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Solution of Volterra Integral Equation in Metric Spaces via New Fixed Point Theorem

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Abstract. The aim of this article is to study the existence of coincidences and fixed points of generalized hybrid contractions involving single-valued mappings and left total relations in the context of complete metric spaces. Some special cases are also discussed to derive some well known results of the literature. Finally, some examples and applications are also presented to verify the effectiveness and applicability of our main results.

1. Introduction and Preliminaries

One of the simplest and most useful results in fixed point theory is the Banach contraction principle [9], a powerful tool in analysis for establishing existence and uniqueness of solution of problems in different fields. Over the years, this principle has been generalized in numerous directions in different spaces. These generalizations have been obtained either by extending the domain of the mapping or by considering a more general contractive condition on the mappings.

Very recently, Jleli and Samet [24] introduced a new type of contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.1. *Let $\psi : (0, \infty) \to (1, \infty)$ be a function satisfying:*

(ψ_1) ψ is nondecreasing;

(ψ_2) for each sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \psi(\alpha_n) = 1$ if and only if $\lim_{n\to\infty}(\alpha_n) = 0$;

(ψ_3) there exists $0 < k < 1$ and $l \in (0, \infty]$ such that $\lim_{\alpha\to 0^+} \frac{\psi(\alpha)-1}{\alpha^k} = l$.

A mapping $F : X \to X$ is said to be JS-contraction if there exist the function ψ satisfying (ψ_1)-(ψ_3) and a constant $\alpha \in (0, 1)$ such that for all $x, y \in X$,

$$d(Fx, Fy) \neq 0 \Longrightarrow \psi(d(Fx, Fy)) \leq [\psi(d(x, y))]^{\alpha}. \tag{1.1}$$

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Theorem 1.2. *[24] Let (X, d) be a complete metric space and $F : X \to X$ be a JS-contraction, then F has a unique fixed point.*

To be consistent with Samet et al. [24], we denote by Ψ the set of all functions $\psi : (0, \infty) \to (1, \infty)$ satisfying the above conditions.

Hussain et al. [18] modified and extended the above result and proved the following fixed point theorem for ψ-contractive condition in the setting of complete metric spaces.

Theorem 1.3. *[18] Let (X, d) be a complete metric space and $F : X \to X$ be a self-mapping. If there exist a function $\psi \in \Psi$ and positive real numbers $\alpha, \beta, \gamma, \delta$ with $0 \leq \alpha + \beta + \gamma + 2\delta < 1$ such that*

$$\psi(d(Fx, Fy)) \leq [\psi(d(x, y))]^{\alpha} \cdot [\psi(d(x, Fx))]^{\beta} \cdot [\psi(d(y, Fy))]^{\gamma} \cdot [\psi((d(x, Fy) + d(y, Fx))]^{\delta} \tag{1.2}$$

for all $x, y \in X$, then F has a unique fixed point.

Hybrid fixed point theory is a recent growth in the scope of fixed point theorems for contracting single-valued and multivalued mappings in metric spaces. Indeed, the study of such mappings was initiated during 1980-90 by Beg et al. [10], Hadzic [15], Kaneko [25], Kubiak [26], Azam [8] and Hussain et al.[16]. Functional inclusions, optimization theory, fractal graphics and discrete dynamics for set-valued operators are the fields in which hybrid fixed point theory has potential applications. For more details in this direction, we refer the reader to (see [3, 4, 6, 11–14, 17, 20–23, 27]).

Let A and B be arbitrary nonempty sets. A relation R from A to B is a subset of $A \times B$ and is denoted by $R : A \rightsquigarrow B$. The statement $(x, y) \in R$ is read "x is R-related to y", and is denoted by xRy. A relation $R : A \rightsquigarrow B$ is called left-total if for all $x \in A$ there exists a $y \in B$ such that xRy that is R is a multivalued function. A relation $R : A \rightsquigarrow B$ is called right-total if for all $y \in B$ there exists an $x \in A$ such that xRy. A relation $R : A \rightsquigarrow B$ is known as functional, if xRy, xRz implies that $y = z$, for $x \in A$ and $y, z \in B$. A mapping $F : A \to B$ is a relation from A to B which is both functional and left-total.

For $R : A \rightsquigarrow B$, $E \subset A$ we define

$$R(E) = \{y \in B : xRy \text{ for some } x \in E\}.$$

$$\text{dom}(R) = \left\{x \in A : R(\{x\}) \neq \phi\right\},$$

$$\text{Range}(R) = \{y \in B : y \in R(\{x\}) \text{ for some } x \in \text{dom}(R)\}.$$

For convenience, we denote $R(\{x\})$ by $R\{x\}$. The class of relations from A to B is denoted by $\mathcal{R}(A, B)$. Thus the collection $\mathcal{M}(A, B)$of all mappings from A to B is a proper sub collection of $\mathcal{R}(A, B)$. An element $w \in A$ is called coincidence point of $F : A \to B$ and $R : A \rightsquigarrow B$ if $Fw \in R\{w\}$. In the following we always suppose that X is nonempty set and (Y, d) is a metric space. For $R : X \rightsquigarrow Y$ and $u, v \in \text{dom}(R)$, we define

$$D(R\{u\}, R\{v\}) = \inf_{uRx, vRy} d(x, y).$$

The aim of this paper is to prove coincidence fixed point results of a pair of self mappings and left total relation satisfying a generalized ψ-contractive condition in the framework of complete metric spaces.

2. Main Results

Now we state and prove our main results of this section.

Theorem 2.1. *Let X be a nonempty set and (Y, d) be a metric space. Let $F : X \to Y$ be single-valued mapping, $R : X \rightsquigarrow Y$ be such that R is left-total, Range(F) $\subseteq$ Range(R) and Range(F) or Range(R) is complete. If there exist a mapping $\psi \in \Psi$ and a constant $k \in (0, 1)$ such that*

$$\psi(d(Fx, Fy)) \leq [\psi(D(R\{x\}, R\{y\}))]^{k} \tag{2.1}$$

for all $x, y \in X$. Then there exists $w \in X$ such that $Fw \in R\{w\}$.

Proof. Let $x_0 \in X$ be an arbitrary, but fixed element. We define the sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Range(R)$. Let $y_1 = Fx_0$, since $Range(F) \subseteq Range(R)$. We may choose $x_1 \in X$ such that $x_1 R y_1$, since R is left-total. Let $y_2 = Fx_1$, since $Range(F) \subseteq Range(R)$. If $Fx_0 = Fx_1$, then we have $x_1 R y_2$. This implies that x_1 is the required point that is $Fx_1 \in R\{x_1\}$. So we assume that $Fx_0 \neq Fx_1$, then from (2.1) we get

$$1 < \psi(d(y_1, y_2)) = \psi(d(Fx_0, Fx_1)) \leq [\psi(D(R\{x_0\}, R\{x_1\}))]^k. \tag{2.2}$$

We may choose $x_2 \in X$ such that $x_2 R y_2$, since R is left-total. Let $y_3 = Fx_2$, since $Range(F) \subseteq Range(R)$. If $Fx_1 = Fx_2$, then we have $x_2 R y_3$. This implies that $Fx_2 \in R\{x_2\}$ and x_2 is the coincidence point. So $Fx_1 \neq Fx_2$, then from (2.1), we get

$$1 < \psi(d(y_2, y_3)) = \psi(d(Fx_1, Fx_2)) \leq [\psi(D(R\{x_1\}, R\{x_2\}))]^k. \tag{2.3}$$

By induction, we can construct sequences $\{x_n\} \subset X$ and $\{y_n\} \subset Range(R)$ such that

$$y_n = Fx_{n-1} \text{ and } x_n R y_n \tag{2.4}$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $Fx_{n_0-1} = Fx_{n_0}$. Then $x_{n_0} R y_{n_0+1}$. Thus $Fx_{n_0} \in R\{x_{n_0}\}$ and the proof is finished. So we suppose now that $Fx_{n-1} \neq Fx_n$ for every $n \in \mathbb{N}$. Then from (2.2),(2.3) and (2.4), we deduce that

$$1 < \psi(d(y_n, y_{n+1})) = \psi(d(Fx_{n-1}, Fx_n)) \leq [\psi(D(R\{x_{n-1}\}, R\{x_n\}))]^k \tag{2.5}$$

for all $n \in \mathbb{N}$. Since $x_n R y_n$ and $x_{n+1} R y_{n+1}$, therefore by the definition of D, we get $D(R\{x_{n-1}\}, R\{x_n\}) \leq d(y_{n-1}, y_n)$.Thus from (2.5), we have

$$1 < \psi(d(y_n, y_{n+1})) \leq [\psi(d(y_{n-1}, y_n))]^k \tag{2.6}$$

which further implies that

$$1 < \psi(d(y_n, y_{n+1})) \leq [\psi(d(y_{n-1}, y_n))]^k \leq [\psi(d(y_{n-2}, y_{n-1}))]^{k^2} \leq ... \leq [\psi(d(y_0, y_1))]^{k^n}. \tag{2.7}$$

From (2.7), we obtain

$$\lim_{n\to\infty} \psi(d(y_n, y_{n+1})) = 1. \tag{2.8}$$

Then from (ψ_2), we get

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.9}$$

From the condition (ψ_3), there exist $0 < k < 1$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\psi(d(y_n, y_{n+1})) - 1}{d(y_n, y_{n+1})^k} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$|\frac{\psi(d(y_n, y_{n+1})) - 1}{d(y_n, y_{n+1})^k} - l| \leq B$$

for all $n > n_1$. This implies that

$$\frac{\psi(d(y_n, y_{n+1})) - 1}{d(y_n, y_{n+1})^k} \geq l - B = \frac{l}{2} = B$$

for all $n > n_1$. Then

$$n(d(y_n, y_{n+1}))^k \leq An[\psi(d(y_n, y_{n+1})) - 1]$$

for all $n > n_1$, where $A = \frac{1}{B}$. Now we suppose that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$B \leq \frac{\psi(d(y_n, y_{n+1})) - 1}{(d(y_n, y_{n+1}))^k}]$$

for all $n > n_1$. This implies that

$$n(d(y_n, y_{n+1}))^k \leq An[\psi(d(y_n, y_{n+1})) - 1]$$

for all $n > n_1$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_1 \in \mathbb{N}$ such that

$$n(d(y_n, y_{n+1}))^k \leq An[\psi(d(y_n, y_{n+1})) - 1]$$

for all $n > n_1$. Thus by (2.7), we get

$$n(d(y_n, y_{n+1}))^k \leq An([\psi(d(y_0, y_1))]^{k^n} - 1).$$

Letting $n \to \infty$ in the above inequality, we obtain

$$\lim_{n\to\infty} n(d(y_n, y_{n+1}))^k = 0.$$

Thus, there exists $n_2 \in \mathbb{N}$ such that

$$d(y_n, y_{n+1}) \leq \frac{1}{n^{1/k}} \tag{2.10}$$

for all $n > n_2$. Now we prove that $\{y_n\}$ is a Cauchy sequence. For $m > n > n_2$ we have,

$$d(y_n, y_m) \leq \sum_{i=n}^{m-1} d(y_i, y_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}. \tag{2.11}$$

Since, $0 < k < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore, $d(y_n, y_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{y_n\}$ is a Cauchy sequence in $Range(R)$. Completeness of $Range(R)$ ensures that there exist $z \in Range(R)$ such that, $y_n \to z$ as $n \to \infty$. Now since R is left-total, so wRz for some $w \in X$. Now

$$\begin{aligned} 1 &< \psi(d(y_n, Fw)) = \psi(d(Fx_{n-1}, Fw)) \leq [\psi(D(R\{x_{n-1}\}, R\{w\}))]^k \\ &\leq [\psi(d(y_{n-1}, z))]^k. \end{aligned}$$

Since $\lim_{n\to\infty} d(y_{n-1}, z) = 0$, so by (ψ_2), we have $\lim_{n\to\infty} \psi(d(y_{n-1}, z)) = 1$. This implies that $\lim_{n\to\infty} \psi(d(y_n, Fw)) = 1$, which further implies that $\lim_{n\to\infty} d(y_n, Fw) = 0$. Hence $d(z, Fw) = 0$. It follows that $z = Fw$. Hence $Fw \in R\{w\}$. In the case when $Range(F)$ is complete. Since $Range(F) \subseteq Range(R)$, so there exists an element $z^* \in Range(R)$ such that $y_n \to z^*$. The remaining part of the proof is same as in previous case. □

Example 2.2. *Let* $X = Y = \mathbb{R}$, $d(x, y) = \left|x - y\right|$. *Define* $F : \mathbb{R} \to \mathbb{R}$, $R : \mathbb{R} \rightsquigarrow \mathbb{R}$ *as follows:*

$$Fx = \begin{cases} 0 & \text{if } x \in \mathbb{Q}' \\ 1 & \text{if } x \in \mathbb{Q}, \end{cases}$$

$$R = (\mathbb{Q} \times [0, 3]) \cup (\mathbb{Q}' \times [7, 9])$$

Then $\text{Range}(F) = \{0, 1\} \subset \text{Range}(R) = [0, 3] \cup [7, 9]$. *Let* $\psi(t) = e^{\sqrt{t}}$.

For $x \in \mathbb{Q}$, $y \in \mathbb{Q}'$ or either $y \in \mathbb{Q}$, $x \in \mathbb{Q}'$, we have $d(Fx, Fy) \neq 0$ implies

$$\psi(d(Fx, Fy)) \leq [\psi(D(R\{x\}, R\{y\}))]^k$$

with $k = \frac{1}{2}$. Thus all conditions of the above theorem are satisfied and 1 is the coincidence point of F and R.
From Theorem 2.1, we deduce the following result immediately.

Theorem 2.3. *Let X be a nonempty set and (Y, d) be a metric space. Let $F, R : X \to Y$ be two mappings such that $Range(F) \subseteq Range(R)$ and $Range(F)$ or $Range(R)$ is complete. If there exist a mapping $\psi \in \Psi$ and a constant $k \in (0, 1)$ such that*

$$\psi(d(Fx, Fy)) \leq [\psi(d(Rx, Ry))]^k$$

for all $x, y \in X$. Then F and R have a coincidence point in X. Moreover, if either F or R is injective, then R and F have a unique coincidence point in X.

Proof. By Theorem 2.1, we obtain that there exists $w \in X$ such that $Fw = Rw$, where,

$$Rw = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Fx_{n-1}, x_0 \in X.$$

For uniqueness, assume that $w_1, w_2 \in X$, $w_1 \neq w_2$, $Fw_1 = Rw_1$ and $Fw_2 = Rw_2$. Then

$$1 < \psi(d(Fw_1, Fw_2)) \leq [\psi(d(Rw_1, Rw_2))]^k$$

for any $k \in (0, 1)$. If R or F is injective, then

$$d(Rw_1, Rw_2) > 0$$

and

$$1 < \psi(d(Rw_1, Rw_2)) = \psi(d(Fw_1, Fw_2)) \leq [\psi(d(Rw_1, Rw_2))]^k < \psi(d(Rw_1, Rw_2)),$$

a contradiction. Thus proved. □

Corollary 2.4. *[24] Let (X, d) be a complete metric space and $F : X \to X$ be a self mapping. If there exist a function ψ and a constant $k \in (0, 1)$ such that for all $x, y \in X$, ,*

$$d(Fx, Fy) \neq 0 \Longrightarrow \psi(d(Fx, Fy)) \leq [\psi(d(x, y))]^k.$$

then F has a unique fixed point.

Proof. Choosing $X = Y$ and $R = I$ (the identity mapping on X). □

Corollary 2.5. *Let $F : X \to Y$, $R : X \rightsquigarrow Y$ be such that R is left-total,* $\mathrm{Range}(F) \subseteq \mathrm{Range}(R)$ *and* $\mathrm{Range}(F)$ *or* $\mathrm{Range}(R)$ *is complete. If there exists some $k \in [0, 1)$ such that for all $x, y \in X$*

$$d(Fx, Fy) \leq kD(R\{x\}, R\{y\}).$$

Then there exists $w \in X$ such that $Fw \in R\{w\}$.

Proof. Consider the mapping $\psi(t) = e^{\sqrt{t}}$, for $t > 0$. Then obviously ψ satisfies (ψ_1)-(ψ_3). From Theorem 2.1, we obtain the desired conclusion. □

Corollary 2.6. *Let X be nonempty set and (Y, d) be a metric space. $F, R : X \to Y$ be two mappings such that* $\mathrm{Range}(F) \subseteq \mathrm{Range}(R)$ *and* $\mathrm{Range}(F)$ *or* $\mathrm{Range}(R)$ *is complete. If there exists some $k \in [0, 1)$ such that for all $x, y \in X$*

$$d(Fx, Fy) \leq kd(Rx, Ry).$$

Then R and F have a coincidence point in X. Moreover, if either F or R is injective, then R and F have a unique coincidence point in X .

Proof. Consider the mapping $\psi(t) = e^{\sqrt{t}}$, for $t > 0$. Then obviously F satisfies (ψ_1)-(ψ_3). From Theorem 2.3, we obtain the desired conclusion. □

Remark 2.7. *If in the above Corollary we choose $X = Y$, and $R = I$ (the identity mapping on X), we obtain the Banach contraction theorem.*

Note that the family Ψ consists of a large class of functions. For example, if we take

$$\psi(t) = 2 - \frac{2}{\pi}\arctan(\frac{1}{t^{\beta}})$$

where $0 < \beta < 1$ and $t > 0$, we can obtain the following result from our main Theorem.

Theorem 2.8. *Let X be a nonempty set and (Y, d) be a metric space. Let $F : X \to Y$ be single-valued mapping, $R : X \rightsquigarrow Y$ be such that R is left-total, $Range(F) \subseteq Range(R)$ and $Range(F)$ or $Range(R)$ is complete. If there exist a mapping $\psi \in \Psi$ and a constant $\beta, k \in (0, 1)$ such that*

$$2 - \frac{2}{\pi}\arctan(\frac{1}{d(Fx, Fy)^{\beta}}) \leq [2 - \frac{2}{\pi}\arctan(\frac{1}{D(R\{x\}, R\{y\})^{\beta}})]^{k}$$

for all $x, y \in X$. Then there exists $w \in X$ such that $Fw \in R\{w\}$.

Theorem 2.9. *Let X be a nonempty set and (Y, d) be a metric space. Let $F, R : X \to Y$ be two mappings such that $Range(F) \subseteq Range(R)$ and $Range(F)$ or $Range(R)$ is complete. If there exist a mapping $\psi \in \Psi$ and a constant $\beta, k \in (0, 1)$ such that*

$$2 - \frac{2}{\pi}\arctan(\frac{1}{d(Fx, Fy)^{\beta}}) \leq [2 - \frac{2}{\pi}\arctan(\frac{1}{d(Rx, Ry)^{\beta}})]^{k}$$

for all $x, y \in X$. Then F and R have a coincidence point in X. Moreover, if either F or R is injective, then R and F have a unique coincidence point in X.

Corollary 2.10. *Let (X, d) be a complete metric space. Let $F : X \to X$ be a self mapping. If there exist a mapping $\psi \in \Psi$ and a constant $\beta, k \in (0, 1)$ such that*

$$2 - \frac{2}{\pi}\arctan(\frac{1}{d(Fx, Fy)^{\beta}}) \leq [2 - \frac{2}{\pi}\arctan(\frac{1}{d(x, y)^{\beta}})]^{k}$$

for all $x, y \in X$. Then F has a unique fixed point in X.

Example 2.11. *Consider the sequence*

$S_1 = 1 \times 2$
$S_2 = 1 \times 2 + 3 \times 4$
$S_3 = 1 \times 2 + 3 \times 4 + 5 \times 6$
$S_n = 1 \times 2 + 3 \times 4 + \ldots + (2n - 1)(2n) = \frac{n(n+1)(4n-1)}{3}$.

Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x, y) = \left|x - y\right|$. Then (X, d) is a complete metric space. Define the mapping $F : X \to X$ by,

$$F(S_1) = S_1, \quad F(S_n) = S_{n-1}, \quad \text{for all } n \geq 2.$$

Clearly, the Banach contraction is not satisfied. In fact, we can check easily that

$$\lim_{n\to\infty}\frac{d(F(S_n), F(S_1))}{d(S_n, S_1)} = 1$$

Let us consider the mapping $\psi : (0,\infty) \to (1,\infty)$ defined by

$$\psi(t) = e^{\sqrt{te^t}}.$$

We can easily show that $\psi \in \Psi$. Now we shall prove that F satisfies the conditions of Corollary 2.4, that is $d(F(S_n), F(S_m)) \neq 0$ implies that

$$d(F(S_n), F(S_m)) \neq 0 \Longrightarrow e^{\sqrt{d(F(S_n),F(S_m))e^{d(F(S_n),F(S_m))}}} \leq e^{k\sqrt{d(S_n,S_m)e^{d(S_n,S_m)}}}$$

for some $k \in (0,1)$. The above condition is equivalent to

$$d(F(S_n), F(S_m))e^{d(F(S_n),F(S_m))} \leq k^2 d(S_n, S_m)e^{d(S_n,S_m)}.$$

So, we have to check that

$$\frac{d(F(S_n), F(S_m))e^{d(F(S_n),F(S_m))}}{d(S_n, S_m)e^{d(S_n,S_m)}} \leq k^2$$

for some $k \in (0,1)$. We consider two cases,

Case 01. For $1 = n$ and $m > 2$, we have

$$\begin{aligned}
&\frac{d(F(S_1), F(S_m))e^{d(F(S_1),F(S_m))-d(S_1,S_m)}}{d(S_1, S_m)}\\
= \; &\frac{4m^3 - 9m^2 + 5m - 6}{4m^3 + 3m^2 - m - 6}e^{-2(2m-1)m}\\
\leq \; &e^{-1}
\end{aligned}$$

Case 02. For $m > n > 1$, we have

$$\begin{aligned}
&\frac{d(F(S_m), F(S_n))e^{d(F(S_m),F(S_n))-d(S_m,S_n)}}{d(S_m, S_n)}\\
= \; &\frac{(2m-3)(2m-2) + (2n-1)(2n)}{(2m-1)(2m) + (2n+1)(2n+2)}e^{2(2n-1)n-2(2m-1)m}\\
\leq \; &e^{-1}
\end{aligned}$$

with $k = e^{-\frac{1}{2}}$. Thus all the conditions of Corollary 2.4 are satisfied and F has a unique fixed point. In this example S_1 is a unique fixed point of F.

Now we discuss the existence and uniqueness of solution of a general class of the following Volterra type integral equation under various assumptions on the functions involved. Let $C[0,\Theta]$ denote the space of all continuous functions on $[0,\Theta]$, where $\Theta > 0$ and for an arbitrary $\|x\|_\lambda = \sup_{t\in[0,\Theta]}\{|x(t)|\, e^{-\lambda t}\}$, where $\lambda > 0$ is taken arbitrary . Note that $\|\cdot\|_\lambda$ is a norm equivalent to supremum norm and $(C([0,\Theta],\mathbb{R}), \|\cdot\|_\lambda)$ endowed with the metric d_λ defined by

$$d_\lambda(x,y) = \sup_{t\in[0,\Theta]}\{\left|x(t) - y(t)\right| e^{-\lambda t}\}$$

for all $x, y \in C([0,\Theta],\mathbb{R})$ is a Banach space.

Consider the integral equation:

$$(fy)(t) = \int_0^t K(t,s,hx(s))ds + g(t) \tag{2.12}$$

where $x : [0,\Theta] \to \mathbb{R}$ is unknown, $g : [0,\Theta] \to \mathbb{R}$ and $h, f : \mathbb{R} \to \mathbb{R}$ are given functions. The kernel K of the integral equation is defined on $[0,\Theta] \times [0,\Theta] \times \mathbb{R}$.

Theorem 2.12. *Assume that the following conditions are satisfied:*

(i) $K : [0,\Theta] \times [0,\Theta] \times \mathbb{R} \to \mathbb{R}$, $g : [0,\Theta] \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous
(ii) $\int_0^t K(t,s,.) : \mathbb{R} \to \mathbb{R}$ is increasing, for all $t,s \in [0,\Theta]$,
(iii) there exists $\lambda \in (0,+\infty)$ such that

$$|K(t,s,hx(s)) - K(t,s,hy(s)))| \leq \lambda |hx(s) - hy(s)|$$

for all $t,s \in [0,\Theta]$ and $hx, hy \in \mathbb{R}$.
(iv) If f is injective, there exists $k \in (0,1)$ such that for all $x,y \in \mathbb{R}$;

$$\left|hx - hy\right| \leq k^2 \left|fx - fy\right|$$

and $\{fx : x \in \mathbb{R}\}$ is complete. Then there exist $w \in C([0,\Theta],\mathbb{R})$ such that for $x_0 \in \mathbb{R}$,

$$fw(t) = \lim_{n\to\infty} fx_n(t) = \lim_{n\to\infty}\left[g(t) + \int_0^t K(t,s,hx_{n-1}(s))ds]\right]$$

and w is the unique solution of (2.12).

Proof. Let $X = Y = C([0,\Theta],\mathbb{R})$ and

$$d_\lambda(x,y) = \sup_{t\in[0,\Theta]} \{\left|x(t) - y(t)\right| e^{-\lambda t}\}$$

for all $x,y \in X$. Let $F,R : X \to X$ be defined as follows:

$$(Fx)(t) = g(t) + \int_0^t K(t,s,hx(s))ds \text{ and } Rx = fx.$$

Then by assumptions $RX = \{Rx : x \in X\}$ is complete. Let $x^* \in FX$, then $x^* = Fx$ for $x \in X$ and $x^*(t) = Fx(t)$. By assumptions there exists $y \in X$ such that $Fx(t) = fy(t)$, hence $RX \subseteq FX$. Since

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &= \left|\int_0^t [K(t,s,hx(s))]ds - \int_0^t [K(t,s,hy(s))]ds\right| \\
&\leq \int_0^t \left|K(t,s,hx(s)) - K(t,s,hy(s))\right| ds \\
&\leq \int_0^t \lambda \left|hx(s) - hy(s)\right| ds \\
&\leq \int_0^t \lambda k^2 \left|fx(s) - fy(s)\right| ds \\
&= \int_0^t \lambda k^2 \left|(Rx)(s) - (Ry)(s)\right| e^{-\lambda s} e^{\lambda s} ds \\
&\leq \lambda k^2 |\left|Rx - Ry\right||_\lambda \int_0^t e^{\lambda s} ds \\
&\leq \lambda k^2 |\left|Rx - Ry\right||_\lambda \frac{e^{\lambda t}}{\lambda} \\
&= k^2 |\left|Rx - Ry\right||_\lambda e^{\lambda t}.
\end{aligned}$$

This implies that

$$|(Fx)(t) - (Fy)(t)|e^{\lambda t} \leq k^2 \left| \left|(Rx - Ry\right| \right|_\lambda$$

or equivalently,

$$d_\lambda(Fx, Fy) \leq k^2 d_\lambda(Rx, Ry).$$

Taking exponential, we have

$$e^{d_\lambda(Fx,Fy)} \leq e^{k^2 d_\lambda(Rx,Ry)}$$

Now, we observe that mapping $\psi : (0, \infty) \to (1, \infty)$ defined by

$$\psi(t) = e^{\sqrt[k]{t}}.$$

for each $t \in [0, \Theta]$ and $k \in (0, 1)$. Thus all conditions of Theorem 2.1 are satisfied. Hence, there exists a unique $w \in X$ such that

$$fw(t) = \lim_{n\to\infty} Rx_n(t) = \lim_{n\to\infty} Fx_{n-1}(t) = F(w)(t),\ x_0 \in X$$

for all t, which is the unique solution of (2.12). □

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