-p}(A\sharp_t B)\|^2 \quad \text{(by Lemma 2.3)} \\ &\leq \tfrac{1}{4}\|\Phi(A\nabla_t B) + Mm(1 + Q(t)(\log \tfrac{M'}{m'})^2)\Phi^{-1}(A\sharp_t B)\|^{2p} \quad \text{(by Lemma 2.4)} \\ &\leq \tfrac{1}{4}\|\Phi(A\nabla_t B) + Mm(1 + Q(t)(\log \tfrac{M'}{m'})^2)\Phi((A\sharp_t B)^{-1})\|^{2p} \quad \text{(by Lemma 2.1)} \\ &\leq \tfrac{1}{4}(M + m)^{2p}. \quad \text{(by Lemma 2.6)} \end{aligned}$$

Thus we obtain

$$\|\Phi^p(A\nabla_t B)\Phi^{-p}(A\sharp_t B)\| \leq (\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log \frac{M'}{m'})^2)})^p,$$

which is equivalent to (2.3).

Next we prove (2.4). Compute

$$\begin{aligned} &\|\Phi^p(A\nabla_t B)M^p m^p(1 + Q(t)(\log \tfrac{M'}{m'})^2)^p(\Phi(A)\sharp_t\Phi(B))^{-p}\| \\ &\leq \tfrac{1}{4}\|\Phi^p(A\nabla_t B) + M^p m^p(1 + Q(t)(\log \tfrac{M'}{m'})^2)^p(\Phi(A)\sharp_t\Phi(B))^{-p}\|^2 \\ &\leq \tfrac{1}{4}\|\Phi(A\nabla_t B) + Mm(1 + Q(t)(\log \tfrac{M'}{m'})^2)(\Phi(A)\sharp_t\Phi(B))^{-1}\|^{2p} \\ &\leq \tfrac{1}{4}\|\Phi(A\nabla_t B) + Mm(1 + Q(t)(\log \tfrac{M'}{m'})^2)(\Phi(A\sharp_t B))^{-1}\|^{2p} \quad \text{(by Lemma 2.2)} \\ &\leq \tfrac{1}{4}\|\Phi(A\nabla_t B) + Mm(1 + Q(t)(\log \tfrac{M'}{m'})^2)\Phi((A\sharp_t B)^{-1})\|^{2p} \\ &\leq \tfrac{1}{4}(M + m)^{2p}. \end{aligned}$$

Thus we obtain

$$\|\Phi^p(A\nabla_t B)(\Phi(A)\sharp_t\Phi(B))^{-p}\| \le \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p,$$

which completes the proof. □

Remark 2.8. Letting $p = 1$ and $t = \frac{1}{2}$ in Theorem 2.7, we thus get (1.5) and (1.6) by (2.3) and (2.4), respectively.

Lemma 2.9. [8] For any bounded operator X,

$$|X| \le tI \Leftrightarrow \|X\| \le t \Leftrightarrow \begin{bmatrix} tI & X \\ X^* & tI \end{bmatrix} \ge 0 \quad (t \ge 0)$$

Theorem 2.10. Let $0 < mI \le A \le m'I \le M'I \le B \le MI$ and $p \ge 1$. Then for every positive unital linear map Φ,

$$|\Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) + \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B)| \le 2\left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p I \tag{2.5}$$

and

$$\Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) + \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B) \le 2\left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p I, \tag{2.6}$$

where $t \in [0,1]$, $h = \frac{M}{m}$ and $Q(t)$ is from (1.15).

Proof. By computation, one can have

$$\begin{aligned}
&\|\Phi^p(A\nabla_t B)M^p m^p(1 + Q(t)(\log\tfrac{M'}{m'})^2)^p\Phi^p((A\sharp_t B)^{-1})\| \\
&\le \tfrac{1}{4}\|\Phi^p(A\nabla_t B) + M^p m^p(1 + Q(t)(\log\tfrac{M'}{m'})^2)^p\Phi^p((A\sharp_t B)^{-1})\|^2 \quad \text{(by Lemma 2.3)} \\
&\le \tfrac{1}{4}\|\Phi(A\nabla_t B) + Mm(1 + Q(t)(\log\tfrac{M'}{m'})^2)\Phi((A\sharp_t B)^{-1})\|^{2p} \quad \text{(by Lemma 2.4)} \\
&\le \tfrac{1}{4}(M+m)^{2p}, \quad \text{(by Lemma 2.6)}
\end{aligned}$$

which is equivalent to

$$\|\Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1})\| \le \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p. \tag{2.7}$$

By (2.7) and Lemma 2.9 we obtain

$$\begin{bmatrix} \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p I & \Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) \\ \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B) & \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p I \end{bmatrix} \ge 0$$

and

$$\begin{bmatrix} \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p I & \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B) \\ \Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) & \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p I \end{bmatrix} \ge 0.$$

Summing up the two operator matrices above, we get

$$\begin{bmatrix} 2\left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p I & X \\ X^* & 2\left(\frac{K(h,2)}{4^{\frac{1}{p}-1}(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^p I \end{bmatrix} \ge 0,$$

where we denote that $X = \Phi^p(A\nabla_t B)\Phi^p((A\sharp_t B)^{-1}) + \Phi^p((A\sharp_t B)^{-1})\Phi^p(A\nabla_t B)$. It is easy to see that X is self-adjoint. Utilizing Lemma 2.9 again, we thus obtain (2.5) and (2.6). □

Remark 2.11. Putting $t = 0$ in Theorem 2.10, we obtain (1.9) and (1.10) by (2.5) and (2.6), respectively.

Next, we give improvements of (1.3) and (1.4).

Theorem 2.12. Let $0 < mI \le A \le m'I \le M'I \le B \le MI$ and $p \ge 2$. Then for every positive unital linear map Φ,

$$\Phi^{2p}(A\nabla_t B)\frac{(K(h,2)(M^2+m^2))^{2p}}{16M^{2p}m^{2p}(1+Q(t)(\log\frac{M'}{m'})^2)^{2p}}\Phi^{2p}(A\sharp_t B) \tag{2.8}$$

and

$$\Phi^{2p}(A\nabla_t B) \le \frac{(K(h,2)(M^2+m^2))^{2p}}{16M^{2p}m^{2p}(1+Q(t)(\log\frac{M'}{m'})^2)^{2p}}(\Phi(A)\sharp_t\Phi(B))^{2p}, \tag{2.9}$$

where $t \in [0,1]$, $h = \frac{M}{m}$ and $Q(t)$ is from (1.15).

Proof. It is easy to to verify that

$$mI \le \Phi(A\nabla_t B) \le MI.$$

Thus we obtain

$$m^2 I \le \Phi^2(A\nabla_t B) \le M^2 I.$$

Therefore

$$(M^2 I - \Phi^2(A\nabla_t B))(m^2 - \Phi^2(A\nabla_t B))\Phi^{-2}(A\nabla_t B) \le 0.$$

That is equivalent to

$$M^2m^2\Phi^{-2}(A\nabla_t B) + \Phi^2(A\nabla_t B) \le (M^2+m^2)I. \tag{2.10}$$

Taking $p = 1$ in (2.3), we have

$$\Phi^2(A\nabla_t B) \le \left(\frac{K(h,2)}{(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^2\Phi^2(A\sharp_t B),$$

which is equivalent to

$$\Phi^{-2}(A\sharp_t B) \le \left(\frac{K(h,2)}{(1+Q(t)(\log\frac{M'}{m'})^2)}\right)^2\Phi^{-2}(A\nabla_t B). \tag{2.11}$$

Thus we compute

$$\begin{aligned}
&\|\Phi^p(A\nabla_t B)M^p m^p\Phi^{-p}(A\sharp_t B)\| \\
&\le \frac{1}{4}\left\|\frac{K^{\frac{p}{2}}(h,2)}{(1+Q(t)(\log\frac{M'}{m'})^2)^{\frac{p}{2}}}\Phi^p(A\nabla_t B) + \frac{(1+Q(t)(\log\frac{M'}{m'})^2)^{\frac{p}{2}}M^p m^p}{K^{\frac{p}{2}}(h,2)}\Phi^{-p}(A\sharp_t B)\right\|^2 \\
&\le \frac{1}{4}\left\|\frac{K(h,2)}{(1+Q(t)(\log\frac{M'}{m'})^2)}\Phi^2(A\nabla_t B) + \frac{(1+Q(t)(\log\frac{M'}{m'})^2)M^2m^2}{K(h,2)}\Phi^{-2}(A\sharp_t B)\right\|^p \\
&\le \frac{1}{4}\left\|\frac{K(h,2)}{(1+Q(t)(\log\frac{M'}{m'})^2)}\Phi^2(A\nabla_t B) + \frac{K(h,2)M^2m^2}{(1+Q(t)(\log\frac{M'}{m'})^2)}\Phi^{-2}(A\nabla_t B)\right\|^p \quad \text{(by (2.11))} \\
&= \frac{1}{4}\left\|\frac{K(h,2)}{(1+Q(t)(\log\frac{M'}{m'})^2)}(\Phi^2(A\nabla_t B) + M^2m^2\Phi^{-2}(A\nabla_t B))\right\|^p \\
&\le \frac{1}{4}\frac{(K(h,2)(M^2+m^2))^p}{(1+Q(t)(\log\frac{M'}{m'})^2)^p}, \quad \text{(by (2.10))}
\end{aligned}$$

which is equivalent to (2.8). The proof of (2.9) is similar, we omit the details. □

Theorem 2.13. Let $0 < mI \leq m'A^{-1} \leq A \leq MI$ and $m' > 1$ and let X and Y be two isometries such that $X^*Y = 0$. For every 2-positive linear map Φ, we have

$$||(\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(X^*AX)|| \leq \frac{(M-m)^p(M^\alpha+m^\alpha)^{\frac{p}{\alpha}}}{2^{2+\frac{p}{2}}(Mm)^{\frac{3}{4}}(M+m)^{\frac{p}{2}}(1+\frac{(\log m')^2}{8})^{\frac{p}{2}}} \tag{2.12}$$

for $1 \leq \alpha \leq 2$ and $p \geq 2\alpha$.

Proof. By (1.13), we obtain

$$(\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX))^2 \leq (\frac{(M-m)^2}{2\sqrt{Mm}(M+m)(1+\frac{(\log m')^2}{8})})^2\Phi^2(X^*AX),$$

By L-H inequality [4, p. 112], we have

$$(\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX))^\alpha \leq (\frac{(M-m)^2}{2\sqrt{Mm}(M+m)(1+\frac{(\log m')^2}{8})})^\alpha\Phi^\alpha(X^*AX).$$

Thus we get

$$\begin{aligned}
&||\frac{(M-m)^p}{2^{\frac{p}{2}}(M+m)^{\frac{p}{2}}(1+\frac{(\log m')^2}{8})^{\frac{p}{2}}}M^{\frac{p}{4}}m^{\frac{p}{4}}(\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX))^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(X^*AX)|| \\
&\leq \frac{1}{4}||(\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX))^{\frac{p}{2}} + (\frac{(M-m)^2}{2(M+m)(1+\frac{(\log m')^2}{8})}\sqrt{Mm}\Phi^{-1}(X^*AX))^{\frac{p}{2}}||^2 \\
&\leq \frac{1}{4}||(\Phi(X^*AY)\Phi^{-1}(Y^*AY)\Phi(Y^*AX))^\alpha + \frac{(M-m)^{2\alpha}}{2^\alpha(M+m)^\alpha(1+\frac{(\log m')^2}{8})^\alpha}M^{\frac{\alpha}{2}}m^{\frac{\alpha}{2}}\Phi^{-\alpha}(X^*AX))||^{\frac{p}{\alpha}} \\
&\leq \frac{1}{4}||(\frac{(M-m)^2}{2\sqrt{Mm}(M+m)(1+\frac{(\log m')^2}{8})})^\alpha\Phi^\alpha(X^*AX) + \frac{(M-m)^{2\alpha}}{2^\alpha(M+m)^\alpha(1+\frac{(\log m')^2}{8})^\alpha}M^{\frac{\alpha}{2}}m^{\frac{\alpha}{2}}\Phi^{-\alpha}(X^*AX))||^{\frac{p}{\alpha}} \\
&= \frac{(M-m)^{2p}}{2^{2+p}M^{\frac{p}{2}}m^{\frac{p}{2}}(M+m)^p(1+\frac{(\log m')^2}{8})^p}||\Phi^\alpha(X^*AX) + M^\alpha m^\alpha\Phi^{-\alpha}(X^*AX))||^{\frac{p}{\alpha}} \\
&\leq \frac{(M-m)^{2p}(M^\alpha+m^\alpha)^{\frac{p}{\alpha}}}{2^{2+p}M^{\frac{p}{2}}m^{\frac{p}{2}}(M+m)^p(1+\frac{(\log m')^2}{8})^p},
\end{aligned}$$

which completes the proof. □

Based on (2.12), we thus get an improvement of (1.14).

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